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Some results on the open subset intersection graph of a product topological space

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Abstract

In the recent paper, R. A. Muneshwar et al., introduced a graph structure called open subset intersection graph $\gamma(\tau)$ on a topological space (X, τ). In this paper, we study some important results of a graph $\gamma(\tau)$ of a product topological space $(X \times Y, \tau)$. We also determine relationship between diameter, girth, clique number, chromatic number, domination number etc. of an open subset intersection graph of a topological space $(X \times Y, \tau)$, (X, τ_X) and (Y, τ_Y). Moreover, we proved that, if (X, τ_X) and (Y, τ_Y) are discrete topological space then $\omega(\gamma(\tau_X \times \tau_Y)) = \omega(\gamma(\tau_X)) * \omega(\gamma(\tau_Y)) - 2$ and $\chi(\gamma(\tau_X \times \tau_Y)) = \chi(\gamma(\tau_X)) * \chi(\gamma(\tau_Y)) - 2$ and

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domination number of $\gamma(\tau_X \times \tau_Y)$ is 2. We also determine diameter and girth of intersection Graph of Product Topology on $X \times Y$, for different values of *m* and *n*.

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1. Introduction

If *R* is a commutative ring with unity then the zero divisor graph of *R* was firstly introduced by Beck[2], which is defined as, if *R* is any ring then $G(R)$ denotes the zero divisor graph of *R* whose vertex set is $V = R$, such that any distinct vertices *x* and *y* are adjacent if and only if $x \cdot y = 0$. Graphs of several algebraic structures have been defined in recent decades and can be found in [1, 3, 4, 5, 6, 7, 13, 14, 19]. We choose to offer a new concept of graph of topological space (X, τ), since graph theory has such a wide range of applications and along with this, we also discussed certain key aspects of these graphs, which may be found in [17,18,19]. R. A. Muneshwar et al. [12], introduced a graph structure called as intersection graph of a topology τ and derived some important results on this graph. The reader can refer to [17, 18, 19], for undefined terms. In [9, 11], authors introduced and discussed some properties of the graph $\gamma(\tau)$ of τ. An open subset intersection graph $\gamma(\tau)$ of τ, is defined as follows:

Definition 1.1 : [9] Open Subset Intersection Graph of a Topological Space:

Let (X, τ) be the topological space with topology τ and ϕ denote the empty set. We define a Open Subset Intersection Graph of a Topological Space (X, τ) is denoted by $\gamma(\tau)$ and defined as $\gamma(\tau) = (V, E)$, where $V = \{U_k \in \tau \mid U_k \neq \phi \ \& \ X\}$ and $E = \{(U_1, U_2) \mid U_1 \cap U_2 \neq \phi\}$.

Example 1.2 : Let $(X = \{a_1, a_2, a_3\}, \tau)$ be the discrete topological space with $\tau = \{U_1 = \{a_1\},\ U_2 = \{a_2\},\ U_3 = \{a_3\},\ U_{12} = \{a_1, a_2\},\ U_{13} = \{a_1, a_3\},\ U_{23} = \{a_2, a_3\}\}$ then corrosponding open subset intersection graph shown in Figure 1.

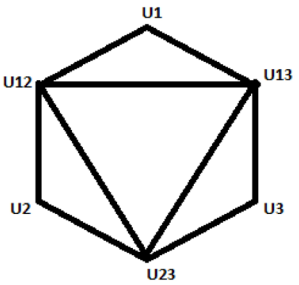


Figure 1

The Graph $\gamma(\tau)$ of a Discrete Topological Space (X, τ) with $|X| = 4$

2. Diameter and Girth of Intersection Graph of Product Topology on $X \times Y$

In this section we investigate the diameter and girth of intersection graph of product topology on $X \times Y$.

Theorem 2.1: *If (X, τ_X) and (Y, τ_Y) are discrete topological spaces with $|X| = m$ and $|Y| = n$ then $\gamma(\tau_X \times \tau_Y)$ is connected and diam $(\gamma(\tau_X \times \tau_Y)) = 2$, where $\tau_X \times \tau_Y$ is a product topology on $X \times Y$.*

Proof: Suppose $|X| = m$ and $|Y| = n$, where $m > n$ and $U_1 \times V_1$ & $U_2 \times V_2$ any two open subsets of $X \times Y$.

Claim: $\gamma(\tau_X \times \tau_Y)$ is connected graph.

To prove this result, the following three cases may arises.

Case 1: If $(U_1 \times V_1) \cap (U_2 \times V_2) \neq \phi$, then $(U_1 \times V_1) \sim (U_2 \times V_2)$. Therefore $(U_1 \times V_1) \sim (U_2 \times V_2)$ be an required path from $(U_1 \times V_1)$ to $(U_2 \times V_2)$ and hence $dist((U_1 \times V_1), (U_2 \times V_2)) = 1$.

Case 2: If $(U_1 \times V_1) \cup (U_2 \times V_2) \neq X \times Y$, then $(U_1 \times V_1) \subset (U_1 \times V_1) \cup (U_2 \times V_2)$ and $(U_2 \times V_2) \subset (U_1 \times V_1) \cup (U_2 \times V_2)$. Therefore $(U_1 \times V_1) \cap \{(U_1 \times V_1) \cup (U_2 \times V_2)\} \neq \phi$ and $(U_1 \times V_1) \cap \{(U_1 \times V_1) \cup (U_2 \times V_2)\} \neq \phi$ and hence $(U_1 \times V_1) \sim (U_1 \times V_1) \cup (U_2 \times V_2) \sim (U_2 \times V_2)$ be an required path from $(U_1 \times V_1)$ to $(U_2 \times V_2)$. Therefore $dist((U_1 \times V_1), (U_2 \times V_2)) = 2$.

Case 3: If $(U_1 \times V_1) \cap (U_2 \times V_2) = \phi$ and $(U_1 \times V_1) \cup (U_2 \times V_2) = X \times Y$. As $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) = \phi$ then $(U_2 \cap U_2) = \phi$ or $(V_1 \cap V_2) = \phi$. Also $(U_1 \times V_1) \cup (U_2 \times V_2) = (U_1 \cup U_2) \times (V_1 \cup V_2) = X \times Y$ then $(U_1 \cup U_2) = X$ and $(V_1 \cup V_2) = Y$.

Subcase -I If $(U_2 \cap U_2) = \phi$ and $(V_1 \cup V_2) = X$ then we can choose $(a_1, b_1) \in (U_1 \times V_1)$, & $(a_2, b_2) \in (U_2 \times V_2)$. If $U_3 \times V_3 = \{(a_1, b_1)\}$ and $U_4 \times V_4 = \{(a_2, b_2)\}$ then $\{(U_3 \times V_3) \cup (U_4 \times V_4)\} \cap (U_1 \times V_1) \neq \phi$, $\{(U_3 \times V_3) \cup (U_4 \times V_4)\} \cap (U_2 \times V_2) \neq \phi$. Thus $(U_1 \times V_1) \sim (U_3 \times V_3) \cup (U_4 \times V_4) \sim (U_2 \times V_2)$ be an required path from and hence $dist(U_1 \times V_1, U_2 \times V_2) = 2$.

Thus, the graph $\gamma(\tau_X \times \tau_Y)$ is connected and diam $(\gamma(\tau_X \times \tau_Y)) = 2$.

Theorem 2.2 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m, |Y| = n$ *resp. then following are hold.*

1. *If* $m = 2, n = 1$ *then* $girth(\gamma(\tau))) = \infty$.
2. *If* $m = 2, n = 3$ *then* $girth(\gamma(\tau))) = 3$.
3. *If* $m = n = 3$ *then* $girth(\gamma(\tau_X \times \tau_Y)) = 3$.
4. *If* $m = 2, n = 2$ *then* $girth(\gamma(\tau_X \times \tau_Y)) = 2$.
5. *If* $m \geq 4$ *or* $m \geq 4$ *then* $girth(\gamma(\tau_X \times \tau_Y)) = 3$.

Proof: Let τ_X and τ_Y are discrete topologies defined on $X = \{a_1, a_2, ..., a_m\}$ and $Y = \{b_1, b_2, ..., b_n\}$ respectively, where $m > n$. Let $U_i = \{a_1, a_2.., a_i\}$ and $V_j = \{b_1, b_2, ..., b_j\}$, are open subsets of X and Y respectively, where $i = 1, 2, ...m-1$, $j = 1, 2, ...n-1$. Following three cases might occur in order to demonstrate this theorem:

Case 1: If $m = 2, n = 1$ then by Corollary 3.7[8], the graph $\gamma(\tau)$ is a null graph and hence $girth(\gamma(\tau)) = \infty$.

Case 2: If $m = n = 3$ with $X = \{a_1, a_2, a_3\}$ and $Y = \{b_1, b_2, b_3\}$. Let $U_1 = \{a_1\}$, $U_2 = \{a_1, a_2\}$ are two open subsets of X and $V_1 = \{b_1\}$, $V_2 = \{b_1, b_2\}$ are two open subsets of Y. Then we have $(U_1 \times V_1) \cap (U_1 \times V_2) = (U_1 \times V_1) \neq \phi$, $(U_1 \times V_2) \cap (U_2 \times V_2) = (U_1 \times V_1) \neq \phi$ & $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \times V_1) \neq \phi$ and hence $(U_1 \times V_1) \sim (U_1 \times V_2)$, $(U_1 \times V_2) \sim (U_2 \times V_2)$ & $(U_1 \times V_1) \sim (U_2 \times V_2)$. Therefore $U_1 \times V_1 \sim U_1 \times V_2 \sim U_2 \times V_2 \times U_1 \times V_1$ is a triangle and hence in this case girth of the graph $\gamma(\tau_X \times \tau_Y)$ is 3.

Case 3: If $m = 2, n = 3$, with $X = \{a_1, a_2\}$ and $Y = \{b_1, b_2, b_3\}$. Let $U_1 = \{a_1\}$, $U_2 = \{a_2\}$ are two open subsets of X and $V_1 = \{b_1\}$, $V_2 = \{b_1, b_2\}$ are two open subsets of Y. Then we have $(U_1 \times V_1) \cap (U_1 \times V_2) = (U_1 \times V_1) \neq \phi$, $(U_1 \times V_2) \cap (X \times V_2) = (U_1 \times V_2) \neq \phi$ & $(U_1 \times V_1) \cap (X \times V_2) = (U_1 \times V_2) \neq \phi$ and hence $(U_1 \times V_1) \sim (U_1 \times V_2)$, $(U_1 \times V_2) \sim (X \times V_2)$ & $(U_1 \times V_1) \sim (X \times V_2)$. Therefore $U_1 \times V_1 \sim U_1 \times V_2 \sim X \times V_2 \times U_1 \times V_1$ is a triangle and hence in this case girth of the graph $\gamma(\tau_X \times \tau_Y)$ is 3.

Case 4: If $m \geq 4$ or $n \geq 4$. For our convenience, suppose $m \geq 4$ and a, b, c be three distinct elements in X.

Consider the following three distinct subcases in order to demonstrate this theorem in the case.

Sub case I: If $U_1 = \{a\}$ then there exist open sets $U_2 = \{a, b\}$, $U_3 = \{a, b, c\}$ of X. Then there exist open subsets $(U_1 \times Y)$, $(U_2 \times Y)$, $(U_3 \times Y)$ in $X \times Y$ such that $(U_1 \times Y) \cap (U_2 \times Y) \neq \phi$, $(U_2 \times Y) \cap (U_3 \times Y) \neq \phi$ and $(U_3 \times Y) \cap (U_1 \times Y) \neq \phi$. Hence $U_1 \times Y \sim U_2 \times Y \sim U_3 \times Y \sim U_1 \times Y$, which is triangle.

Sub case II: If $U_1 = \{a, b\}$ and since $|X| \geq 4$, , then there exist two open subsets $U_2 = \{a\}$ and $U_{3=\{a,b,c\}}$ of X. Here there exist open subsets $U_1 \times Y$, $U_2 \times Y$, $U_3 \times Y$ in $X \times Y$ such that $(U_1 \times V_1) \cap (U_1 \times V_2) = (U_1 \times V_1) \neq \phi$, $(U_1 \times V_2) \cap (X \times V_2) = (U_1 \times V_2) \neq \phi$ & $(U_1 \times V_1) \cap (X \times V_2) = (U_1 \times V_2) \neq \phi$ and hence $(U_1 \times V_1) \sim (U_1 \times V_2)$, $(U_1 \times V_2) \sim (X \times V_2)$ & $(U_1 \times V_1) \sim (X \times V_2)$. Therefore $U_1 \times V_1 \sim U_1 \times V_2 \sim X \times V_2 \times U_1 \times V_1$.

Sub case III: If $|U_1| = k \geq 3$ and since $|X| \geq 4$, there exist two open subsets U_2 and U_3 of X with $|U_2| = k-1$, $|U_3| = k+1$ such that $U_2 \subset U_1 \subset U_3$. Then there exist open subsets $U_2 \times Y$, $U_1 \times Y$, $U_3 \times Y$ in $X \times Y$ and $(U_1 \times Y) \cap (U_2 \times Y) \neq \phi$, $(U_2 \times Y) \cap (U_3 \times Y) \neq \phi$ & $(U_3 \times Y) \cap (U_1 \times Y) \neq \phi$. Hence $U_1 \times Y \sim U_2 \times Y \sim U_3 \times Y \sim U_1 \times Y$, which is triangle.

Case 5: If $m = 2, n = 2$, with $X = \{a_1, a_2\}$ and $Y = \{b_1, b_2\}$. Let $U_1 = \{a_1\}$, $U_2 = \{a_2\}$ are two open subsets of X and $V_1 = \{b_1\}$, $V_2 = \{b_2\}$ are two open subsets of Y. Then $(U_1 \times V_1) \cap (U_1 \times Y) \neq \phi$, $(U_1 \times Y) \cap (X \times V_1) \neq \phi$, $(X \times V_1) \cap (U_1 \times V_1) \neq \phi$ and hence $U_1 \times V_1 \sim U_1 \times Y \sim X \times V_1 \sim U_1 \times V_1$ which is triangle and hence in this case girth of the graph $\gamma(\tau_X \times \tau_Y)$ is 3. □

Theorem 2.3 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m, |Y| = n$. *If* $m \geq 4$, *or* $n \geq 4$ *then the graph* $\gamma(\tau_X \times \tau_Y)$ *is triangulated.*

Proof: Let $X = \{a_1, a_2, ..., a_m\}$ and $Y = \{b, b_2, ..., b_n\}$, where $m \geq 2$, $m \geq 2$. Let $U_i = \{a_1, a_2, ..., a_i\}$, for $i = 1, 2, ...m-1$, is open subsets of X and $V_j = \{b_1, b_2, ..., b_j\}$, for $j = 1, 2, ...n-1$, is open subsets of Y. Let $U \times V$ be any open subset of $X \times Y$. We'll demonstrate the result in the three different circumstances listed below.

Case 1: If $U_1 = \{a_1\}$ and since $|X| \geq 4$, there exist two open subsets $U_2 = \{a_1, a_2\}$ and $U_3 = \{a_1, a_2, a_3\}$ of X. Then there exist open subsets $U_1 \times Y$, $U_2 \times Y$, $U_3 \times Y$ in $X \times Y$ such that $(U_1 \times Y) \cap (U_2 \times Y) \neq \phi$, $(U_2 \times Y) \cap (U_3 \times Y) \neq \phi$, and $(U_3 \times Y) \cap (U_1 \times Y) \neq \phi$. Hence $U_1 \times Y \sim U_2 \times Y \sim U_3 \times Y \sim U_1 \times Y$, which is triangle.

Case 2: If $U_1 = \{a_1, a_2\}$ and since $|X| \geq 4$, there exist two open subsets $U_2 = \{a_1\}$ and $U_3 = \{a_1, a_2, a_3\}$ of X. Here there exist open subsets $U_1 \times Y$, $U_2 \times Y$, $U_3 \times Y$ in $X \times Y$ such that $(U_1 \times Y) \cap (U_2 \times Y) \neq \phi$, $(U_2 \times Y) \cap (U_3 \times Y) \neq \phi$, and $(U_3 \times Y) \cap (U_1 \times Y) \neq \phi$. Hence $U_1 \times Y \sim U_2 \times Y \sim U_3 \times Y \sim U_1 \times Y$, which is triangle.

Case 3: If $|U_1| = k \geq 3$ and since $|X| \geq 4$, there exist two open subsets U_2 and U_3 of X with $|U_2| = k-1$, $|U_3| = k+1$ such that $U_2 \subset U_1 \subset U_3$. Therefore, there exist open subsets $U_1 \times Y$, $U_2 \times Y$, $U_3 \times Y$ in $X \times Y$ such that $(U_1 \times Y) \cap (U_2 \times Y) \neq \phi$, $(U_2 \times Y) \cap (U_3 \times Y) \neq \phi$, and $(U_3 \times Y) \cap (U_1 \times Y) \neq \phi$. Hence $U_1 \times Y \sim U_2 \times Y \sim U_3 \times Y \sim U_1 \times Y$, which is triangle.

Thus, the graph $\gamma(\tau_X \times \tau_Y)$ is triangulated. □

3. Clique Number and Chromatic Number of Intersection Graph of Product Topology on X × Y

The clique number and chromatic number of $\gamma(\tau_X \times \tau_Y)$ of $X \times Y$ are examined in this section .

Theorem 3.1 : *Let* (X, τ_X) *&* (Y, τ_Y) *are discrete topological space with* $|X| = m$, $|Y| = n$ *and* $U_1 \times V_1 = \{a_1\} \times \{b_1\}$ *be an open subset of* $X \times Y$. *If* $Z = \{U \times V \subset X \times Y \mid U_1 \times V_1 \subset U \times V\}$, *then* Y *is a clique in* $\gamma(\tau)$.

Proof: Let τ_X & τ_Y are discrete topologies defined on $X = \{a_1, a_2, ..., a_m\}$ & $Y = \{b_1, b_2, ..., b_n\}$ respectively, where $m > n$. As $U_1 \times V_1 = \{a_1\} \times \{b_1\}$ be an open subset of $X \times Y$ and $Z = \{U \times V \subset X \times Y \mid U_1 \times V_1 \subset U \times V\}$ then $(U \times V) \cap (U' \times V') \neq \phi$, $\forall\, U \times V$, $U' \times V' \in Z$. Hence, $(U \times V) \sim (U' \times V')$, $\forall\, U \times V$, $U' \times V' \in Z$, and thus Z is clique in the graph $\gamma(\tau_X \times \tau_Y)$.

Theorem 3.2 : *If* (X, τ_X) *&* (Y, τ_Y) *are discrete topological space with* $|X| = m$, *and* $|Y| = n$ *then* $\omega(\gamma(\tau_X \times \tau_Y)) = 2^{m+n-2} - 2 = \omega(\gamma(\tau_X)) * \omega(\gamma(\tau_Y)) - 2$, $\forall\ m, n$.

Proof: Let τ_X and τ_Y are discrete topologies defined on $X = \{a_1, a_2, ..., a_m\}$ and $Y = \{b_1, b_2, ..., b_n\}$ respectively, where $m > n$. As $U_1 \times V_1 = \{a_1\} \times \{b_1\}$ be an open subset of $X \times Y$ and $Z = \{U \times V \subset X \times Y \mid U_1 \times V_1 \subset U \times V\}$ then by Theorem 3.1, Z is clique in the graph $\gamma(\tau_X \times \tau_Y)$.

Claim: Z is a maximal clique in the graph $\gamma(\tau_X \times \tau_Y)$.

If possible $M = Z \cup \{U \times V\}$ is clique in the graph $\gamma(\tau_X \times \tau_Y)$, where $U \times V \notin Z$, then $a_1 \notin U$, $b_1 \notin V$ and hence $(U \times V) \cap \{a_1\} \times \{b_1\} = \phi$, a contradiction. Therefore, M is not clique and hence Z is a maximal clique in the graph $j(\tau_X \times \tau_Y)$. Thus, $\omega(\gamma(\tau_X \times \tau_Y)) = 2^{m+n-2} - 2 = \omega(\gamma(\tau_X)) * \omega(\gamma(\tau_Y)) - 2$, $\forall$ m, n. □

Theorem 3.3 : *If* (X, τ_X) *&* (Y, τ_Y) *are discrete topological space with* $|X| = m$, *and* $|Y| = n$ *then* $\chi(\gamma(\tau_X \times \tau_Y)) = 2^{m+n-2} - 2 = \chi(\gamma(\tau_X) * \chi(\gamma(\tau_Y)) - 2$.

Proof: Let τ_X and τ_Y are discrete topologies defined on $X = \{a_1, a_2, ..., a_m\}$ and $Y = \{b_1, b_2, ..., b_n\}$ respectively, where $m > n$. As $U_1 \times V_1 = \{a_1\} \times \{b_1\}$ be an open subset of $X \times Y$ and $Z = \{U \times V \subset X \times Y \mid U_1 \times V_1 \subset U \times V\}$ then by Theorem 3.1, Z is a maximal clique in the graph $\gamma(\tau_X \times \tau_Y)$. Hence $\chi(\gamma(\tau_X \times \tau_Y)) \geq 2^{m+n-2} - 2$.

Let $U_i = \{a_1, a_2, ..., a_i\}$, for $i = 1, 2, ...n-1$ are open subsets of X. Let $Z = \{U_1 \times V_1 \subset U_2 \times V_1 \subset U_3 \times V_1 \subset ... \subset U_{n-1} \times V_1 \subset X \times V_1 \subset X \times V_2 \subset X \times V_3 \subset ... \subset X \times V_{n-1}\}$ be a ascending chain of non-trivial proper open subsets of $(X \times Y, \tau)$. Then clearly Z is clique in $\gamma(\tau_X \times \tau_Y)$. As clique number of $\gamma(\tau_X \times \tau_X)$ is $2n-2$, then we have $\chi(\gamma(\tau_X \times \tau_Y)) \geq 2^{m+n-2} - 2$.

Claim: We demonstrate a $2^{m+n-2} - 2$ colouring of $\gamma(\tau_X \times \tau_Y)$:

Now label the vertex $U_i \times V_1$ and $U_{n-1} \times V_j$ by the colour C_{i1} and $C_{n-1,j}$ respectively. To demonstrate a $2^{m+n-2} - 2$ colouring of $\gamma(\tau_X \times \tau_Y)$, we start with any vertex $U \times V$ of $\gamma(\tau_X \times \tau_Y)$. If $U \subset X$ and $V \subset Y$ then there exist open subsets U, V such that $|U| = p$ and $|V| = q$, where $1 < p < (m-1)$, $1 < q < (n-1)$. As $U \subset X$ then there exist $a_i \notin U$ such that $(U \times V) \cap (U_i \times V_1) = \phi$ and hence $(U \times V) \nsim (U_i \times V_1)$. Also as $V \subset Y$ then there exist $b_j \notin V$ such that $(U \times V) \cap (X \times V_j) = \phi$ and hence $U \times V \nsim X \times V_j$. Thus vertex $U \times V$ is labled with colour $C_{m,1}$. In order for neighbouring vertices to have different colours, there is therefore no need to label $U \times V$ with a new colour and we hence have, $\chi(\gamma(\tau_X \times \tau_Y)) \leq 2^{m+n-2} - 2$. Thus $\chi(\gamma(\tau_X \times \tau_X)) = 2^{m+n-2} - 2 = \chi(\gamma(\tau_X) * \chi(\gamma(\tau_Y)) - 2$. □

Theorem 3.4 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$. *If* $m + n - 2 \geq 1$ *then the graph* $\gamma(\tau_X \times \tau_Y)$ *is not planar.*

Proof: As $|X| = m$ and $|Y| = n$, where $m > n \geq 4$ then by Theorem 3.2, the clique number of the graph $\omega(\gamma(\tau_X \times \tau_Y)) = 2^{m+n-2} - 2 = \omega(\gamma(\tau_X))\omega(\gamma(\tau_Y))$, $\forall\ m, n$. If $m+n-2 \geq 1$ then $\omega(\gamma(\tau_X \times \tau_Y) \geq 5$, and hence complete graph K_5 is subgraph of the graph $\gamma(\tau_X \times \tau_Y)$. Thus, if $m+n-2 \geq 1$ then the graph $\gamma(\tau_X \times \tau_Y)$ is not planar. □

Theorem 3.5 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$. *If* $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$ *then Z is dominating set.*

Proof: Let $|X| = m$ and $|Y| = n$ resp and $Z = \{U_1 \times Y,\ X \times V_1 |\ |U_i| = (m-1), |V_j| = (n-1)\}$. If $U \times V = \{a_1, a_2, ...a_p\} \times \{b_1, b_2, ...b_q\} \in \tau, \forall\ p < m$, $q < n$, be any open subset of $X \times Y$, then there exist at least one element $U_1 \times Y$ or $X \times V_1$ of Z, such that $(U \times V) \cap (U_1 \times Y) \neq \phi$ or $(U \times V) \cap (X \times V_1) \neq \phi$ and hence $U \times V \sim U_1 \times Y$ or $U \times V \sim X \times V_1$. Thus Z is dominating set. □

Theorem 3.6 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$. *If* $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$ *then Z is minimal dominating set.*

Proof: If $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$ then by Theorem 3.5, Z is dominating set.

Let $Z_i = Z - \{U_i \times Y\}; \forall\ i$ or $Z = Z - \{X \times V_j\}; \forall j$ then there exist an open subset, $U \times V = U_i \times Y$ of $X \times Y$ or $U \times V = X \times V_j$, $\forall\ i, j$ of $X \times Y$ such that $(U \times V) \cap (U_i \times Y) = \phi$ or $(U \times V) \cap (X \times V_j) = \phi, \forall\ i, j$ and hence $U \times V \nsim U_i \times Y$ or $U \times V \nsim X \times V_j, \forall i, j$. Thus Z is minimal dominating set. □

Theorem 3.7 : *If* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$ *then domination number of* $\gamma(\tau_X \times \tau_Y)$ *is 2.*

Proof: As $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$ then by Theorem 3.6, Z is minimal dominating set and hence domination number of $\gamma(\tau_X \times \tau_Y)$ is 2.

Theorem 3.8 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$. *If* $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$ *then Z is totally domination set.*

Proof: As $Z = \{U_1 \times Y, X \times V_1 \,||\, U_i| = (m-1), |V_j| = (n-1)\}$, then by Theorem 3.6, Z is minimal dominating set. By Theorem 3.2[8], any two elements of Z are adjacent and therefore Z is clique in the graph $\gamma(\tau_X \times \tau_Y)$. Hence Z is totally dominating set in $\gamma(\tau_X \times \tau_Y)$. □

4. Degree of vertex and Intersection Graph of Product Topology on X × Y

In this section we investigate the degree of vertex and $\gamma(\tau_X \times \tau_Y)$ of product topology on $X \times Y$.

Theorem 4.1: *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$, *where* $m > n$. *If* $|U \times V| = pq$, *where* $|U| = p$ *and* $|V| = q$ *then number of vertices of* $deg(U \times V) = (2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2) * (2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2)$, *is equal to* $(m!/(m-p)!p!)(n!/(n-q)!q!)$.

Proof: Suppose $|X| = m$ and $|Y| = n$, where $m > n \geq 4$. Let $U = \{a_1, a_2, ...a_p\}$ and $V = \{b_1, b_2, ...b_q\}$ are any open subset of X and Y resp.. Let $U \times V$ be any open subset of $X \times Y$. If $U_1 \times V_1 \sim U \times V$ then we have $(U_1 \times V_1) \cap (U \times V) \neq \phi$. Since $(U_1 \times V_1) \cap (U \times V) = (U \cap U_1) \times (V \cap V_1) \neq \phi$ and hence $(U \cap U_1) \neq \phi$, $(V \cap V_1) \neq \phi$.

Case 1: If $(U \cap U_1) \neq \phi$ then by Theorem 7.1[2], number of such U_1 is equals to $(2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2)$.

Case 2: If $(V \cap V_1) \neq \phi$ then by Theorem 7.1[2], number of such V_1 is equals to $(2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2)$.

Hence number of possibilities of $U_1 \times V_1$ is equal to $(2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2) * (2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2)$. Thus $deg(U \times V) = (2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2) * (2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2) = deg(U)deg(V)$, is equal to $(m!/(m-p)!p!)(n!/(n-q)!q!)$. □

Theorem 4.2 : *Let* (X, τ_X) *and* (Y, τ_Y) *are discrete topological space with* $|X| = m$ *and* $|Y| = n$. *Then* $deg(\gamma(\tau_X \times \tau_Y)) = \Sigma_1^{(n-1)} (m!/((m-p)!p!))(n!/((n-q)!q!)) \{(2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)\}$.

Proof: As $|U \times V| = pq$, where $|U| = p$ and $|V| = q$ then by Theorem 4.1, we have $deg(U \times V) = (2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2) * (2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2)$, and number of such subsets is

equal to $m!/(m-p)!p!n!/(n-q)!kq!$. Thus $deg(\gamma(\tau_X \times \tau_Y)) = \Sigma_1^{(m-1)}\Sigma_1^{(n-1)}$ $(m!/((m-p)!p!))$ $(n!/((n-q)!q!))$ $\{(2^k + 2^{(m-k)} - 4) + (2^{(m-k)} - 1)(2^k - 2) * (2^k + 2^{(n-k)} - 4) + (2^{(n-k)} - 1)(2^k - 2)\}$. □

5. Conclusion

In this study, we looked at several significant findings from the graph $\gamma(\tau_X \times \tau_Y)$ of a product topological space $(X \times Y, \tau)$. Furthermore, we discovered a relationship between diameter, girth, clique number, chromatic number, domination number etc. of an open subset intersection graphs $\gamma(\tau_X)$, $\gamma(\tau_Y)$ and $\gamma(\tau_X \times \tau_Y)$. Moreover, we proved that, if (X, τ_X) and (Y, τ_Y) are discrete topological space with $|X| = m$ and $|Y| = n$, then $\omega(\gamma(\tau_X \times \tau_Y)) = \omega(\gamma(\tau_X)) * \omega(\gamma(\tau_Y)) - 2$, $\chi(\gamma(\tau_X \times \tau_Y)) = \chi(\gamma(\tau_X)) * \chi(\gamma(\tau_Y)) - 2$ and Then $deg(\gamma(\tau_X \times \tau_Y)) = \Sigma_1^{(n-1)}$ $(m!/((m-p)!p!))$ $(n!/((n-q)!q!))$ $\{(2^p - 2) + (2^q - 2) + (2^p - 2)(2^q - 2) + (2^{m-p} - 2) + (2^{n-q} - 2) + (2^{m-p} - 2)(2^{n-q} - 2)\}$.

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