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On the solution of quadratic programming problems

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Abstract

The problems of quadratic programming are examined in this paper, particularly, the convex quadratic programming problems. Karush-Kuhn-Tucker (KKT) optimality conditions are formulated for the considered convex quadratic programming problems, and a necessary and sufficient condition for solvability and for optimal solution of this problem is stated and proved.

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1. Introduction. Statement of the problem

Consider the following general quadratic programming problem

$$\min\left\{Q(x) \equiv \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_i x_j + \sum_{j=1}^{n} p_j x_j + q\right\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_{kj}x_j \le b_k, \quad k = 1,\ldots,s \tag{2}$$

$$\sum_{j=1}^{n} a_{kj}x_j = b_k, \quad k = s+1,\ldots,m \tag{3}$$

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$$x_j \geq 0, \quad j = 1, \ldots, l \quad (l \leq n). \tag{4}$$

The objective function $Q(\mathrm{x})$ is a quadratic function. Coefficient $\frac{1}{2}$ before $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_i x_j$ is taken for the sake of convenience (see the explanation below), and $q = const.$ can be omitted (that is, can be equal to zero) because q does not significantly change the behavior of the quadratic function $Q(\mathrm{x})$.

The feasible region X defined by the constraints (2)–(4) is a convex closed polyhedral set and it is the same as the feasible region of the linear programming problem.

Minimum solution of the quadratic programming problem can be obtained at any feasible point – at vertex, on edge, on a face, or at an interior point of the feasible polyhedral set.

Quadratic programming problems and related problems are considered in many works, for example, in [1]–[18], etc. Quadratic programming is discussed, for example, in the books of Bazaraa, Sherali and Shetty [1], Fletcher [4], Stefanov [15], Taha [16], Zaichenko [18], etc. An $O(n)$ algorithm for quadratic knapsack problems is proposed in [Brucker 2]. Convex quadratic programming with one constraint and bounded variables is considered in [Dussault, Ferland and Lemaire 3]. A polynomially bounded algorithm for a singly constrained quadratic program is suggested in [Helgason, Kennington and Lall 5]. Benchmarking large-scale distributed convex quadratic programming algorithms are proposed in [Kozma, Conte and Diehl 6]. Algorithms for bound constrained quadratic programming problems are suggested by Moré and Toraldo [7]. An algorithm for a singly constrained class of quadratic programs subject to upper and lower bounds is proposed in [Pardalos and Kovoor 8], and algorithms for the solution of quadratic knapsack problems are suggested in [Pardalos, Ye and Han 9]. The continuous quadratic knapsack problem is studied in [Robinson, Jiang and Lerme 10]. Convergent polynomial algorithms for solving the quadratic problem for projecting the current iterate onto some feasible regions are proposed, for example, in [Stefanov 11, 12]. The solvability of quadratic programming problem with a feasible region defined as a Minkowski sum of a compact set and finitely generated convex closed cone is studied in [Stefanov 13], and the Cauchy-Schwarz inequality approach for solving a quadratic optimization problem is proposed in [Stefanov 14]. An exterior point polynomial-time algorithm for convex quadratic programming is suggested in [Tian 17].

2. Preliminaries

In this section, some preliminary concepts and results are presented.

The function $Q_0(\mathrm{x}) \equiv Q_0(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_i x_j$

is called the *quadratic form*.

Let

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

be a quadratic symmetric matrix, that is, $c_{ij} = c_{ji}$ for all $i = 1,\ldots,n, j = 1,\ldots,n.$

As it is known, the quadratic form $Q_0(\mathrm{x})$ can be expressed as follows

$$Q_0(\mathrm{x}) = \langle C\mathrm{x}, \mathrm{x}\rangle = \mathrm{x}^T C\mathrm{x},$$

where $\langle .,. \rangle$ denotes the scalar (inner, dot) product of vectors, and

$$\mathrm{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a column vector.

This discussion shows that with each quadratic form, a quadratic symmetric matrix is connected, and vice versa, each quadratic symmetric matrix generates a quadratic form.

Recall that a quadratic symmetric matrix C is said to be

- *positive semidefinite* or *nonnegative definite* if $\langle C\mathrm{x}, \mathrm{x}\rangle \geq 0$ for all $\mathrm{x} \in \mathbb{R}^n$,
- *positive definite* if $\langle C\mathrm{x}, \mathrm{x}\rangle > 0$ for all $\mathrm{x} \in \mathbb{R}^n, \mathrm{x} \neq 0$,
- *negative semidefinite* or *nonpositive definite* if $-C$ is positive semidefinite,
- *negative definite* if $-C$ is positive definite.

In these four cases, the quadratic form $Q_0(\mathrm{x}) = \langle C\mathrm{x}, \mathrm{x}\rangle$, generated by the quadratic symmetric matrix C, is called

- *positive semidefinite* or *nonnegative definite*,
- *positive definite*,

- *negative semidefinite* or *nonpositive definite*,
- *negative definite*,

respectively.

Example 1 : Let $\mathrm{x} = (x_1, x_2)$. Then

1. $Q_1(\mathrm{x}) = x_1^2 - 2x_1x_2 + x_2^2$ is a positive semidefinite quadratic form,
2. $Q_2(\mathrm{x}) = x_1^2 + x_2^2$ is a positive definite quadratic form,
3. $Q_3(\mathrm{x}) = x_1^2 - x_2^2$ is a non-definite quadratic form.

1. The quadratic form $Q_1(\mathrm{x})$ is generated by the matrix

$$C_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Indeed,

$$\begin{aligned} \langle C_1\mathrm{x}, \mathrm{x}\rangle &= \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (x_1, x_2) \right\rangle = \\ &= \langle (x_1 - x_2, -x_1 + x_2), (x_1, x_2)\rangle = \\ &= (x_1 - x_2)x_1 + (-x_1 + x_2)x_2 = \\ &= x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2 \geq 0. \end{aligned}$$

Therefore, matrix C_1 (and the quadratic form $Q_1(x)$ generated by C_1) is positive semidefinite by definition.

However, when $x_1 = x_2$, even when $(x_1, x_2) \neq (0,0)$, we have $\langle C\mathrm{x}, \mathrm{x}\rangle = (x_1 - x_2)^2 = 0$. Hence, matrix C_1 (and the quadratic form $Q_1(\mathrm{x})$ generated by C_1) is not positive definite.

2. The quadratic form $Q_2(\mathrm{x})$ is generated by the matrix

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Indeed,

$$\langle C_2\mathrm{x}, \mathrm{x}\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (x_1, x_2) \right\rangle = \langle (x_1, x_2), (x_1, x_2)\rangle = x_1^2 + x_2^2 > 0$$

for every $(x_1, x_2) \neq (0,0)$.

Therefore, matrix C_2 (and the quadratic form $Q_2(\mathrm{x})$ generated by C_2) is positive definite by definition.

3. The quadratic form $Q_3(\mathrm{x})$ is generated by the matrix

$$C_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Indeed,

$$\langle C_3\mathrm{x}, \mathrm{x}\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (x_1, x_2) \right\rangle = \langle (x_1, -x_2), (x_1, x_2)\rangle = x_1^2 - x_2^2,$$

and this expression can be positive, negative, or equal to zero, depending on the absolute values of x_1 and x_2.

Therefore, matrix C_3 (and the quadratic form $Q_3(x)$ generated by C_3) is non-definite.

If C is a nonzero $n \times n$ quadratic symmetric matrix and $\mathrm{p}, \mathrm{x} \in \mathbb{R}^n$, then the quadratic function (1) can be presented as follows

$$Q(\mathrm{x}) = \frac{1}{2}\langle C\mathrm{x}, \mathrm{x}\rangle + \langle \mathrm{p}, \mathrm{x}\rangle + q.$$

As pointed out at the beginning, q does not have much effect on the behavior of the quadratic function $Q(\mathrm{x})$, especially on its extreme properties, that is why, without loss of generality, we can assume that $q = 0$.

The first and the second derivative of $Q(\mathrm{x})$ can be calculated as follows, respectively

$$Q'(\mathrm{x}) = \left(\frac{\partial Q(\mathrm{x})}{\partial x_1}, \ldots, \frac{\partial Q(\mathrm{x})}{\partial x_n} \right)^T + \mathrm{p} = \left(\sum_{j=1}^{n} c_{1j}x_j, \ldots, \sum_{j=1}^{n} c_{nj}x_j \right) + \mathrm{p} = C\mathrm{x} + \mathrm{p} \in \mathbb{R}^n,$$

$$Q''(\mathrm{x}) = C \in \mathbb{R}^{n \times n}.$$

It is known that the quadratic function $Q(\mathrm{x})$ is

- convex if $Q''(\mathrm{x}) = C$ is a positive semidefinite matrix,
- strictly convex if $Q''(\mathrm{x}) = C$ is a positive definite matrix.

The expression $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_i x_j = \langle Cx, x\rangle$ in (1) is taken with coefficient $\frac{1}{2}$ in order to have $Q''(x) = C$ instead of $Q''(x) = 2C$.

In [13], the following theorem is stated and proved.

Theorem 1 : *(Equivalent formulation) Let the quadratic function* $Q(x) = \frac{1}{2}\langle Cx, x\rangle + \langle p, x\rangle$ *be bounded below (bounded above, respectively) in the set* $X = G + K$, *where C is an* $n \times n$ *symmetric matrix,* $p, x \in \mathbb{R}^n$, $\langle .,. \rangle$ *denotes the scalar (inner) product in* $\mathbb{R}^n$, *G is a compact set, K is a finitely generated convex closed cone, and "+" denotes the Minkowski sum of G and K.*

Then $Q(x)$ *attains its infimum (supremum, respectively) on X, that is, optimization problem* $\min(\max)\{Q(x) : x \in X\}$ *has optimal solution under the assumptions of this theorem.*

3. Main results

It is known that each quadratic programming problem can be presented in one of the following forms.

Standard form.

$$\min\left\{Q(x) = \frac{1}{2}\langle Cx, x\rangle + \langle p, x\rangle\right\}$$
$$X_1 : \quad Ax \le b, x \ge 0, \tag{5}$$

Canonical form.

$$\min\left\{Q(x) = \frac{1}{2}\langle Cx, x\rangle + \langle p, x\rangle\right\}$$
$$X_2 : \quad Ax = b, x \ge 0, \tag{6}$$

General form.

$$\min\left\{Q(x) = \frac{1}{2}\langle Cx, x\rangle + \langle p, x\rangle\right\}$$
$$X_3 : \quad Ax \le b, \tag{7}$$

where $A_{m\times n}, b \in \mathbb{R}^m, x \in \mathbb{R}^n$.

If $C = (c_{ij})_{n\times n}$ is a positive semidefinite or positive definite matrix then problem (1)–(4) (problem (5), problem (6), problem (7), respectively) is a *convex* quadratic programming problem.

All three problems (5), (6), and (7) have the same Lagrangian function

$$L(\mathrm{x},\lambda) = Q(\mathrm{x}) + \langle \lambda, A\mathrm{x} - \mathrm{b}\rangle =$$

$$= \frac{1}{2}\langle C\mathrm{x},\mathrm{x}\rangle + \langle \mathrm{p},\mathrm{x}\rangle + \langle \lambda, A\mathrm{x} - \mathrm{b}\rangle =$$

$$= \frac{1}{2}\langle C\mathrm{x},\mathrm{x}\rangle + \langle \mathrm{p},\mathrm{x}\rangle + \langle A^T\lambda,\mathrm{x}\rangle - \langle \lambda,\mathrm{b}\rangle,$$

and the differences are only in the nonnegativity requirements for $\lambda = (\lambda_1,\ldots,\lambda_m)$, namely $\lambda \geq 0$ for problems (5) and (7) which have inequality constraints, whereas components of λ are unrestricted in sign for problem (6) which has equality constraints.

We have

$$L'_{\mathrm{x}}(\mathrm{x},\lambda) = C\mathrm{x} + \mathrm{p} + A^T\lambda \stackrel{\text{def}}{=} \mathrm{v}, \quad \mathrm{v} \in \mathbb{R}^n, \tag{8}$$

$$L'_{\lambda}(\mathrm{x},\lambda) = A\mathrm{x} - \mathrm{b} \stackrel{\text{def}}{=} -\mathrm{y}, \quad \mathrm{y} \in \mathbb{R}^m. \tag{9}$$

Let C be positive semidefinite or positive definite matrix, that is, let (5), (6), and (7) be convex quadratic programming problems.

The KKT optimality conditions for problems (5), (6), and (7) are formulated below.

KKT conditions for problem (5).

(I) $A\mathrm{x} + \mathrm{y} = \mathrm{b}$ according to (9)

(II) $C\mathrm{x} - \mathrm{v} + \mathrm{A}^T\lambda = -\mathrm{p}$ according to (8)

(III) $\mathrm{x} \geq 0, \mathrm{v} \geq 0, \mathrm{y} \geq 0, \lambda \geq 0$

(from $\mathrm{x} \geq 0$ according to (5), $\lambda \geq 0, L'_{\mathrm{x}}(\mathrm{x},\lambda) \equiv \mathrm{v} \geq 0, L'_{\lambda}(\mathrm{x},\lambda) \equiv -\mathrm{y} \leq 0$)

(IV) $\langle \mathrm{x},\mathrm{v}\rangle + \langle \lambda,\mathrm{y}\rangle = 0$

(from $\langle \mathrm{x}, L'_{\mathrm{x}}(\mathrm{x},\lambda)\rangle + \langle \lambda, L'_{\lambda}(\mathrm{x},\lambda)\rangle = 0 + 0 = 0$).

KKT conditions for problem (6).

(I) $A\mathrm{x} = \mathrm{b}$ according to (9) with $\mathrm{y} = 0$ due to $L'_{\lambda}(\mathrm{x},\lambda) \equiv -\mathrm{y} = 0$

(II) $C\mathrm{x} - \mathrm{v} + A^T\lambda = -\mathrm{p}$ according to (8)

(III) $\mathrm{x} \geq 0, \mathrm{v} \geq 0$

(from $\mathrm{x} \geq 0$ according to (6), $L'_{\mathrm{x}}(\mathrm{x},\lambda) \equiv \mathrm{v} \geq 0$)

(IV) $\langle x, v\rangle = 0$

(from $\langle x, L'_x(x,\lambda)\rangle = \langle x, v\rangle = 0$).

KKT conditions for problem (7).

(I) $Ax + y = b$ according to (9)

(II) $Cx + A^T\lambda = -p$ according to (8) with $v = 0$ due to $L'_x(x,\lambda) \equiv v = 0$

(III) $y \geq 0, \lambda \geq 0$

(from $L'_\lambda(x,\lambda) \equiv -y \leq 0$)

(IV) $\langle \lambda, y\rangle = 0$

(from $\langle \lambda, L'_\lambda(x,\lambda)\rangle = \langle \lambda, -y\rangle = 0$).

Note that conditions (I), (II) and (III) are linear whereas condition (IV) is nonlinear with respect to x, v, y, λ.

Also note that conditions $\langle x, v\rangle = 0, \langle \lambda, y\rangle = 0$ and the condition $\langle x, v\rangle + \langle \lambda, y\rangle = 0$ are equivalent only when conditions (III) are satisfied; that is why we can present condition (IV) for problem (5) in the form $\langle x, v\rangle + \langle \lambda, y\rangle = 0$ instead of in the form $\langle x, v\rangle = 0$ and $\langle \lambda, y\rangle = 0$.

Theorem 2 : *If the quadratic function* $Q(x)$ *is convex, then conditions* (I), (II), (III) *are necessary and sufficient conditions for solvability of the corresponding quadratic programming problem* (5),(6) *or* (7), *respectively, that is, if the system* (I), (II), (III) *is solvable, then the corresponding quadratic programming problem* (5),(6) *or* (7) *is also solvable, and vice versa.*

In order that x *be optimal solution to the corresponding quadratic programming problem, in addition to conditions* (I), (II), (III), x *must also satisfy the corresponding condition* (IV).

Proof: *Necessity.* If the corresponding quadratic programming problem (5), (6) or (7) is solvable then conditions (I), (II), (III), (IV) for this problem are satisfied. In particular, the corresponding system (I), (II), (III) is solvable.

Sufficiency. Since (5), (6), (7) are different forms of the quadratic programming problem (1)–(4), it is sufficient to prove the statement for only one of these three problems, for example, for problem (5).

Let system (I), (II), (III) for problem (5) be solvable and $\bar{x}, \bar{v}, \bar{\lambda}, \bar{y}$ be its solution. From conditions (I), (III) for problem (5) and definition of the feasible region X_1 it follows that $\bar{x} \in X_1$. Consider an arbitrary $x \in X_1$.

From a theorem for convex functions and from conditions (I), (II), (III) for problem (5) we have

$$Q(x) - Q(\bar{x}) \geq \langle Q'(\bar{x}), x - \bar{x} \rangle =$$

$$= \langle C\bar{x} + p, x - \bar{x} \rangle \overset{(II)}{=} \langle \bar{v} - A^T \bar{\lambda}, x - \bar{x} \rangle =$$

$$= \langle \bar{v}, x \rangle - \langle \bar{v}, \bar{x} \rangle - \langle A^T \bar{\lambda}, x \rangle + \langle A^T \bar{\lambda}, \bar{x} \rangle =$$

$$= \langle \bar{v}, x \rangle - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, Ax \rangle + \langle \bar{\lambda}, A\bar{x} \rangle \overset{(I)}{=}$$

$$\overset{(I)}{=} \langle \bar{v}, x \rangle - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, b - y \rangle + \langle \bar{\lambda}, b - \bar{y} \rangle =$$

$$= \langle \bar{v}, x \rangle - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, b \rangle + \langle \bar{\lambda}, y \rangle + \langle \bar{\lambda}, b \rangle - \langle \bar{\lambda}, \bar{y} \rangle \overset{(III)}{\geq}$$

$$\overset{(III)}{\geq} - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, \bar{y} \rangle,$$

that is,

$$Q(x) \geq Q(\bar{x}) - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, \bar{y} \rangle \quad \forall x \in X_1. \tag{10}$$

The right-hand side of (10) is a *concrete* value because $\bar{x}, \bar{v}, \bar{\lambda}, \bar{y}$ are vectors (points) with fixed components.

Since x has been arbitrarily chosen in the feasible region X_1 of problem (5), then inequality (10) means that the objective function $Q(x)$ of problem (5) is bounded below in the feasible region X_1 of this problem.

From Theorem 1 it follows that the quadratic minimization problem (5) is solvable.

If x^* is an optimal solution of problem (5), that is, $x^* \in X_1$ and the objective function $Q(x)$ attains its minimum in the feasible region X_1 for x^*, then from (10), which is satisfied for each $x \in X_1$, including for $x^* \in X_1$, we obtain

$$Q(\bar{x}) \geq Q(x^*) \overset{(10)}{\geq} Q(\bar{x}) - \langle \bar{v}, \bar{x} \rangle - \langle \bar{\lambda}, \bar{y} \rangle. \tag{11}$$

Let $\bar{x}, \bar{v}, \bar{\lambda}, \bar{y}$ satisfy condition (IV) for problem (5) as well, that is, $\langle \bar{x}, \bar{v} \rangle + \langle \bar{\lambda}, \bar{y} \rangle = 0$. Then (11) imply

$$Q(\bar{x}) \geq Q(x^*) \geq Q(\bar{x}).$$

Then it follows that

$$Q(\bar{x}) = Q(x^*). \tag{12}$$

However, $\bar{x}, x^* \in X_1$ and x^* is an optimal solution of problem (5) by assumption. From this assumption and from (12) it follows that $\bar{x}$ is an optimal solution of problem (5) as well. □

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