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On eigenvalue, singular value and norm of symmetric matrices

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Abstract

Symmetric matrices play an important role in data science. In this paper, we present some new results on the eigenvalues, singular values, the spectral and Euclidean norms of real symmetric matrices in form $A = [x_i + x_j]_{i,j=1}^{n}$.

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1. Introduction

There has been a growing interest among researchers in studying the symmetric matrices. The symmetric matrices have been discussed in so many papers and books. People are particularly interested in the spectral properties (for example, in the eigenvalues, singular values and norms), in the structure of the inverse, in several kinds of factorizations, and in efficiently solving linear systems with such coefficient matrices. It is well known that symmetric matrices in data science are widely used during modeling, for example, similarity matrices, adjacency matrices, Laplacian matrices, covariance matrices, correlation matrices, and kernel matrices [2]. A detailed description of these matrices and extensive bibliographies can be found in [5], [6], [9] and [10].

The set of n-order Hermitian matrices (or real symmetric matrices) is a normed linear space. Gähler [4] proposed a mathematical structure,

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called 2-normed space, as a generalization of normed linear spaces. Laith, Akram and Anas [8] defined the concept of $(\beta, \mathcal{E}_0)$ -symmetric 2-normed space, where $\beta > 0, \mathcal{E}_0 \in \mathcal{X}, X$ is 2 - normed space, with examples and basic properties. Also, in that paper, they discussed the new results of this concept. Jiang, Chunfeng and Yu-Hong [7] considered the problem of computing several extreme eigenvalues and their corresponding eigenvectors of a real symmetric matrix $A \in \mathbb{R}^{n\times n}$. They emphasize that this classical problem plays an important role in many fields of scientific and engineering computing and it is equivalent to some optimization problems by using the variational principles. The MIN matrix $[\min(i,j)]$ and the MAX matrix $[\max(i,j)]$ are clearly square and symmetric. da Fonseca [3] studied the eigenvalues of certain MIN and MAX matrices via their matrix inverses. İpek [6] dealt with rank, trace, eigenvalues and norms of the matrix $C_x = (x_i x_j)_{i,j=1}^{n}$, where x_i are its components of any real sequence (x_n).

In this paper, we obtain some new results on the eigenvalues, singular values, the spectral and Euclidean norms of a real symmetric matrix of the form $A = [x_i + x_j]_{i,j=1}^{n}$. When $x_i = i, i = 1,2,\ldots,n,$ for the elements of the matrix A, the matrix A becomes the n -by- n real symmetric Hankel matrix.

Square Hankel matrices are matrices of the form

$$H_n = (a_{j+k-1})_{j,k=1}^{n} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{pmatrix}.$$

Each of this type of $n\times n$ Hankel matrices depends on $2n-1$ parameters, which are usually complex numbers. Such matrices are currently emerging in plenty of applications [1].

The content of this paper are as follows: In Section 2, we give some basic needed known results for matrices and their norms. In Section 3, we obtain some new results on the eigenvalues, singular values, spectral and Euclidean norms of a real symmetric matrix $A = [x_i + x_j]_{i,j=1}^{n}$. In Section 4, we introduce two numerical examples illustrated Theorem 1 and Thereom 2, respectively. In Section 5, we present the conclusions of this study.

2. Preliminaries

We recall some basic definitions and properties of the spectral and Euclidean norms of matrices. For a comprehensive exposition of the subject we refer the reader to [5], [10] and [12].

Let $\mathbb{C}^{n\times n}$ denote the space of $n\times n$ complex matrices. The conjugate $\overline{A}$ of $A=[a_{ij}]\in\mathbb{C}^{n\times n}$ is the matrix such that $\overline{A}_{ij}=\overline{a}_{ij}$, $1\le i,j\le n$. The transpose of A is the $n\times n$ matrix A^{T} such that $A^{\mathrm{T}}_{ij}=a_{ji}$, $1\le i,j\le n$. The conjugate transpose of A is the $n\times n$ matrix A^* such that $A^*=(\overline{A^{\mathrm{T}}})=(\overline{A})^{\mathrm{T}}$. When A is a real matrix $(A\in\mathbb{R}^{n\times n})$, $A^*=A^{\mathrm{T}}$. A matrix A is Hermitian if $A^*=A$. If A is a real matrix $(A\in\mathbb{R}^{n\times n})$, we say that A is symmetric if $A^{\mathrm{T}}=A$. The trace of A is the sum of its diagonal elements $tr(A)=a_{11}+\cdots+a_{nn}$.

The matrix A^TA is symmetric and its eigenvalues are nonnegative. Because they are nonnegative, we can write the eigenvalues of A^TA in decreasing order of magnitude as the squares of nonnegative real numbers, say as $\sigma_1^2\ge\sigma_2^2\ge\cdots\ge\sigma_n^2$. It follows that the numbers $\sigma_1,\sigma_2,\ldots,\sigma_n$ are uniquely determined by A. The numbers $\sigma_1,\sigma_2,\ldots,\sigma_p$ are called the singular values of the matrix A.

A matrix $A\in\mathbb{C}^{n\times n}$ is called normal if $A^*A=AA^*$. If $A\in\mathbb{R}^{n\times n}$ and $A^T=A$, we have $A^*A=AA^*=A^2$. Hence real matrices for which $A^T=A$, called symmetric, are normal. $\sigma_i=|\lambda_i|$ for all $i=1,\ldots,n$ if and only if A is normal.

The Euclidean norm can also be written in the form of the trace function as follows:

$$\|A\|_E=\sqrt{\sum_{i,j=1}^{n}|a_{ij}|^2}=\sqrt{\sum_{i}^{n}\sigma_i^2(A)}=\sqrt{tr(AA^*)}. \tag{1}$$

The spectral norm $\|\cdot\|_2$ is defined on $\mathbb{C}^{n\times n}$ by

$$\|A\|_2=\sigma_1(A).$$

From $\|A\|_2=\sigma_{\max}(A)$ and $\|A\|_F=\sqrt{\sum_i^n\sigma_i^2(A)}$ one gets a relationship between the spectral norm and the Euclidean norm:

$$\|A\|_2\le\|A\|_E$$

and the equality holds if and only if the matrix A is a rank-1 matrix or a zero matrix, since $\sqrt{\sum_i^n\sigma_i^2(A)}=\sigma_{\max}(A)$ in this case.

Throughout this paper, we use $A^T,\|A\|_E$ and $\|A\|_2$ to denote transpose, the Euclidean norm and the spectral norm of A, respectively.

For any $n\ge2$, we consider the n-by-n real symmetric matrix

$$A=[x_i+x_j]_{i,j=1}^n=\begin{bmatrix} x_1+x_1 & x_1+x_2 & x_1+x_3 & \cdots & x_1+x_n \\ x_2+x_1 & x_2+x_2 & x_2+x_3 & \cdots & x_2+x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n+x_1 & x_n+x_2 & x_n+x_3 & \cdots & x_n+x_n \end{bmatrix}. \tag{2}$$

In this study, we find the eigenvalues and singular values of matrix A in (2) and we also calculate the spectral and Euclidean norms of this matrix.

3. Main Results

With the following theorem, we give the eigenvalues and singular values of the matrix considered in (2).

Theorem 3.1 : *Except for* $n-2$ *zeroes, the eigenvalues and singular values of A in (2), respectively, are*

$$\lambda_1 = \alpha + \sqrt{\beta n}, \quad \lambda_2 = \alpha - \sqrt{\beta n}$$

and

$$\sigma_1 = \left|\alpha + \sqrt{\beta n}\right|, \quad \sigma_2 = \left|\alpha - \sqrt{\beta n}\right| \tag{3}$$

where $i^2 = -1$, $\alpha = \sum_{k=1}^{n} x_k$ *and* $\beta = \sum_{k=1}^{n} x_k^2$.

Proof: We write the matrix A given in (2) as follows:

$$A = \begin{bmatrix} x_1 + x_1 & x_1 + x_2 & x_1 + x_3 & \cdots & x_1 + x_n \\ x_2 + x_1 & x_2 + x_2 & x_2 + x_3 & \cdots & x_2 + x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n + x_1 & x_n + x_2 & x_n + x_3 & \cdots & x_n + x_n \end{bmatrix}$$
$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

We know that A has n real eigenvalues, counting multiplicity. Let

$$v = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \text{ and } e = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} \in \mathbb{R}^n.$$

Except for $n-2$ zeroes, the eigenvalues of A are the same as those of

$$B = \begin{bmatrix} e^T v & e^T e \\ v^T v & v^T e \end{bmatrix}.$$

We, respectively, get the real numbers $e^T v$, $e^T e$, $v^T v$ and $v^T e$ as follows:

$$e^T v = \sum_{k=1}^{n} x_k,$$

$$e^T e = \sum_{k=1}^{n} 1 = n,$$

$$v^T v = \sum_{k=1}^{n} x_k^2,$$

$$v^T e = \sum_{k=1}^{n} x_k.$$

Let

$$\alpha = \sum_{k=1}^{n} x_k \text{ and } \beta = \sum_{k=1}^{n} x_k^2. \tag{4}$$

So, the B matrix will be in the form of

$$B = \begin{bmatrix} \alpha & n \\ \beta & \alpha \end{bmatrix}.$$

The eigenvalues of the B matrix are

$$\lambda_1 = \alpha + \sqrt{\beta n}, \quad \lambda_2 = \alpha - \sqrt{\beta n}.$$

Since A is a symmetric matrix, A is a normal matrix. Thus, since A is a normal matrix, $\sigma_i = |\lambda_i|$ for all $i = 1, \ldots, n$. The proof of the theorem is completed with this result.

Theorem 3.2 : *The spectral norm and Euclidean norms of A in (2), respectively, are*

$$\| A \|_2 = \left| \alpha + \sqrt{\beta n} \right|$$

and

$$\| A \|_E = \sqrt{2(\alpha^2 + \beta n)}$$

with $\alpha = \sum_{k=1}^{n} x_k$ *and* $\beta = \sum_{k=1}^{n} x_k^2$.

Proof: Since the spectral norm of A is defined by the largest singular value of A we obtain from Theorem 1 ,

$$\| A \|_2 = \left| \alpha + \sqrt{\beta n} \right|.$$

The Euclidean norm of A is from (1) and (3)

$$\begin{aligned} \| A \|_E &= \sqrt{\sum_i^n \sigma_i^2(A)} \\ &= \sqrt{2(\alpha^2 + \beta n)}. \end{aligned}$$

Consequently, the proof of the theorem is completed.

4. Numerical examples

The following example illustrates Theorem 1.

Example 4.1 : We consider the 3 -by-3 real skew-symmetric Toeplitz matrix

$$C = [i + j]_{i,j=1}^n = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}. \tag{5}$$

The values of α and β in (4) for the C matrix are calculated as follows:

$$\alpha = \sum_{k=1}^3 x_k = 6, \tag{6}$$

$$\beta = \sum_{k=1}^3 x_k^2 = 14, \tag{7}$$

where $x_1 = 1, x_2 = 2, x_3 = 3$. Except for $n - 2 = 3 - 2 = 1$ zero, the eigenvalues λ_1, λ_2 and singular values σ_1, σ_2 of C from Theorem 3.1, respectively, are

$$\lambda_1 = \alpha + \sqrt{\beta n} = 6 + \sqrt{42} = 12.4807, \quad \lambda_2 = \alpha - \sqrt{\beta n} = 6 - \sqrt{42} = -0.4807,$$

and

$$\sigma_1 = \left| \alpha + \sqrt{\beta n} \right| = 6 + \sqrt{42} = 12.4807, \quad \sigma_2 = \left| \alpha - \sqrt{\beta n} \right| = \left| 6 - \sqrt{42} \right| = 0.4807.$$

Using a computer program, we get the eigenvalues and singular values of the C matrix as

$$\lambda_1 = 6+\sqrt{42}, \quad \lambda_2 = 6-\sqrt{42}, \quad \lambda_3 = 0$$

and

$$\sigma_1 = 6+\sqrt{42}, \quad \sigma_2 = \left|6-\sqrt{42}\right|, \quad \sigma_3 = 0,$$

respectively,

The following example illustrates Theorem 3.2.

Example 4.2 : We consider the 3 -by-3 real skew-symmetric Toeplitz matrix in 5. Using a computer program, we find the spectral and Euclidean norms of C as $\| C \|_2 = 6+\sqrt{42} = 12.4807$ and $\| C \|_E = 2\sqrt{39} = 12.489,$ respectively. For $x_1 = 1, x_2 = 2, x_3 = 3$ and $n = 3,$ we verify Theorem (3.2) by (6) and (7).

5. Conclusion

We have obtained the eigenvalues, singular values, the spectral and Euclidean norms of A in (2). As the examples show, the results obtained in Theorem 1 and Thereom 2 to calculate the eigenvalues, singular values, the spectral and Euclidean norms of A are simply applicable.

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