Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
Vol. 44 (2023), No. 5, pp. 835–843
DOI : 10.47974/JIOS-1126

Calculating different indices of Idempotent graph of ring $\mathbb{Z}_n$

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Abstract

By Idempotent graph of $\mathbb{Z}_n$ we mean the graph whose vertices are all elements of $\mathbb{Z}_n$ such that distinct vertices u and v are adjacent if and only if $u + v$ is an idempotent element of $\mathbb{Z}_n$, denoted by $G_{Id}(\mathbb{Z}_n)$. In this paper, we compute some kind of topological indices of Idempotent graph of $\mathbb{Z}_n$.

Subject Classification: *[2010] 05C25, 92E10.*

Keywords: *Zagreb index, Randić index, GA index, Wiener index.*

1. Introduction

In this paper, G is a simple and undirected graph without loops. We denote the set of vertices and edges, by $V(G)$ and $E(G)$, respectively. Also, d_u, $d_G(u,v)$, K_n and P_n are denoteted to indicate degree of the vertex $u \in G$, the distance between u and v in G, a complete graph of order n and a path graph, respectively. The graph G is connected if there exists a path between every two arbitrary vertices.

Let R be an associative ring with $1 \neq 0$. By $Id(R)$, we mean the set of all idempotents of R. The Idempotent graph of R denoted by $G_{Id}(R)$, as a simple graph with the vertex set equal to R such that two vertices u and v are adjacent if and only if $u + v \in G_{Id}(R)$ ([13]). $G_{Id}(R)$ is a complete graph

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if and only if R is Boolean (a ring that every element of it is idempotent). For given rings R and S, if $R \cong S$ as rings, then $G_{Id}(R) \cong G_{Id}(S)$ as graphs.

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. Topological indices are used for example in the development of quantitative structure-activity relationships (QSARs) in which the biological activity or other properties of molecules are correlated with their chemical structure. The first and second Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajstic [8] and the third Zagreb index was introduced by Fath-Tabar [5]. They are defined as:

$$M_1(G) = \sum_{u \in V(G)} (d_u)^2$$

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v$$

$$M_3(G) = \sum_{uv \in E(G)} | d_u - d_v | .$$

The first Zagreb index can also be expressed as a sum over the edges of G [8]:

$$M_1(G) = \sum_{uv \in E(G)} [d_u + d_v].$$

When the sum of the degrees of the vertices of G run over the edges of the complement of G, we get Zagreb coindices. These were introduced by Doslic in [2] in 2008 to express vertex-weighted Wiener polynomials of composite graphs and are defined by

$$\overline{M_1(G)} = \sum_{uv \not\in E(G)} [d_u + d_v]$$

$$\overline{M_2(G)} = \sum_{uv \not\in E(G)} d_u d_v .$$

There is a common confusion that the Zagreb coindices of G are the Zagreb indices of the complement of G. Indeed the sums run over the edges of G but the degrees are with respect to G.

Gutman [6, 7] has recently proposed to consider the multiplicative variants of Zagreb indices as:

$$\Pi_1(G) = \prod_{u \in V(G)} (d_u)^2$$

$$\Pi_2(G) = \prod_{uv \in E(G)} d_u d_v.$$

The product-connectivity index, also called Randić index $\chi(G)$ of a graph G and is defined such as:

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

This topological index was first proposed by Randić [12] in 1975. The geometric arithmetic index is another topological index based on degrees of vertices defined by Vukičevič and B. Furtula [14]:

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

Recently, Estrada et al. [4] introduced atom-bond connectivity (ABC) index, which has been applied up to study the stability of alkanes and the strain energy of cycloalkanes. This index is defined as follows:

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

A number of paper have been written on calculating some topological indices of graphs [1, 3, 9, 10, 11]. In this paper, we calculate some topological indices of Idempotent graph of $\mathbb{Z}_n$ (The ring of Integers modolo n).

2. Preliminaries

Throughout the article we set $G = G_{Id}(\mathbb{Z}_n)$, $deg(u) = d_u$, $deg(v) = d_v$ and $\mathbb{Z}_n = \prod_{i=1}^{k} \mathbb{Z}_{p_i}^{\alpha_i}$, where p_i's are distinct prime numbers. G is a connected graph by [13]. Clearly, $|Id(\mathbb{Z}_n)| = 2^k$. The following propositions are useful tools needed for main results.

Proposition 2.1 : For any positive integer n, we have the followings:

1. If $2u \in Id(\mathbb{Z}_n)$, then $d_u = 2^k - 1$ and the number of vertices of d_u is 2^k.
2. If $2u \notin Id(\mathbb{Z}_n)$, then $d_u = 2^k$ and the number of vertices of d_u is $n - 2^k$.

Proof:

1. Let $u=(u_1,u_2,\ldots,u_k)\in V(G)$ such that $2u\in Id(\mathbb{Z}_n)$ then $d_u=2^k-1$ by Proposition 2.2 of [13]. Since $\mathbb{Z}_n=\prod_{i=1}^{k}\mathbb{Z}_{p_i}^{\alpha_i}$ and $\mathbb{Z}_{p_i}^{\alpha_i}$ has no nontrivial idempotent, then $2u\in Id(\mathbb{Z}_n)$ if and only if for each i, $2u_i\in\{0,1\}$. Thus there are 2^k vertices of $d_u=2^k-1$ in G.
2. Since the number of vertices of G is n, the proof is done by Proposition 2.2 of [13] and case 1.
   □

Corollary 2.2 : *For a positive integer n,* $|E(G)|=(n-1)2^{k-1}$. *Particularly, if n is prime number, then* $|E(G)|=n-1$.

Proof: Clearly, there is $\frac{2^k(2^k-1)+(n-2^k)2^k}{2}=2^{k-1}(n-1)$ edges in G by Propsition 2.1. □

Proposition 2.3 : *Let* $\mathbb{Z}_n=\prod_{i=1}^{k}\mathbb{Z}_{p_i}^{\alpha_i}$ *and* $u,v\in V(G)$. *Then the following statements hold:*

1. If for each i, $\mathbb{Z}_{p_i^{\alpha_i}}\neq\mathbb{Z}_2$ and $2u,2v\in Id(\mathbb{Z}_n)$, then u,v are not adjacent.
2. If for each i, $\mathbb{Z}_{p_i^{\alpha_i}}=\mathbb{Z}_2$ then G is a complete graph.
3. If $\mathbb{Z}_n$ has $\underbrace{\mathbb{Z}_2\times\ldots\times\mathbb{Z}_2}_{s\ times}$, $0\leq s\leq k$ as a quotient and $2u\in Id(\mathbb{Z}_n)$ then u is adjacent to 2^s-1 vertices of d_u in G.

Proof:

1. Suppose $2u\in Id(\mathbb{Z}_n)$. Then all vertices of d_u are corresponds to a $(0,a_i)$-vector of length k such that for $1\leq i\leq k$, $0\neq a_i\in\mathbb{Z}_{p_i^{\alpha_i}}$ and $2a_i\in\{0,1\}$. Note that, since for each i, $\mathbb{Z}_{p_i^{\alpha_i}}$ has no nontrivial idempotent, then each a_i is unique. Since for each i, $\mathbb{Z}_{p_i^{\alpha_i}}\ncong\mathbb{Z}_2$, $a_i\neq 1$. Now, let two distinct vertices b and c of d_u are adjacent in G. Clearly, b and c have at least one different component. Without loss of generality, we suppose that $b_1\neq 0$ and $c_1=0$. Therefore, $a_1\in\{0,1\}$, contradiction.
2. It's clear.
3. It follows from (1) and (2). □

In the following, by $d_u\times d_v$ we mean the set of all $uv\in E(G)$ such that $deg(u)=d_u$ and $deg(v)=d_v$.

Proposition 2.4 : For a positive integer n, let $\mathbb{Z}_n$ has $\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{s\ times}$, $0 \leq s \leq k$ as a quotient. Then we have:

1. If $2u, 2v \in Id(\mathbb{Z}_n)$, then the number of $d_u \times d_v$ is $A = (2^{k-1})(2^s - 1)$.
2. If $2u \in Id(\mathbb{Z}_n), 2v \not\in Id(\mathbb{Z}_n)$, then the number of $d_u \times d_v$ is $B = 2^k(2^k - 2^s)$.
3. If $2u, 2v \notin Id(\mathbb{Z}_n)$, then the number of $d_u \times d_v$ is $C = 2^{k-1}(n-1) - A - B$.

Particularly, if $n = 2$, then $A = 1$, $B = 0$ and $C = 0$. Also, if $n = p^\alpha \neq 2$, then $A = 0$, $B = 2$ and $C = n - 3$.

Proof: Let $\mathbb{Z}_n = (\prod_{i=1}^{s}(\mathbb{Z}_2))(\prod_{i=s+1}^{k}(\mathbb{Z}_{p_i}^{\alpha_i}))$ for $k \geq 1$ and uv be an arbitrary edge of $E(G)$.

1. If $2u, 2v \in Id(\mathbb{Z}_n)$, then $d_u = d_v = 2^k - 1$ by Proposition 2.1. Therefore, the number of $d_u \times d_v$ is
$$A = \frac{2^k(2^s - 1)}{2} = 2^{k-1}(2^s - 1)$$
by proposition 2.3(3).
2. If $2u \in Id(\mathbb{Z}_n), 2v \not\in Id(\mathbb{Z}_n)$, then $d_u = 2^k - 1$ $d_v = 2^k$ by Proposition 2.1. Therefore, the number of $d_u \times d_v$ is
$$B = 2^k[(2^k - 1) - (2^s - 1)] = 2^k(2^k - 2^s).$$
3. Assume $2u, 2v \notin Id(\mathbb{Z}_n)$. Since $|E(G)| = 2^{k-1}(n-1)$ by corollary 2.2, then
$$C = |E(G)| - A - B = 2^{k-1}(n-1) - A - B$$
by case (1) and (2). $\square$

3. Topological indices of G

In this section, we calculate some kind of topological indices of G.

Theorem 3.1 : *For a positive integer n, the first Zagreb index of G is* $(n-2)2^{2k} + 2^k$. *Particularly,*

$$M_1(G) = \begin{cases} 2 & \text{if } n = 2 \\ 4p^\alpha - 6 & \text{if } n = p^\alpha \neq 2 \end{cases}$$

Proof: There are 2^k vertices of degree $2^k - 1$ and $n - 2^k$ vertices of degree 2^k by Proposition 2.1. Therefore,

$$M_1(G) = 2^k(2^k - 1)^2 + (n - 2^k)2^{2k} = (n - 2)2^{2k} + 2^k.$$

□

Theorem 3.2 : *For a positive integer n, the second Zagreb index of G is* $A(2^k - 1)^2 + B((2^k - 1)2^k) + C(2^{2k})$. *Particularly,*

$$M_2(G) = \begin{cases} 1 & \text{if } n = 2 \\ 4p^{\alpha} - 8 & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Proof: By Proposition 2.4, we are done. □

Theorem 3.3 : *Let n be a positive integer. The third Zagreb index of G is B. Particularly,*

$$M_3(G) = \begin{cases} 0 & \text{if } n = 2 \\ 2 & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Proof: Suppose that $uv \in E(G)$. If $2u, 2v \in Id(\mathbb{Z}_n)$ or $2u, 2v \notin Id(\mathbb{Z}_n)$, then $|d_u - d_v| = 0$ by Proposition 2.1. If $2u \notin Id(\mathbb{Z}_n)$ and $2v \in Id(\mathbb{Z}_n)$, then $|d_u - d_v| = 1$ by Proposition 2.1. Therefore, by the definition of third Zagreb index and Proposition 2.4, we are done. □

Corollary 3.4 : *Let n be a positive integer. Then the first Zagreb coindex of G is* $(2^{k-1}(2^k - 1) - A)(2^{k+1} - 2) + (2^k(n - 2^k) - B)(2^{k+1} - 1) + ((n - 2^k)(n - 2^k - 1) - 2C)2^k$. *Particularly,*

$$\overline{M_1(G)} = \begin{cases} 0 & \text{if } n = 2 \\ 2p^{2\alpha} - 8p^{\alpha} + 6 & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Proof: Let $u, v \in V(G)$.

1. Suppose $2u, 2v \in Id(\mathbb{Z}_n)$. Then $d_u = d_v = 2^k - 1$. If all of vertices of d_u form a complete subgraph, then the number of $d_u \times d_v$ is $2^{k-1}(2^k - 1)$. If there is non-adjacent vertices u and v, then the number of $d_u \times d_v$ is $2^{k-1}(2^k - 1) - A$ by Proposition 2.4.
2. Suppose $2u \notin Id(\mathbb{Z}_n)$ and $2v \in Id(\mathbb{Z}_n)$. Then $d_u = 2^k, d_v = 2^k - 1$. If each vertex of d_u is adjacent to each vertex of d_v, then there is

$(n-2^k)2^k$ edges of $d_u \times d_v$. If there is non-adjacent vertices u and v, then the number of $d_u \times d_v$ is $2^k(n-2^k)-B$ by Proposition 2.4.

3. Suppose $2u, 2v \notin Id(\mathbb{Z}_n)$. Then $d_u = d_v = 2^k$. If each vertex of d_u form a complete subgraph, then the number of $d_u \times d_v$ is $\frac{(n-2^k)(n-2^k-1)}{2}$. If there is non-adjacent vertices u and v, then the number of $d_u \times d_v$ is $\frac{(n-2^k)(n-2^k-1)}{2} - C$ by Proposition 2.4. □

Corollary 3.5 : *Let n be a positive integer. The second Zagreb coindex of G is* $(2^{k-1}(2^k-1)-A)(2^k-1)^2 + ((n-2^k)2^k - B)2^k(2^k-1) + ((n-2^k)(n-2^k-1)-2C)2^k$. *Particularly,*

$$\overline{M_2}(G) = \begin{cases} 0 & \text{if } n = 2 \\ 2p^{2\alpha} - 10p^{\alpha} + 13 & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Proof: The proof is similar to Corollary 3.4. □

The following Theorems are calculated by Proposition 2.4.

Theorem 3.6 : *Let n be a positive integer. Then the first multiplicative Zagreb index of G is* $(2^k-1)^{2^{k+1}}(2)^{2k(n-2^k)}$. *Particularly,*

$$\Pi_1(G) = \begin{cases} 1 & \text{if } n = 2 \\ 4^{(p^{\alpha}-2)} & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Theorem 3.7 : *Let n be a positive integer. Then the second multiplicative Zagreb index of G is* $(2^k-1)^{(2A+B)}2^{k(2C+B)}$. *Particularly,*

$$\Pi_2(G) = \begin{cases} 1 & \text{if } n = 2 \\ 2^{(2p^{\alpha}-4)} & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Theorem 3.8 : *Let n be a positive integer. Then the Randić index of G is* $A\left(\frac{1}{2^k-1}\right) + B\frac{1}{\sqrt{2^{2k}-2^k}} + \frac{C}{2^k}$. *Particularly,*

$$\chi(G) = \begin{cases} 1 & \text{if } n = 2 \\ \frac{2}{\sqrt{2}} + \frac{p^{\alpha}-3}{2} & \text{if } n = p^{\alpha} \neq 2 \end{cases}$$

Theorem 3.9 : *Let n be a positive integer. Then the GA index of G is* $A + B\frac{2\sqrt{2^{2k}-2^k}}{2^{k+1}-1} + C$. *Particularly,*

$$GA(G) = \begin{cases} 1 & \text{if } n = 2 \\ \frac{4\sqrt{2}}{3} + (p^\alpha - 3) & \text{if } n = p^\alpha \neq 2 \end{cases}$$

Theorem 3.10 : *Let n be a positive integer. Then the ABC index of G is* $A\frac{\sqrt{2^{k+1}-4}}{2^k-1} + B\sqrt{\frac{2^{k+1}-3}{2^{2k}-2^k}} + C\frac{\sqrt{2^{k+1}-2}}{2^k}$. *Particularly,*

$$ABC(G) = \begin{cases} 0 & \text{if } n = 2 \\ \frac{p^\alpha - 1}{\sqrt{2}} & \text{if } n = p^\alpha \neq 2 \end{cases}$$

Example 3.11 : The Values of $M_1(G_{Id}(\mathbb{Z}_n))$, $M_2(G_{Id}(\mathbb{Z}_n))$, $M_3(G_{Id}(\mathbb{Z}_n))$, $\overline{M_1}(G_{Id}(\mathbb{Z}_n))$, $\overline{M_2}(G_{Id}(\mathbb{Z}_n))$, $\Pi_1(G_{Id}(\mathbb{Z}_n))$, $\Pi_2(G_{Id}(\mathbb{Z}_n))$, $\chi(G_{Id}(\mathbb{Z}_n))$, $GA(G_{Id}(\mathbb{Z}_n))$ and $ABC(G_{Id}(\mathbb{Z}_n))$ for $n = 2, 6, 9, 15$ is computed in Table 1

Table 1

n	2	6	9	15
$M_1(G_{Id}(\mathbb{Z}_n))$	2	68	30	212
$M_2(G_{Id}(\mathbb{Z}_n))$	1	114	28	400
$M_3(G_{Id}(\mathbb{Z}_n))$	0	8	2	12
$\overline{M_1(G_{Id}(\mathbb{Z}_n))}$	0	24	96	508
$\overline{M_2(G_{Id}(\mathbb{Z}_n))}$	0	36	85	972
$\Pi_1(G_{Id}(\mathbb{Z}_n))$	1	$9^4 \times (16)^2$	4^7	$9^4 \times (16)^{11}$
$\Pi_2(G_{Id}(\mathbb{Z}_n))$	1	$3^{12} \times 8$	2^{14}	$3^{12} \times 4^{44}$
$\chi(G_{Id}(\mathbb{Z}_n))$	1	2.97	4.41	7.46
$GA(G_{Id}(\mathbb{Z}_n))$	1	9.9	6.88	27.87
$ABC(G_{Id}(\mathbb{Z}_n))$	0	6.4	5.65	17.53

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Received June 2021

Rrevised December 2021