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Some properties of bases intersection graph-I

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Abstract

In the recent paper Iram Tahleel Jaleel Ahmed, Suryakant M Jogdand , introduced the graphical structure of vector space over a finite field $\mathbb{F}$ called as "Bases Intersection Graph" and discussed some basic properties of it. In the present paper, we continue the study of bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ over a finite field $\mathbb{F}$ regarding some important properties. It is shown that the graph is regular graph, Eulerian graph etc. Further it is shown that graph

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is never bipartite. In the last, some properties of 2 dimensional vector space over a field of 3 elements were discussed.

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1. Introduction

Initially the concept of zero divisor graph were introduced by Beck [1] in 1988. Concept includes study of zero divisor graph of a commutative ring with unity. After Beck,Anderson and Livingston [2] gives a details regarding properties of zero divisor graph over a commutative ring with identity element. Nowadays it is most interesting and attractive topic for researchers to define algebraic structure in terms of graph and viceversa. In this regard a lot of research is done e.g. in [3], Akbari and Habibi define inclusion ideal graph, in [4, 5, 6, 7], A.Das defines Non zero component graph, subspace inclusion graph etc. Parastoo Malakooti Rad and Lotf Ali Mahdavi [8] defines the intersection graph of nontrivial submodules of M, denoted by G(M), which is an undirected simple graph whose vertices are in one-to-one correspondence with all nontrivial submodules of M and two distinct vertices are adjacent if and only if the intersection of corresponding submodules is a nonzero submodule of M. Later on in this connection R.A.Muneshwar and K.L.Bondar [9, 10, 11] defines open subset inclusion graph of a topological space and discussed some properties of these graphs such as girth, diameter, completeness, connectivity, clique, chromatic numbers, independent set etc. For sake of understanding of undefined terms reader may refer [13]

Bases intersection graph

In [11] authors introduced the definition of bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ of a vector space $\mathbb{V}$ over a field $\mathbb{F}$, as follows :

Definition 2.1 : Let $\mathbb{V}$ be any finite dimensional vector space over a field F. We define a bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V}) = (\mathcal{V}, E)$, where $\mathcal{V} =$ the collection of all bases of $\mathbb{V}$ over F and for $B_1, B_2 \in \mathcal{V}, B_1 \sim B_2$ or $(B_1, B_2) \in E$ if and only if $B_1 \cap B_2 \neq \phi$.

Theorem 2.2 : *If $\mathbb{V}$ is a vector space over a field F and W is a subspace of $\mathbb{V}$ then $\mathcal{I}_{\mathcal{B}}(W)$ is a subgraph of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.*

Proof: Let $\mathbb{V}$ is a vector space over a field F and W is a subspace of $\mathbb{V}$. Let B_1 and B_2 are bases of W such that $B_1 \cap B_2 \neq \phi$, therefore by definition of $\mathcal{I}_{\mathcal{B}}(W)$, $B_1 \sim B_2$. Now as B_1 and B_2 are bases of W, they contained linearly independent vectors of W and hence $\mathbb{V}(\because W \subseteq \mathbb{V})$. Therefore by bases extension theorem, these linearly independent set B_1 and B_2 can be extended to form a bases B_1' and B_2' of $\mathbb{V}$ such that $B_1' \cap B_2' \neq \phi$, and hence $B_1' \sim B_2'$. This proves that $\mathcal{I}_{\mathcal{B}}(W)$ is a subgraph of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.

Corollary 2.3 : *If $\mathbb{V}$ is a vector space over a field F and $dim(\mathbb{V}) = 1$, then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is an null graph.*

Proof: We have $dim(\mathbb{V}) = 1$ then every basis of $\mathbb{V}$ contains only one nonzero vector. Thus any two distinct bases B_i, B_j of a vector space $\mathbb{V}$ dose not have any vector in common, i.e $B_i \cap B_j = \phi$, for all $i \neq j$ and $B_i, B_j \in B$. This shows that $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is null graph.

Corollary 2.4 : *If $\mathbb{V}$ is a vector space over a field F then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not complete graph.*

Proof: Let $\mathbb{V}$ is a vector space over a field F and $dim(\mathbb{V}) = n$. If $dim(\mathbb{V}) = 1$, then it is obvious from Corollary 3.3. Let $dim(\mathbb{V}) = n > 1$, and let $B_1 = \{e_1, e_2, e_3, \ldots, e_n\}$ be any basis of $\mathbb{V}$. Thus $B_2 = \{e_1 + e_2, e_2 + e_3, \ldots, e_n + e_1\}$ is also a basis of $\mathbb{V}$, such that $B_1 \cap B_2 = \phi$, and hence $B_1 \nsim B_2$. This shows that $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not complete graph.

Lemma 2.5 : *Let $\mathbb{V}$ be a finite dimensional vector space over a field F and $dim(\mathbb{V}) \geq 2$, then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is connected and $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) \geq 2$.*

Proof: Let $dim(\mathbb{V}) = n \geq 2$, and let $B_1 = \{u_1, u_2, \ldots, u_n\}$, $B_2 = \{v_1, v_2, \ldots, v_n\}$ are bases of $\mathbb{V}$. Then we have two cases.

Case 1: If $B_1 \cap B_2 \neq \phi$, then by definition of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$, we have, $B_1 \sim B_2$ be a required path from B_1 to B_2 and $dist(B_1, B_2) = 1$.

Case 2: If $B_1 \cap B_2 = \phi$, then there exists $u_i \in B_1$, and $v_j \in B_2$, for some i, j, such that, $u_i \neq \alpha v_j$, and hence u_i and v_j are linearly independent vectors of $\mathbb{V}$. Therefore by Basis extension theorem, we can construct a basis B_3 as, $B_3 = \{u_i, v_j, w_3, w_4, w_5, \ldots, w_n\}$ such that, $B_1 \cap B_3 = u_i \neq \phi$ and $B_2 \cap B_3 = v_j \neq \phi$ and hence $B_1 \sim B_3$ and $B_2 \sim B_3$. Thus $B_1 \sim B_3 \sim B_2$ be a required path between B_1 and B_2 and $dist(B_1, B_2) = 2$.

Hence $\mathcal{I}_{\mathcal{B}}(V)$ is connected and $diam(\mathcal{I}_{\mathcal{B}}(V)) \geq 2$.

Theorem 2.6 : *If* $\mathbb{V}$ *is a vector space over a field F and* $dim(\mathbb{V}) \geq 2$, *then* $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = 2$.

Proof: Let B_1 and B_2 be any two bases then by the lemma 2.5, either $B_1 \sim B_2$ or there exist $B_3 \in \mathcal{B}$ such that, $B_1 \sim B_2 \sim B_3$. Therefore $dist(B_1, B_2) \leq 2$ and hence $diam(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = 2$.

Theorem 2.7 : *Let* $\mathbb{V}$ *be an n-dimensional vector space over a field F, then*

$$girth(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \begin{cases} 3 & \text{if } n > 1. \\ \infty & \text{if } n = 1. \end{cases}$$

Proof: Let $\mathbb{V}$ be an n-dimensional vector space over a field F.

Case I: If $n = 1$ then by Corollary 3.2, the graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is null graph and hence $girth(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \infty$.

Case -II: If $n > 1$, then consider $B_1 = \{u_1, u_2, u_3, \ldots, u_n\}$ be a basis of $\mathbb{V}$, then clearly $u_1, u_2, u_3, \ldots, u_n$ are linearly independent vectors of $\mathbb{V}$. In particular, the set $\{u_1\}$ is linearly independent, so by Basis extension theorem, we can extend this upto a basis $B_2 = \{u_1, v_2, v_3, \ldots, v_n\}$ of V. Then clearly $B_1 \cap B_2 = \{u_1\} \neq \phi$, and hence $B_1 \sim B_2$. Choose some u_i in B_1 such that $u_i \neq \alpha v_k, \forall k = 2, 3, \ldots, n$, and let it be u_1 and v_2 in perticular, then clearly u_1 and v_2 are linearly independent vectors in $\mathbb{V}$. Then by Basis extension theorem, we can form a basis B_3 as, $B_3 = \{u_1, v_2, w_3, w_4, \ldots, w_n\}$ such that $B_1 \cap B_2 \neq \phi$, $B_1 \cap B_3 \neq \phi$, $B_2 \cap B_3 \neq \phi$, and hence $B_1 \sim B_2 \sim B_3 \sim B_1$ be the required cycle of length 3.

Theorem 2.8 : *Let* $dim(\mathbb{V}) > 1$, *then* $I_{\mathcal{B}}(\mathbb{V})$ *is triangulated graph.*

Proof: Proof of this theorem follows obviously from Theorem 2.7.

Theorem 2.9 : *Let* $\mathbb{V}$ *be a n-dimensional vector space over a finite field F of order p. Let K be a collection of all bases of a vector space* $\mathbb{V}$ *having one vector in common then K be the maximal clique of* $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ *having clique number*

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{(n-1)!}$$

Proof: Let $\mathbb{V}$ be a finite dimensional vector space over a field F and B be a collection of all bases of a vector space $\mathbb{V}$. . Let K is a collection of all bases of a vector space $\mathbb{V}$ having one vector in common.

$$i.e. K = \{B_i \mid \alpha \in B_i \forall i\} i.e. \cap B_i = \{\alpha\} \tag{1}$$

i.e. $K = \{B_i \mid B_i \cap B_j = \{\alpha\} \neq \phi, \forall i \neq j\}$ then $B_i \cap B_j = \{\alpha\} \neq \phi$ and hence $B_i \sim B_j, \forall B_i, B_j \in K$. Thus K is a clique in $\mathcal{I}_{\mathcal{B}}(V)$. We know that, if V be a finite dimensional vector space over a field F of order p then the number of unordered basis of V is

$$\frac{(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{n!} \tag{2}$$

We have to find the clique number, as clique is the collection of those bases having one vector in common.

$\therefore$ Number of possibilities of selecting one vector in common is $\frac{p^n - 1}{n}$

$\therefore$ The total number of basis having one vector in common is given by

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{\frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{n!}}{\frac{p^n-1}{n}} \tag{3}$$

$$\therefore \qquad \omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{(n-1)!} \tag{4}$$

Which is the required clique number of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$.

Now as $K = \{B_i \mid \alpha \in B_i \forall i\} i.e. \cap B_i = \{\alpha\}$ We will show this K is a maximal clique. If possible suppose $K' = K \cup \{B\}$ is a clique where $B \not\subseteq K$ i.e. $\alpha \not\subseteq B$. As $B \not\subseteq K'$ and K' is a clique. $\Rightarrow B \cap B_i \neq \phi \forall i$ therefore there exist some $\gamma \neq$ such that $\gamma \in B \cap B_i \forall i \Rightarrow \gamma \in B$ and $\gamma \in B_i, \forall i \Rightarrow \alpha, \gamma \in B_i \forall i$. $\Rightarrow K$ contains two vectors in common. Number of possibilities of selecting two vectors in common is $\frac{(p^n - 1)(p^n - p)}{n(n-1)}$

$\therefore$ The total number of basis having one vector in common is given by

$$\frac{\frac{(p^n-1)(p^n-p)(p^n-p^2)\cdots(p^n-p^{n-1})}{n!}}{\frac{(p^n-1)(p^n-p)}{n(n-1)}}$$

which is the number less than number given in equation (3), the contradiction, hence K' is not a clique.

Corollary 2.10 : *Let $\mathbb{V}$ be an n-dimensional vector space over a finite field of order P with $n > 1$ and $p > 2$ then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is not a planar graph.*

Proof: Let $\mathbb{V}$ be a finite dimensional vector space over a field F of order p then

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V})) = \frac{(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{(n-1)!}$$

Then for $n > 1$ and $p > 2$, $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ has a clique of size atleast 6, i.e. there exist a subgraph which is isomorphic to K_5 or $K_{3,3}$. Thus it is not planar.

3. Main results

Now we will prove some more results on bases intersection graph.

Theorem 3.1 : *Let $\mathbb{V}$ be an n dimensional vector space over a finite field* **F**, *then the bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is never bipartite for $n > 1$.*

Proof: It is given that $\mathbb{V}$ is a finite dimensional vector space over a finite field **F**, then by theorem 2.7, girth of the graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is 3. This shows the presence of cycles of length 3 (odd) and hence $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is never bipartite for $n > 1$.

Theorem 3.2 : *Let $\mathbb{V}$ be an n dimensional vector space over a finite field of p element then degree of vertex $\mathrm{B} = \{e_1, e_2, ..., e_n\}$ of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is*

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]...[p^n - p^{n-1} - (n-1)]\}$$

Proof: Let $\mathbf{B} = \{e_1, e_2, ..., e_n\}$ be any basis of a vector space $\mathbb{V}$ over a finite field of p element. We have to find degree of a vertex $\mathbf{B} = \{e_1, e_2, \cdots, e_n\}$ of its bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$. i.e to find number of bases which are adjacent to **B**. This means we have to find those bases of a vector space $\mathbb{V}$ whose intersection with this basis **B** is non empty.

Let $\mathbf{B}_1 = \{e_1, u_2, u_3, \cdots, u_n\}$ such that $e_i \neq u_j$ for all $i, j = 2, 3, \cdots, n$ and hence $\mathbf{B} \cap \mathbf{B}_1 = \{e_1\} \neq \phi$ therefor by definition of bases intersection graph $\mathbf{B} \sim \mathbf{B}_1$.

Therefore number of possibilities to $u_2 = p^n - p - (n-1)$, $u_3 = p^n - p^2 - (n-1)$, $\cdots$, $u_n = p^n - p^{n-1} - (n-1)$. Therefore number of possibilities for $\mathbf{B}_1$ is given by

$$[p^n - p-(n-1)].[p^n - p^2-(n-1)].\cdots.[p^n - p^{n-1}-(n-1)] \tag{5}$$

Now let $\mathbf{B}_2 = \{v_1, e_2, v_3, \cdots, v_n\}$ such that $v_i \neq e_j$ for all i, j.

Clearly by definition of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$, $\mathbf{B} \sim \mathbf{B}_2$. Therefore number of possibilities to $v_1 = p^n - p - (n-1)$, $v_3 = p^n - p^2 - (n-1)$, $\cdots$, $v_n = p^n - p^{n-1} - (n-1)$, Therefore number of possibilities for $\mathbf{B}_2$ is given by

$$[p^n - p-(n-1)].[p^n - p^2-(n-1)].\cdots.[p^n - p^{n-1}-(n-1)] \tag{6}$$

Continuing in this way we get the number of basis which are adjacent to **B**. i.e.

$$\{p^n - p - (n-1)]\cdot [p^n - p^2 - (n-1)]\cdot \ldots\cdot [p^n - p^{n-1} - (n-1)]\}+$$

$$\{p^n - p - (n-1)]\cdot [p^n - p^2 - (n-1)]\cdot \ldots\cdot [p^n - p^{n-1} - (n-1)]\}+$$

$$\ldots+$$

$$\underbrace{\{p^n - p - (n-1)]\cdot [p^n - p^2 - (n-1)]\cdot \ldots\cdot [p^n - p^{n-1} - (n-1)]\}}_{n\cdot \text{ times}}$$

i.e. number of bases which are adjacent to **B** is

$$n[p^n - p-(n-1)].[p^n - p^2-(n-1)].\cdots.[p^n - p^{n-1}-(n-1)] \tag{7}$$

Therefore degree of any vertex of $\mathbf{B} = \{e_1, e_2, ..., e_n\}$ of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]...[p^n - p^{n-1} - (n-1)]\}$$

Corollary 3.3 : *Let $\mathbb{V}$ be a finite dimensional vector space over a finite field then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is a regular graph.*

Proof: it is given that $\mathbb{V}$ is a finite dimensional vector space over a finite field then by theorem we have degree of any vertex $\mathbf{B} = \{e_1, e_2, ..., e_n\}$ of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]...[p^n - p^{n-1} - (n-1)]\},$$

hence $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is a regular graph.

Remark: As $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is a regular graph then minimum degree δ of the graph is same as degree of vertex i.e.

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]\ldots[p^n - p^{n-1} - (n-1)]\}.$$

Theorem 3.4 : L*et* $\mathbb{V}$ *be a finite dimensional vector space over a finite field then the vertex connectivity of* $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ *is*

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]\ldots[p^n - p^{n-1} - (n-1)]\}.$$

Proof: By the corollary 3.3 $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is regular graph with degree

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]\ldots[p^n - p^{n-1} - (n-1)]\}$$

and hence by example 17[9] vertex connectivity of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]\ldots[p^n - p^{n-1} - (n-1)]\}.$$

Theorem 3.5 : *Let* $\mathbb{V}$ *be a finite dimensional vector space over a finite field then* $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ *is an Eulerian graph.*

Proof: We know that for n dimensional vector space define over a finite field of p elements, degree of any vertex of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is given by

$$n\{[p^n - p - (n-1)][p^n - p^2 - (n-1)]\ldots[p^n - p^{n-1} - (n-1)]\}$$

Case I : If n is even then degree of vertex is even.

Case II: If n is odd then the term $[p^n - p - (n-1)]$ is even and hence degree of vertex is even in this case also.

Hence in both the cases degree of each vertex of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is even. Hence by theorem $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is an Eulerian graph.

Theorem 3.6 : $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ *is a complete graph iff* $dim(\mathbb{V}) = 2$ *and* $|\mathbb{F}| = 2.$

Proof: Let $\mathbb{V}$ be a finite dimensional vector space over a finite field. Let $dim(\mathbb{V}) = 2$ and $|\mathbb{F}| = 2,$ then by theorem [.]number of vertices of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ are

$$(2^2 - 1).(2^2 - 2)/2! = 3$$

and by above theorem degree of each vertex is

$$2.[2^2 - 2 - (2-1)] = 2.[4 - 2 - 1] = 2$$

which is one less than the number of vertex. Therefore by theorem $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is a complete graph.

Conversely, suppose $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is a complete graph. Therefor by theorem degree of each vertex of $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is one less than number of vertices, which holds only for the case $dim(\mathbb{V}) = 2$ and $|\mathbb{F}| = 2$ i.e. $n = 2$ and $p = 2$.

Theorem 3.7 : *Let $\mathbb{V}_1$ and $\mathbb{V}_2$ are two finite dimensional vector space defined over the same finite field then $\mathbb{V}_1$ and $\mathbb{V}_2$ are isomorphic if and only if $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ and $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_2)$ are isomorphic as a graph.*

Proof: It is given that $\mathbb{V}_1$ and $\mathbb{V}_2$ are two finite dimensional vector space defined over the same finite field. Firstly suppose $\mathbb{V}_1$ and $\mathbb{V}_2$ are isomorphic as a vector space. Therefore their corresponding bases are also isomorphic and hence their corresponding bases intersection graphs are also isomorphic. Hence $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ and $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_2)$ are isomorphic as a graph.

Conversely, suppose $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ and $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_2)$ are isomorphic as a graph.

Now let $dim(\mathbb{V}_1) = n$ and $dim(\mathbb{V}_2) = m$ and $|\mathbb{F}| = p,$ than the clique number of $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ is given by,

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)) = \frac{(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})}{(n-1)!} \tag{8}$$

similarly, than the clique number of $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ is given by,

$$\omega(\mathcal{I}_{\mathcal{B}}(\mathbb{V}_2)) = \frac{(p^m - p)(p^m - p^2)\cdots(p^m - p^{m-1})}{(m-1)!} \tag{9}$$

and it is given that $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_1)$ and $\mathcal{I}_{\mathcal{B}}(\mathbb{V}_2)$ are isomorphic as a graph than by the property of isomorphic graphs both the equations (8) and (9) are equal and this is possible only for $n = m.$ Thus dimensions of vector spaces $\mathbb{V}_1$ and $\mathbb{V}_2$ are same. And we have vector space having the same dimensions defined over the same finite field are isomorphic. Therefor $\mathbb{V}_1 \sim \mathbb{V}_2.$

4. Some Properties of bases intersection graph of Vector Space $\mathbb{V}$ Having Dimension 2 Defined Over a Finite Field of 3 Elements:

In this section, we discuss some special properties like automorphism, vertex transitivity, edge transitive and independence etc, of $\mathbb{V}$ Having Dimension 2 Defined Over a Finite Field of 3 Elements.

Theorem 4.1 : *If $\mathbb{V}$ is a vector space of dimension 2 defined over a field of 3 elements then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is compete graph.*

Proof: It is given that $\mathbb{V}$ is a vector space of dimension 2 defined over a field of 3 elements then by theorem 3.6 we have $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is complete graph.

Theorem 4.2 : *If $\mathbb{V}$ is a vector space of dimension 2 defined over a field of 3 elements then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is $10-$ regular graph. Moreover order and size of the graph are 24 and 120 respectively.*

Proof: It is given that $\mathbb{V}$ is a vector space of dimension 2 defined over a field of 3 elements, that is from equation (2) for $p=3$ and $n=2$ we have order of graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is $(3^2-1)(3^2-3)/2!=24.$ Also from theorem 3.5 degree of each vertex is $2.[3^2-3-(2-1)]=2.[9-3-1]=10$ and hence graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is $10-$ *regular* graph.Thus, if m is the number of edges in $\mathcal{I}_{\mathcal{B}}(\mathbb{V}),$ then by degree sum formula we have, the size of the graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is $m=120.$

Theorem 4.3 : *If $\mathbb{V}$ is a vector space of dimension 2 defined over a field of 3 elements then $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is edge transitive graph.*

Proof: Let B_1,B_2,B_3,B_3 be the basis of a vector space $\mathbb{V}$ of dimension 2. Without loss of generality let $B_1\cap B_2=\{u_1\}\neq\phi.$ Therefore we have $B_1=\{u_1,v_1\}$ and $B_2=\{u_1,v_2\}$ then clearly $B_1\sim B_2.$ Now define a map ϕ from vertex set $\mathcal{V}$ to itself, defined by $\phi(B)=\phi\{u,v\}=\{2u,v\}$ then clearly, ϕ induces a graph automorphism on $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ and it maps B_1 as $\phi(B_1)=\phi\{u_1,v_1\}=\{2.u_1,v_1\}=B_3$ (say). Similarly $\phi(B_2)=\phi\{u_1,v_2\}=\{2.u_1,v_2\}=B_4(say),$ such that $B_3\cap B_4=\{2.u_1\}\neq\phi$ and hence $B_3\sim B_4.$

5. Conclusion

In this work, we studied various properties of bases intersection graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$. We studied bipartiteness of graph, regularity, degree of vertex, degree of graph, vertex connectivity, minimum degree etc. Further more we proved that the graph is an Eulerian graph, criterion for graph to be complete. We have proved that the graphs are isomorphic iff corresponding vector space are isomorphic. Main result of this paper is that the graph $\mathcal{I}_{\mathcal{B}}(\mathbb{V})$ is edge transitive graph for a vector space $\mathbb{V}$ of dimension 2 defined over a field of 3 element.

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