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On the super (a, d)-H-antimagic total labelings of three graphs

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Abstract

The conception of (a, d)-H-super antimagic total labeling is generated by graph covering, magic square and graph labeling. In this paper we study the (a, d)-H-super antimagic total labelings of the friendship graph F_n, wheel W_n and the graph mG.

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1. Introduction

It is convenient to adopt the following notations. Let Z be the ring of integers. The symbols $[a,b]$ and $[a,b]_k$ denote the set $\{x \mid x \in Z,\ a \le x \le b\}$ and the set $\{x \in Z \mid\ a \le x \le b,\ x \equiv a (\mathrm{mod}\, k)\}$, respectively. If $f: V_1 \to V_2$ is a mapping and $V_3 \subseteq V_1$, we denote $\{f(v) \mid v \in V_3\}$ by $f(V_3)$. If A is a number set, then we denote the sum $\sum\limits_{i \in A} i$ by $\sum A$. In the paper we denote by $V(H)$ and $E(H)$ the set of vertices and the set of edges of a graph H, respectively. A graph H is called a (p, q)-graph if it has $|V(H)| = p$ vertices and $|E(H)| = q$ edges. An H-covering of a graph G is a set of subgraphs of G, say $\{H_1, H_2, \ldots, H_n\}$, such that each edge of G belongs to at least one of the subgraphs $H_i, i \in [1, n]$. Let the simple graph G admits an H-covering $\mathcal{B} = \{H_1, H_2, \ldots, H_n\}$. The graph G is said to be (a, d)-H-antimagic total (simply, (a, d)-H-AMT) if there exists a bijection $f: V(G) \cup E(G) \to [1, |V(G) \cup E(G)|]$ such

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that for all $H' \in \mathcal{B}$, $\sum_{v \in V(H')} f(v) + \sum_{e \in E(H')} f(e)$ (called weight of H' and denoted by $w_f(H')$ or $\sum f(H')$) form an arithmetic progression $a, a+d, a+2d, \ldots, a+(n-1)d$, where a and d are two integers. An (a, d)-H-antimagic total graph G is said to be (a, d)-H-super antimagic total (simply, (a, d)-H-SAMT) if $f(V(G)) = [1, |V(G)|]$. The f is said to be an (a, d)-H-(super) antimagic total labeling of G (simply, (a, d)-H-(S)AMTL).

An (a, d)-H-AMTL of G is an (a, d)-edge-antimagic total labeling (simply, (a, d)-edge-AMTL) of G if $H \cong K_2$. Definitions of (a, d)-edge-AMTL and (a, d)-edge-SAMTL were natural extensions of the concept of magic valuation (see [1] and [2]). Liang [6] and [7] completely settled the problems of the super (a, d)-edge-AMTL of the complete bipartite graph and the G-supermagic of $I(G, H, m)$ and $m \odot G$. Zhu and Liang [3] studied the (a, d)-H-super antimagic total labelings of path chain graph $PCG(H, U, n)$ when $d = 0, 1, 2, 3, 4$. For a survey of various graph labeling problems one may refer to Gallian [8]. In [4] and [5] authors studied the irregular chromatic number of some graphs and the topological index problems, respectively.

Problem 1: (see [9]) Known if $n \geq 5$ is odd, then the wheel W_n has $(a, 0)$-C_3-SAMTL. The open problem is that does the wheel W_{2r} has $(a, 0)$-C_3-SAMTL if $r > 1$?

Problem 2: Bača et al. in [10] proved

(1) if the friendship graph F_n has (a, d)-edge-SAMTL, then $d < 3$;

(2) when $n \in \{1, 3, 4, 5, 7\}$ and $d = 0, 2$ the graph F_n has (a, d)-edge-SAMTL;

(3) the graph F_n has $(a, 1)$-edge-SAMTL. Slamin et al. in [11] obtained that F_n has an $(a, 0)$-edge-SAMTL if and only if $n \in \{1, 3, 4, 5, 7\}$. Nalliah in [12] proved F_n has not $(a, 2)$-edge-SAMTL if n is even and $n \not\equiv 4 \pmod{12}$. Open problem: whether F_n has $(a, 2)$-edge-SAMTL if $n \geq 8$?

Problem 3: The disjoint union of m copies of G is denoted by mG. Bača et al. in [13] proved that if T is tree and G has an $(a, 1)$-T-SAMTL, then mG has also $(b, 1)$-T-SAMTL if $m \geq 1$. They proposed the following problem: if G has an (a, d)-H-SAMTL, whether mG has an (a', d')-H-SAMTL?

In this paper we mainly solve these problems.

2. The (a, d)-C_3-SAMTL of wheel W_n

In this section we'll study Problem 1: (a, d)-C_3-SAMTL of wheel W_n for any integer $n \geq 4$ and $d = 0$ or 1. For any integer $n \geq 3$, the wheel W_n is the $n + 1$-vertex graph obtained by joining a vertex v to each of the n vertices $\{v_1, v_2, \cdots, v_n\}$ of the cycle C_n. The vertex v is called the center of the wheel. Let the subgraph H_i of W_n be the C_3 with vertex set $\{v, v_i, v_{i+1}\}$, $1 \leq i \leq n-1$, and H_n be the C_3 with vertex set $\{v, v_n, v_1\}$. Then $\mathcal{B} = \{H_1, H_2, ..., H_n\}$ is a C_3-covering of W_n.

Theorem 2.1: *Wheel W_{2r} is an $(a, 0)$- C_3-SAMT graph, where $a = 19r + 5$ and $r > 1$.*

***Proof*:** Let $n = 2r$. We construct a mapping $f : V(W_n) \cup E(W_n) \to [1, 3n+1]$ as follows:

When $i \in [1, r]$, $f(v_{2i-1}) = 6r - 2i + 3, f(v_{2i-1}v_{2i}) = 2r + 2i$;

When $i \in [1, r-1]$, $f(v_{2i}) = 6r - 2i, f(vv_{2i-1}) = i, f(vv_{2i}) = r + 1 + i$;

When $i = r$, $f(v_{2i}) = 2r + 1, f(vv_{2i-1}) = 4r - 1, f(vv_{2i}) = r + 1$;

When $i \in [1, r-2]$, $f(v_{2i}v_{2i+1}) = 2r + 2i + 1$;

When $i = r-1$, $f(v_{2i}v_{2i+1}) = r$, and $f(v) = 4r - 1, f(v_{2r}v_1) = 6r$.

Thus when $i \in [1, r]$ there are $f(v_{2i-1}) \in [4r+3, 6r+1]_2$, $f(v_{2i}) \in [4r+2, 6r-2]_2 \cup \{2r+1\}$, $f(v_{2i-1}v_{2i}) \in [2r+2, 4r]_2$, $f(vv_{2i-1}) \in [1, r-1] \cup \{4r-1\}$, $f(vv_{2i}) \in [r+2, 2r] \cup \{r+1\} = [r+1, 2r]$.

When $i \in [1, r-1]$ there are $f(v_{2i}v_{2i+1}) \in [2r+3, 4r-3]_2 \cup \{r\}$, $f(v_{2r}v_1) = 6r, f(v) = 4r + 1$. Therefore f is a bijection from $V(W_n) \cup E(W_n)$ to $[1, 3n + 1]$.

It is easy to verify that each graph $H_i, i \in [1, n]$ in $\mathcal{B}$ satisfies $\sum f(H_i)$ $=19r + 5$. Therefore wheel W_{2r} is a $(19r + 5, 0)$- C_3-SAMT graph for any $r > 1$. □

Theorem 2.2: *Wheel W_n is an $(a, 1)$-C_3-SAMT graph for any integer $n \geq 4$.*

***Proof*:** Let $\mathcal{B} = \{H_1, H_2, ..., H_n\}$ be a C_3-covering of W_n. We construct a mapping $f : V(W_n) \cup E(W_n) \to [1, 3n+1]$ as follows:

$$f(v_iv_{i+1}) = 3n + 2 - i,\ 1 \leq i \leq n-1,\ f(v) = 1, f(v_nv_1) = 2n + 2.$$

We have $f(v_iv_{i+1}) \in [2n+3, 3n+1]$, and $f(v_nv_1) = 2n + 2$. Therefore $f(E(C_n)) =[2n + 2, 3n + 1]$.

(1) When n is odd, let

$$f(v_i)=\begin{cases}\frac{n+i}{2}+1, & i\in[1,n]_2\\ \frac{i}{2}+1, & i\in[2,n-1]_2\end{cases}\quad f(vv_i)=\begin{cases}2n+2-\frac{n+i}{2}, & i\in[1,n]_2\\ 2n+2-\frac{i}{2}, & i\in[2,n-1]_2\end{cases}$$

(2) When n is even, let

$$f(v_i)=\begin{cases}\frac{i+1}{2}+1, & i\in[1,n-1]_2\\ \frac{n+i}{2}+1, & i\in[2,n]_2\end{cases}\quad f(vv_i)=\begin{cases}n+1+\frac{i+1}{2}, & i\in[1,n-1]_2\\ n+1+\frac{n+i}{2}, & i\in[2,n]_2\end{cases}$$

Then f is a bijection from $V(W_n)\cup E(W_n)$ to $[1, 3n+1]$, and $f(V(W_n)) = [1, n + 1]$, $f(E(W_n)) = [n + 2, 3n + 1]$.

It is easy to verify that $\{\sum f(H_1),\ \sum f(H_2),\ \cdots,\ \sum f(H_n)\} = [6n + 9, 7n + 8]$ is a set of consecutive integers. Summarizing the above results, we have proved that W_n is an $(a, 1)$-C_3-SAMT graph, where $n\geq 4$ and $a = 6n + 9$. □

3. On the (a, 2)-edge-SAMTL of F_n

The friendship (or Dutch windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle C_3 of length 3 with a common vertex (called center of F_n). On the (a, d)-edge-SAMT problem of friendship graph F_n, Bača et al. in [10] and [11] have solved the problem for the case d = 0, 1. In this section we'll solve this problem for the case d = 2. This implies that the (a, d)-edge-SAMT problem of F_n has been completely solved.

Lemma 3.1: *(See* [14]*) Let* $\gcd(a_1, a_2) = d > 1$. *If diophantine equation* $a_1x + a_2y = n$ (I) *has a particular solution* (x_0, y_0), *then general solution of* (I) is

$$\begin{cases}x=x_0+\frac{a_2}{d}t\\ y=y_0-\frac{a_1}{d}t\end{cases}\quad t=0,\pm1,\pm2,\cdots$$

Lemma 3.2: [14]

(1) *The diophantine equation* (I) *has a solution in integers* x, y *if and only if* $\gcd(a_1, a_2)$ *divides* n.

(2) *Integers a and b are relatively prime if and only if there exist integers s and t such that* $sa + tb = 1$.

Let graph G has an (a, d)*-edge-*SAMTL *f. We have*

$$\sum_{v\in V(G)} d(v)f(v)+\sum_{e\in E(G)} f(e)=\sum_{e\in E(G)} w_f(e) \tag{II}$$

Where $d(v)$ is the degree of vertex v on G. We know that friendship graph F_n has $|V(F_n)| = p = 2n + 1$ vertices and $|E(F_n)| = q = 3n$ edges. For center c of F_n we have $d(c) = 2n$, and $d(v) = 2$ for other vertex of F_n.

*If F_n has an $(a, 2)$-edge-*SAMTL $f: V\cup E \to \{1,2,3,\cdots,p+q\}$, *then*

$$\{f(u)+f(v)+f(uv)\mid uv\in E(H)\}=\{a,a+2,a+4,\cdots,a+2(3n-1)\}. \tag{III}$$

If $f(c) = i$, then by (II) *we get equation*

$$2\left(\tfrac{1}{2}p(p+1)-i\right)+2ni+\tfrac{1}{2}(p+1+p+q)q=\tfrac{1}{2}\{a+a+2(q-1)\}q.$$

By computation we get

$$i=\frac{11n^2+27n-6na+4}{4(1-n)} \tag{IV}$$

Since $a\ge 1+2+(p+1)$ *and* $a+(q-1)d\le (p-1)+p+(p+q)$, *there is* $2n+5\le a\le 3n+4$.

Again by (IV) *we can obtain* $1+\frac{n}{4}\le i\le 1+\frac{7n}{4}$.

By (III) *we obtain that lesser m edge weights on F_n are* $a,a+2,a+4,\cdots,a+2(m-1)$, *their sum is $am + m(m$-1), greater m edge weights on F_n are* $a+2(3n-1)$, $a+2(3n-2)$, $\cdots,a+2(3n-m)$, *their sum is* $6nm + am - m(m + 1)$. *It is easy to obtain the following conclusion.*

Lemma 3.3: *Let mapping* $f: V\cup E\to\{1,2,3,\cdots,p+q\}$ *be a labeling on F_n. If sum of lesser m edge weights on F_n is $am + m(m$-1) or sum of greater m edge weights on F_n is $6nm + am - m(m + 1)$, then F_n has not $(a, 2)$-edge-*SAMTL.

*Next, we want to show F_n has not $(a, 2)$-edge-*SAMTL. *The equation* (IV) *is equivalent to the equation* $(4 - 4n)\, i + 6na = 11n^2 + 27n + 4$ (V).

If n is a constant, then (V) *is a binary equation of first order with as two unknown elements i and a.*

Lemma 3.4: *Friendship graph F_n has not $(a, 2)$-edge-*SAMTL *if odd* $n\ge 9$.

***Proof*:** In equation (V) let $i = n + 1$. We can obtain integer solution $a=\frac{5n+9}{2}$. Thus $(i, a)=\left(n+1,\frac{5n+9}{2}\right)$ is a particular solution of (V). Since *gcd* $(4 - 4n, 6n) = 2$, the equation (V) has general solution

$$\begin{cases} i = n+1+\frac{6n}{2}t \\ a = \frac{5n+9}{2} - \frac{4-4n}{2}t \end{cases} \quad t = 0, \pm1, \pm2, \cdots$$

Again by $1+\frac{n}{4} \le i \le 1+\frac{7n}{4}$, we know (V) only has integer solution $(i, a) = \left(n+1, \frac{5n+9}{2}\right)$.

Next we show that F_n has not $(a, 2)$-edge-SAMTL if $i = n + 1$ and $a = \frac{5n+9}{2}$.

When $n \equiv 1 (\bmod 4)$, suppose that F_n has an $(a, 2)$-edge-SAMTL f, then the sum of lesser $\frac{n+5}{2}$ edge weights on F_n is $\frac{3n^2+21n+30}{2}$. But when we actually give vertices of F_n labels, then the sum of lesser $\frac{n+5}{2}$ edge weights on F_n is $\frac{49n^2+338n+413}{32}$. By Lemma 3.3 we obtain F_n has not $(a, 2)$-edge-SAMTL.

When $n \equiv 1 (\bmod 4)$, suppose that F_n has an $(a, 2)$-edge-SAMTL f, then the sum of lesser $\frac{n+5}{2}$ edge weights on F_n is $\frac{3n^2+7n+4}{2}$. Whereas when actually compute the vertex labels, then there is the sum of lesser $\frac{n+1}{2}$ edge weights on F_n is $\frac{49n^2+102n+53}{32}$. By Lemma 3.3 we obtain F_n has not $(a, 2)$-edge-SAMTL.

Summarizing the above results, proof of the theorem is now complete. □

Lemma 3.5: *When $n \ge 16$ and $n \equiv 4 \pmod{12}$, graph F_n has not $(a, 2)$-edge-SAMTL.*

***Proof*:** When $n \equiv 4 \pmod{12}$, if $n = 12k + 4$ and $k \in Z^+$, then $(i, a) = \left(1+\frac{n}{4}, 2n+5\right)$ is a particular solution of equation (V). Since $4(3k + 1) - 3(4k + 1) = 1$, there is $(4k + 1, 3k + 1) = 1$ by Lemma 3.2, there is also $d = (4 - 4n, 6n) = 12$. By Lemma 3.1 we obtain the general solution of equation (V) is

$$\begin{cases} i = 1 + \frac{n}{4} + \frac{6n}{12} t \\ \qquad\qquad\qquad\qquad t = 0, \pm 1, \pm 2, \cdots \\ a = 2n + 5 - \frac{4-4n}{12} t \end{cases}$$

Since $1 + \frac{n}{4} \le i \le 1 + \frac{7n}{4}$ and $2n + 5 \le a \le 3n + 4$, the integer solutions of (V) is:

$$\begin{cases} i = 1 + \frac{n}{4} \\ a = 2n + 5 \end{cases}, \begin{cases} i = 1 + \frac{3n}{4} \\ a = 2n + 5 + \frac{n-1}{3} t, \end{cases}$$

$$\begin{cases} i = 1 + \frac{5n}{4} \\ a = 2n + 5 + \frac{2n-2}{3} t \end{cases}, \begin{cases} i = 1 + \frac{7n}{4} \\ a = 3n + 4 \end{cases}.$$

By Lemma 3.3 we obtain that F_n has not $(a, 2)$-edge-SAMTL for above each solution. Therefore the result is true. □

From Lemmas 3.4 and 3.5 we obtain the following theorem.

Theorem 3.6: *Friendship graph F_n has $(a, 2)$-edge-*SAMTL *if and only if* $n \in \{1, 3, 4, 5, 7\}$.

4. On the (a, d)-H-SAMTL of graph mG

Theorem 4.1: *Let $m, t \ge 1$ be positive integers. If H is a (r, s)-graph and G* a *(p, q)-graphs and the graph G with H-covering $\mathcal{B}_1$ is a (a_n, d_n)-H-*SAMT *graph, then graph mG with H-covering $\mathcal{B}_2$ is (b_n, d_n)-H-*SAMT *graph. There $n \in [1, 4]$, and*

(1) *when* $r \equiv 0 (\text{mod } 2)$, $s \equiv 0 (\text{mod } 2)$, $d_1 \in [0, r + s]_2$.

(2) *when* $r \equiv 1 (\text{mod } 2)$, $s \equiv 1 (\text{mod } 2)$, $d_2 \in [2, r + s]_2$.

(3) *when* $r \equiv 1 (\text{mod } 2)$, $s \equiv 0 (\text{mod } 2)$, $d_3 \in [1, r + s]_2$.

(4) *when* $r \equiv 0 (\text{mod } 2)$, $s \equiv 1 (\text{mod } 2)$, $d_4 \in [1, r + s]_2$.

Proof: Let $\mathcal{B}_1 = \{H^1, H^2, \cdots, H^t\}$. Since every $H^j, j \in [1, t]$ and H are isomorphic, $|V(H^j)| = |V(H)| = r, |E(H^j)| = |E(H)| = s$. Let $V(H^j) = \{v_k^j \mid k \in [1, r]\}$, $E(H^j) = \{e_l^j \mid l \in [1, s]\}$. Again G is an (a, d)-H-SAMT graph with labeling $f' : V(G) \cup E(G) \to \{1, 2, \cdots, p\} \cup \{p + 1, p + 2, \cdots, p + q\}$, therefore

$$\{w_{f'}(H^j) \mid j = 1, 2, \cdots, t\} = \{a, a + d_n, \cdots a + (t - 1) d_n\}, n \in [1, 4]. \qquad (1)$$

For convenience sake, we denote m graph of mG by G_i, $i\in[1,m]$, and let $\mathcal{B}_1=\bigcup_{i=1}^{m}\{H_i^1,H_i^2,\cdots,H_i^t\}$. $V(mG)=\{v_{i,k}^j\mid i\in[1,m],\ k\in[1,r],\ j\in[1,t]\}$, $E(mG)=\{e_{i,l}^j\mid i\in[1,m],\ l\in[1,s],\ j\in[1,t]\}$. For each, $i\in[1,m]$, $j\in[1,t]$, we define the labeling f on H_i^j as follows:

Case 1: When $r\equiv 0 \pmod 2$, $s\equiv 0 \pmod 2$,

$$f(v_{i,k}^j)=\begin{cases} m\,f'(v_k^j)+1-i, & k\in\left[1,\frac{r}{2}-u\right]\\ m(f'(v_k^j)-1)+i, & k\in\left[\frac{r}{2}-u+1,r\right]\end{cases}$$

$$f(e_{i,l}^j)=\begin{cases} m\,f'(e_l^j)+1-i, & l\in\left[1,\frac{s}{2}-v\right]\\ m(f'(e_l^j)-1)+i, & l\in\left[\frac{s}{2}-v+1,s\right]\end{cases}$$

Case 2: When $r\equiv 1 \pmod 2$, $s\equiv 1 \pmod 2$,

$$f(v_{i,k}^j)=\begin{cases} m\,f'(v_k^j)+1-i, & k\in\left[1,\frac{r-1}{2}-u\right]\\ m(f'(v_k^j)-1)+i, & k\in\left[\frac{r-1}{2}-u+1,r\right]\end{cases}$$

$$f(e_{i,l}^j)=\begin{cases} m\,f'(e_l^j)+1-i, & l\in\left[1,\frac{s-1}{2}-v\right]\\ m(f'(e_l^j)-1)+i, & l\in\left[\frac{s-1}{2}-v+1,s\right]\end{cases}$$

Case 3: When $r\equiv 1 \pmod 2$, $s\equiv 0 \pmod 2$,

$$f(v_{i,k}^j)=\begin{cases} m\,f'(v_k^j)+1-i, & k\in\left[1,\frac{r-1}{2}-u\right]\\ m(f'(v_k^j)-1)+i, & k\in\left[\frac{r-1}{2}-u+1,r\right]\end{cases}$$

$$f(e_{i,l}^j)=\begin{cases} m\,f'(e_l^j)+1-i, & l\in\left[1,\frac{s}{2}-v\right]\\ m(f'(e_l^j)-1)+i, & l\in\left[\frac{s}{2}-v+1,s\right]\end{cases}$$

Case 4: When $r\equiv 0 \pmod 2$, $s\equiv 1 \pmod 2$,

$$f(v_{i,k}^j) = \begin{cases} m\,f'(v_k^j)+1-i, & k \in \left[1, \frac{r}{2}-u\right] \\ m(f'(v_k^j)-1)+i, & k \in \left[\frac{r}{2}-u+1, r\right] \end{cases}$$

$$f(e_{i,l}^j) = \begin{cases} m\,f'(e_l^j)+1-i, & l \in \left[1, \frac{s-1}{2}-v\right] \\ m(f'(e_l^j)-1)+i, & l \in \left[\frac{s-1}{2}-v+1, s\right] \end{cases}$$

Here when $r \equiv 0 \pmod 2$, $u \in \left[0, \frac{r}{2}\right]$; when $r \equiv 1 \pmod 2$, $u \in \left[0, \frac{r-1}{2}\right]$; when $s \equiv 0 \pmod 2$, $v \in \left[0, \frac{s}{2}\right]$; when $s \equiv 1 \pmod 2$, $v \in \left[0, \frac{s-1}{2}\right]$.

Thus for vertex labels, we have $\{f(v_{i,k}^j) \mid i \in [1,m]\} = [mf'(v_k^j) - m+1, mf'(v_k^j)]$. Again since $\{f'(v_k^j) \mid j \in [1,t],\ k \in [1,r]\} = [1,p], \{f(v_{i,k}^j) \mid i \in [1,m], j \in [1,t], k \in [1,r]\} = [1,mp]$.

For edge labels, we have $\{f(e_{i,k}^j) \mid i \in [1,m]\} = [mf'(e_l^j) - m+1,\ mf'(e_l^j)]$, Since

$\{f'(e_k^j) \mid j \in [1,t],\ l \in [1,s]\} = [p+1, p+q], \{f(e_{i,k}^j) \mid i \in [1,m],\ j \in [1,t]\ l \in [1,s]\} = [pm+1, (p+q)m]$.

Summarizing the above results, we obtain f is a bijection from $V(mG) \cup E(mG)$ to $[1, (p+q)m]$.

In Case 1, we have $w_f(H_i^j) = mw_{f'}(H_i^j) + h_1 + d_1 i$.
Where

$$h_1 = \left(\frac{r}{2}-u\right) - \left(\frac{r}{2}+u\right)m + \left(\frac{s}{2}-v\right) - \left(\frac{s}{2}+v\right)m, d_1 = 2u+2v \in [0, r+s]_2.$$

In Case 2, we have $w_f(H_i^j) = mw_{f'}(H_i^j) + h_2 + d_2 i$.
Where

$$h_2 = \left(\frac{r-1}{2}-u\right) - \left(\frac{r+1}{2}+u\right)m + \left(\frac{s-1}{2}-v\right) - \left(\frac{s+1}{2}+v\right)m, d_2 = 2u+2v+2 \in [2, r+s]_2.$$

In Case 3, we have $w_f(H_i^j) = mw_{f'}(H_i^j) + h_3 + d_3 i$.
Where

$$h_3 = \left(\frac{r-1}{2}-u\right) - \left(\frac{r+1}{2}+u\right)m + \left(\frac{s}{2}-v\right) - \left(\frac{s}{2}+v\right)m, d_3 = 2u+2v+1 \in [1, r+s]_2.$$

In Case 4, we have $w_f(H_i^j) = mw_{f'}(H_i^j) + h_4 + d_4 i$.
Where

$$h_4 = \left(\frac{r}{2}-u\right) - \left(\frac{r}{2}+u\right)m + \left(\frac{s-1}{2}-v\right) - \left(\frac{s+1}{2}+v\right)m, d_4 = 2u+2v+1 \in [1, r+s]_2.$$

For any, $n \in [1,4]$, By (1) we can get the following results.

For covering $\{H_1^j \mid j \in [1,t]\}$ of G_1, we have $\{\sum f(H_1^j) \mid j \in [1,t]\} =$
$\{ma + h_n + d_n,\ m(a+d_n) + h_n + d_n,\ \cdots, m[a+(t-1)\ d_n] + h_n + d_n\}$.

For covering $\{H_2^j \mid j \in [1,t]\}$ of G_2, we have $\{\sum f(H_2^j) \mid j \in [1,t]\} =$
$\{ma + h_n + 2d_n,\ m(a+d_n) + h_n + 2d_n,\ \cdots, m[a+(t-1)\ d_n] + h_n + 2d_n\}$.

…

For covering $\{H_{m-1}^j \mid j \in [1,t]\}$ of G_{m-1}, we have $\{\sum f(H_{m-1}^j) \mid j \in [1,t]\} =$
$\{ma + h_n + (m-1)d_n,\ m(a+d_n) + h_n + (m-1)d_n,\ \cdots, m[a+(t-1)\ d_n] + h_n + (m-1)d_n\}$.

For covering $\{H_m^j \mid j \in [1,t]\}$ of G_m, we have $\{\sum f(H_m^j) \mid j \in [1,t]\} =$
$\{ma + h_n + md_n,\ m(a+d_n) + h_n + md_n,\ \cdots, m[a+(t-1)\ d_n] + h_n + md_n\}$.

Therefore $\{\sum f(H_i^j) \mid i \in [1,m],\ j \in [1,t]\} = \{ma + h_n + d_n,\ ma + h_n + 2d_n,\ \cdots, m(a+td_n) + h_n\}$. To sum up we get that mG is a (b_n, d_n)-H-SAMT graph, where $b_n = ma + h_n$. □

In above theorem let $f(v_{i,k}^j) = m(f'(v_k^j) - 1) + i, k \in [1,r]$, and $f(e_{i,l}^j) = m f'(e_l^j) + 1 - i, l \in [1,s]$, we can get some consequences of the above theorem.

Corollary 4.2: *Let* $d = |V(H)| - |E(H)|$. *If* G *is an* (a, d)-H-SAMT *graph, then* mG *is a* (b, d)-H-SAMT *graph for any* $m \geq 1$, *where* $b = m(a - |V(H)|) + |V(H)|$.

Corollary 4.3: *If* $|V(H)| = |E(H)|$ *and* G *is an* $(a, 0)$-H-SAMT *graph, then* mG *is a* $(b, 0)$-H-SAMT *graph for any* $m \geq 1$, *where* $b = m(a - |V(H)|) + |V(H)|$.

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