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On the degree sum energy of total transformation graphs of regular graphs

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Abstract

The energy $E(G)$ of a graph G is the sum of absolute values of the eigenvalues of the adjacency matrix of G. This definition of energy was motivated by the large number of results for the Huckel molecular orbital total π-electron energy. Motivated by $E(G)$, The degree sum energy $E_{DS}(G)$ of a simple connected graph G is defined by sum of the absolute values of all

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eigenvalues of degree sum matrix. In this paper, we obtain spectra and degree sum energy of the total transformation graph G^{xyz} of a r-regular graph.

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1. Introduction

Let $G(n, m)$ be a simple connected graph and the vertices of G be labelled as $v_1, v_2, ..., v_n$. The notation $d_G(v_i)$ (or d_i) indicates the number of edges incident to v_i and is called the degree of v_i in G, also $e=v_iv_j$ is an edge of G. If all the vertices of G have the same degree equal to r, then G is called r-regular graph.

A square matrix $A(G)$ of order n is an adjacency matrix of G whose (i, j)-entry is 1 if the vertex v_i is adjacent to v_j, and is 0 otherwise. The eigenvalues of an adjacency matrix $A(G)$ are denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$. Since eigenvalues of an adjacency matrix are real, they are ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Then $E = E_\pi(G) = \sum_{i=1}^{n} |\lambda_i|$ is the energy of G [6].

The degree sum matrix has been introduced by H. S. Ramane *et. al*. [7]. Let G (n, m) is a simple graph with vertices $v_1, v_2, ..., v_n$ and d_i is the degree of v_i, where i=1, 2, … n. $DS(G) = [d_{ij}]$ is the degree sum matrix for G where,

$$d_{ij} = \begin{cases} d_i + d_j, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases}$$

Characteristic polynomial for the degree sum matrix is given as,

$\phi(G{:}\ \gamma) = det(\gamma I - DS(G)) = |\gamma I - DS(G)| = \gamma^n + c_1\gamma^{n-1} + c_2\gamma^{n-2} + \cdots + c_n = 0$, here I being the identity matrix of order n. Since $DS(G)$ is real symmetric matrix, the roots of $\phi(G{:}\ \gamma) = 0$ are real and it can be ordered as $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, with respective multiplicities $k_1, k_2, \ldots, k_n$ then the spectrum can be written as,

$$spectra(G) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

where γ_1 is largest and γ_n is smallest eigen value. Degree sum energy of G is given as,

$$E_{DS}(G) = \sum_{i=1}^{n} |\gamma_i|$$

Bounds for degree sum energy is then modified by Hosamani [5]. Similarly, eccentricity sum energy is defined and studied in [2, 3].

In 2001, Wu and Meng [12] gave new transformations G^{xyz} for G generalizing the concept of the total graph. The total-block graph $T_B(G)$ [11] was introduced by Kulli. Moreover, the concept of the transformation graph G^{xyz} is a special case of the block-transformation graph $G^{\alpha\beta\gamma}$ [1].

Definition 1.1 [10] : For graph $G(V: E)$ on $n \geq 3$ vertices and x, y, z be three variables taking values + or – . Total transformation graph G^{xyz} for G, is a graph having $V(G)$ U $E(G)$ as a vertex set and for $\alpha, \beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if

(1) $\alpha, \beta \in V(G)$, α and β are adjacent in G if $x = +$ and are nonadjacent in G if $x = -$.

(2) $\alpha, \beta \in E(G)$, α and β are adjacent in G if $y = +$ and are nonadjacent in G if $y = -$.

(3) $\alpha \in V(G)$ and $\beta \in E(G)$, α and β are incident in G if $z = +$ and are not incident in G if $z = -$.

We obtain eight graphical transformations for G. For a given graph G, G^{-++}, G^{+++}, G^{+-+}, G^{++-} and its complement G^{+--}, G^{---}, G^{-+-}, G^{--+} are the eight transformation graphs. For more results on total transformation graphs one can refer [9, 10, 12].

Here, we obtain spectra and degree sum energy for the total transformation graph G^{xyz}, where G is r-regular graph. For more results on regular graph one can refer works of Anirban Banerjee et. al. [13].

II. Preliminary Results

Theorem 2.1 [8] : *If G is a r-regular graph with n vertices, then G has only one positive eigen value equal to 2(n – 1)r.*

Proposition 2.2 [9] : For a graph $G(n, m)$, let $u, v \in V(G)$ and $e = uv \in E(G)$. Then degree of corresponding vertices in G^{xyz} are:

(a) $d_{G^{+++}}(u) = 2d_G(u)$ and $d_{G^{+++}}(e) = d_G(u) + d_G(v)$.

(b) $d_{G^{++-}}(u) = m$ and $d_{G^{++-}}(e) = d_G(u) + d_G(v) + n - 4$.

(c) $d_{G^{+-+}}(u) = 2d_G(u)$ and $d_{G^{+-+}}(e) = m - d_G(u) - d_G(v) + 3$.

(d) $d_{G^{-++}}(u) = n - 1$ and $d_{G^{-++}}(e) = d_G(u) + d_G(v)$.

(e) $d_{G^{---}}(u) = n + m - 2d_G(u) - 1$ and $d_{G^{---}}(e) = n + m - d_G(u) - d_G(v) - 1.$

(f) $d_{G^{--+}}(u) = n - 1$ and $d_{G^{--+}}(e) = m - d_G(u) - d_G(v) + 3.$

(g) $d_{G^{-+-}}(u) = n + m - 1 - 2d_G(u)$ and $d_{G^{-+-}}(e) = n + d_G(u) + d_G(v) - 4.$

(h) $d_{G^{+--}}(u) = m$ and $d_{G^{+--}}(e) = m + n - d_G(u) - d_G(v) - 1.$

Proposition 2.3: For a r-regular graph $G(n, m)$, let $u, v \in V(G)$ and $e = uv \in E(G)$. Then the degrees of corresponding vertices in G^{xyz} are:

(a) $d_{G^{+++}}(u) = 2r$ and $d_{G^{+++}}(e) = 2r.$

(b) $d_{G^{++-}}(u) = \frac{nr}{2}$ and $d_{G^{++-}}(e) = 2r + n - 4.$

(c) $d_{G^{+-+}}(u) = 2r$ and $d_{G^{+-+}}(e) = \frac{nr}{2} - 2r + 3.$

(d) $d_{G^{-++}}(u) = n - 1$ and $d_{G^{-++}}(e) = 2r.$

(e) $d_{G^{---}}(u) = n + \frac{nr}{2} - 2r - 1$ and $d_{G^{---}}(e) = n + \frac{nr}{2} - 2r - 1.$

(f) $d_{G^{--+}}(u) = n - 1$ and $d_{G^{--+}}(e) = \frac{nr}{2} - 2r + 3.$

(g) $d_{G^{-+-}}(u) = n + \frac{nr}{2} - 1 - 2r$ and $d_{G^{-+-}}(e) = n + 2r - 4.$

(h) $d_{G^{+--}}(u) = \frac{nr}{2}$ and $d_{G^{+--}}(e) = \frac{nr}{2} + n - 2r - 1.$

Proof: For a r-regular graph $G(n, m)$, we have $d_G(v_i) = r,\ i = 1,2, \ldots, n$ and has $m = nr/2$ edges. Hence the proof follows from Proposition 2.2.

Lemma 2.4 [4] : *If a and b are scalars then,*

$$\left|aI + b(J - I)\right|_{(n\times n)} = \begin{vmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{vmatrix}_{n\times n} = (a-b)^{n-1}[a + (n-1)b],$$

where I is the identity matrix and J is the matrix with all the entries as 1.

Lemma 2.5 [4]: If a , b , c and d are real numbers then the determinant of order $n_1 + n_2$ of the form, $\begin{vmatrix} (\gamma+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 x n_2} \\ -dJ_{n_2 x n_1} & (\gamma+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$ can be expressed as,

$$(\gamma+a)^{n_1-1}(\gamma+b)^{n_2-1}\{[\gamma-(n_1-1)a][\gamma-(n_2-1)b] - n_1 n_2 cd\}$$

Lemma 2.6 : *If a, b, c and d are real numbers and* $\gamma_1, \gamma_2, \ldots, \gamma_{n_1+n_2}$ *are the eigenvalues such that,* $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_{n_1+n_2}$ *of characteristic polynomial of the determinant* $\begin{vmatrix} (\gamma+a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 x n_2} \\ -dJ_{n_2 x n_1} & (\gamma+b)I_{n_2} - bJ_{n_2} \end{vmatrix}$ *of order* $n_1 + n_2$ *, then the eigenvalues are given by,*

$$Spectra = \begin{cases} -a & , (n_1-1) \text{ times} \\ -b & , (n_2-1) \text{ times} \\ [a(n_1-1)+b(n_2-1) \pm \sqrt{[a(n_1-1)-b(n_2-1)]^2+4n_1n_2cd}]/2 & \end{cases}$$

Further,

Case (*i*): If $(n_1-1)(n_2-1)ab \le n_1n_2cd$,

$$\gamma_1 = \sum_{i=2}^{n_1+n_2} \gamma_i = [a(n_1-1)+b(n_2-1)+\sqrt{\left[a(n_1-1)-b(n_2-1)\right]^2+4n_1n_2cd}]/2$$

Case (*ii*): If $(n_1-1)(n_2-1)ab > n_1n_2cd$,

$$\gamma_1+\gamma_2 = \sum_{i=3}^{n_1+n_2} \gamma_i = a\ (n_1-1)+b\ (n_2-1)$$

Where, $\gamma_1 = [a(n_1-1)+b(n_2-1)+\sqrt{[a(n_1-1)-b(n_2-1)]^2+4n_1n_2cd}]/2$
and

$$\gamma_2 = [a(n_1-1)+b(n_2-1)-\sqrt{[a(n_1-1)-b(n_2-1)]^2+4n_1n_2cd}]/2$$

III. Main results

In this section we have obtained degree sum energy of total transformation graphs. By the definition of total transformation graphs of a graph G with n vertices and $m = nr/2$ edges, G^{xyz} have $m + n$ vertices. Let the vertices be labelled as v_i, $i = 1, 2, \ldots, n$ and the edges as e_j, $j = 1, 2, \ldots, m$.

Theorem 3.1 : *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{+++}) = 4r(nr+2n-2)$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{+++} and Proposition 2.2 and 2.3 we have characteristic polynomial of degree sum matrix is $\left|\lambda I - DS(G^{+++})\right| = 0$.

$$\text{Now, } |\lambda I - ES(G^{+++})| = \begin{vmatrix} \lambda & -4r & -4r & \cdots & -4r & -4r \\ -4r & \lambda & -4r & \cdots & -4r & -4r \\ -4r & -4r & \lambda & \cdots & -4r & -4r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -4r & -4r & -4r & \cdots & \lambda & -4r \\ -4r & -4r & -4r & \cdots & -4r & \lambda \end{vmatrix}_{(m+n)\times(m+n)} \tag{1}$$

By using Lemma 2.4,

$$|\lambda I - ES(G^{+++})| = (\lambda + 4r)^{m+n-1}[\lambda - (m+n-1)4r] = 0.$$

$$spectra(ES(G^{+++})) = \begin{pmatrix} -4r & 4r(m+n-1) \\ m+n-1 & 1 \end{pmatrix}$$

Hence $E_{DS}(G^{+++}) = 8r(m+n-1) = 4r(nr+2n-2)$.

Theorem 3.2 : *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{++-}) = nr(n-1) + (nr-2)(n+2r-4)$$
$$+\sqrt{[nr(n-1)-(nr-2)(n+2r-4)]^2 + (n^2r/2)(nr+2n+4r-8)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{++-} and from Proposition 2.2 and 2.3 we have,

$$\left|\lambda I - DS(G^{++-})\right| =$$

$$\begin{vmatrix} \lambda & -2m & \cdots & -2m & -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) \\ -2m & \lambda & \cdots & -2m & -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2m & -2m & \cdots & \lambda & -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) \\ -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) & \lambda & -(2n+4r-8) & \cdots & -(2n+4r-8) \\ -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) & -(2n+4r-8) & \lambda & \cdots & -(2n+4r-8) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(m+n+2r-4) & -(m+n+2r-4) & \cdots & -(m+n+2r-4) & -(2n+4r-8) & -(2n+4r-8) & \cdots & \lambda \end{vmatrix}$$

i.e., $$\left|\lambda I - DS(G^{++-})\right| = \begin{vmatrix} (\lambda I - 2m(J-I))_{n\times n} & -(m+n+2r-4)J_{n\times m} \\ -(m+n+2r-4)J_{m\times n} & (\lambda I-(2n+4r-8)(J-I))_{m\times m} \end{vmatrix} \quad ...(2)$$

Expanding by using lemma 2.5 we obtain,

$$\left|\lambda I - DS(G^{++-})\right| = (\lambda + 2m)^{n-1}(\lambda + 2n + 4r - 8)^{m-1}[\lambda^2 - \lambda[2m(n-1)$$
$$+(m-1)(2n+4r-8)] + 2m(n-1)(m-1)(2n+4r-8)$$
$$-nm(m+n+2r-4)^2] = 0$$

$$Spectra(DS(G^{++-})$$
$$= \begin{Bmatrix} -2m & ,(n-1)\ times \\ -(2n+4r-8) & ,(m-1)\ times \\ m(n-1)+(m-1)(n+2r-4) \pm \sqrt{[m(n-1)-(m-1)(n+2r-4)]^2 + nm(m+n+2r-4)^2} & \end{Bmatrix}$$

In this,

$$\lambda_1 = [m(n-1) + (m-1)(n+2r-4)$$
$$+\sqrt{[m(n-1)-(m-1)(n+2r-4)]^2 + nm(m+n+2r-4)^2}\,]/2$$

is the largest and only one positive eigenvalue.

Hence,

$$E_{DS}(G^{++-}) = 2\lambda_1 = 2m(n-1)+(m-1)(2n+4r-8) + \sqrt{[2m(n-1)-(m-1)(2n+4r-8)]^2 + 4nm(m+n+2r-4)^2}$$

Therefore,

$$E_{DS}(G^{++-}) = nr(n-1)+(nr-2)(n+2r-4) + \sqrt{[nr(n-1)-(nr-2)(n+2r-4)]^2 + (n^2r/2)(nr+2n+4r-8)^2}$$

Example 1 : For $K_{3,3}$, $n = 6$, $m = 9$, $r = 3$

From the Theorem 3.2,

$$E_{DS}(K_{3,3}^{++-}) = 90+128+\sqrt{1444+54(34)^2} = 470.7212$$

Characteristic polynomial of

$$DS(K_{3,3}^{++-}) = |\lambda I - DS(K_{3,3}^{++-})| = \begin{vmatrix} ((\lambda I - 18(J-I))_{6\times 6} & -17J_{6\times 9} \\ -17J_{9\times 6} & ((\lambda I - 16(J-I))_{9\times 9} \end{vmatrix}$$

Let $a = 18$, $b = 16$, $c = 17$, $d = 17$ $n_1 = 6$, $n_2 = 9$.
Here, $(n_1-1)(n_2-1)ab = 11520 \le n_1n_2cd = 15606.$
From case(i) of Lemma 2.6,

$$\gamma_1 = \sum_{i=2}^{n_1+n_2} \gamma_i = [a(n_1-1)+b(n_2-1) + \sqrt{[a(n_1-1)-b(n_2-1)]^2 + 4n_1n_2cd}\,]/2 = 235.360.59$$

From the definition Degree sum energy of $K_{3,3}^{++-}$,

$$E_{DS}(K_{3,3}^{++-}) = \gamma_1 + \sum_{i=2}^{n_1+n_2}|\gamma_i| = 2\gamma_1 = 470.7212$$

Theorem 3.3 : *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{+-+}) = 4r(n-1)+(nr-2)[(nr-4r+6)/2] + \sqrt{[4r(n-1)-(nr-2)[(nr-4r+6)/2]]^2 + (n^2r/2)(nr+6)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{+-+} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{+-+})\right| = \begin{vmatrix} (\lambda I - 4r(J-I))_{n\times n} & -(m+3)J_{n\times m} \\ -(m+3)J_{m\times n} & (\lambda I-(2m-4r+6)(J-I))_{m\times m} \end{vmatrix} \qquad ...(5)$$

Expanding by using lemma 2.5 we obtain,

$$\left|\lambda I - DS(G^{+-+})\right| =$$
$$(\lambda+4r)^{n-1}(\lambda+2m-4r+6)^{m-1}[\lambda^2-\lambda[(m-1)(2m-4r+6)$$
$$+4r(n-1)]+4r(n-1)(m-1)(2m-4r+6)-nm(m+3)^2]=0$$

$$Spectra(DS(G^{+-+})$$
$$=\begin{cases} -4r & ,(n-1)\ times \\ -(2m-4r+6) & ,(m-1)\ times \\ 2r(n-1)+(m-1)(m-2r+3)\pm\sqrt{[r(n-1)-(m-1)(m-2r+3)]^2+nm(m+3)^2} & \end{cases}$$

In this,

$$\lambda_1 = 2r(n-1)+(m-1)(m-2r+3)$$
$$+\sqrt{[r(n-1)-(m-1)(m-2r+3)]^2+nm(m+3)^2}$$

is the largest and only one positive eigenvalue.

Hence

$$E_{DS}(G^{+-+}) = 2\lambda_1 = 4r(n-1)+(m-1)(2m-4r+6)$$
$$+\sqrt{[4r(n-1)-(m-1)(2m-4r+6)]^2+4nm(m+3)^2}$$

Therefore,

$$E_{DS}(G^{+-+}) = 4r(n-1)+(nr-2)[(nr-4r+6)/2]$$
$$+\sqrt{[4r(n-1)-(nr-2)[(nr-4r+6)/2]]^2+(n^2r/2)(nr+6)^2}$$

Theorem 3.4: *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{-++}) = 2(n-1)^2+2r(nr-2)$$
$$+\sqrt{[2(n-1)^2-2r(nr-2)]^2+2n^2r(n+2r-1)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{-++} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{-++})\right| = \begin{vmatrix} (\lambda I-2(n-1)(J-I))_{n\times n} & -(2r+n-1)J_{n\times m} \\ -(2r+n-1)J_{m\times n} & (\lambda I-4r(J-I))_{m\times m} \end{vmatrix} \quad ...(6)$$

Expanding by using lemma 2.5 we obtain,

$$Spectra(DS(G^{-++}) = \begin{cases} -2(n-1) & ,(n-1)\ times \\ -4r & ,(m-1)\ times \\ (n-1)^2+2r(m-1)\pm\sqrt{[(n-1)^2-2r(m-1)]^2+nm(2r+n-1)^2} & \end{cases}$$

Proceeding as given in Theorem (3.2),
Hence,

$$E_{DS}(G^{-++}) = 2(n-1)^2 + 4r(m-1) + \sqrt{[2(n-1)^2 - 4r(m-1)]^2 + 4nm(2r+n-1)^2}$$

Therefore,

$$E_{DS}(G^{-++}) = 2(n-1)^2 + 2r(nr-2) + \sqrt{[2(n-1)^2 - 2r(nr-2)]^2 + 2n^2r(n+2r-1)^2}$$

Theorem 3.5: *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{---}) = (nr+2n-4r-2)(nr+2n-2)$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{---} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{---})\right| = \left|\lambda I - (2m+2n-4r-2)(J-I)\right|_{(m+n)\times(m+n)} \tag{7}$$

By using lemma 2.4,

$$|\lambda I - DS(G^{---})| = (\lambda + 2m+2n-4r-2)[\lambda - (m+n-1)(2m+2n-4r-2)] = 0$$

$$Spectra(DS(G^{---}) = \begin{cases} -2(m+n-2r-1) & ,(m+n-1)\ times \\ 2(m+n-2r-1)(m+n-1) & ,1\ time \end{cases}$$

Hence, $E_{DS}(G^{---}) = 4(m+n-2r-1)(m+n-1)$
Therefore, $E_{DS}(G^{---}) = (nr+2n-4r-2)(nr+2n-2)$

Theorem 3.6: *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{--+}) = 2(n-1)^2 + [(nr-4r+6)(nr-2)/2] + \sqrt{[2(n-1)^2 - ((nr-4r+6)(nr-2)/2)]^2 + (n^2r/2)(nr+2n-4r+4)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{--+} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{--+})\right| = \begin{vmatrix} (\lambda I-2(n-1)(J-I))_{n\times n} & -(m+n-2r+2)J_{n\times m} \\ -(m+n-2r+2)J_{m\times n} & (\lambda I-(2m-4r+6)(J-I))_{m\times m} \end{vmatrix} \quad ...(8)$$

Expanding by using lemma 2.5 we obtain,

$$Spectra(DS(G^{--+}) =$$

$$\begin{cases} -2(n-1) & ,(n-1)\ times \\ -(2m-4r+6) & ,(m-1)\ times \\ (n-1)^2+(m-2r+3)(m-1)\pm\sqrt{[(n-1)^2-(m-2r+3)(m-1)]^2+nm(m+n-2r+2)^2} & \end{cases}$$

Proceeding as given in Theorem (3.2),
Hence,

$$E_{DS}(G^{--+}) = (n-1)^2 + (2m-4r+6)(m-1) + \sqrt{[2(n-1)^2 - (2m-4r+6)(m-1)]^2 + 4nm(m+n-2r+2)^2}$$

Therefore,

$$E_{DS}(G^{--+}) = 2(n-1)^2 + [(nr-4r+6)(nr-2)/2] + \sqrt{[2(n-1)^2 - ((nr-4r+6)(nr-2)/2)]^2 + (n^2r/2)(nr+2n-4r+4)^2}$$

Theorem 3.7: *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{-+-}) = (nr+2n-4r-2)(n-1) + (n+2r-4)(nr-2) + \sqrt{[(nr+2n-4r-2)(n-1) - (n+2r-4)(nr-2)]^2 + (n^2r/2)(nr+4n-10)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{-+-} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{-+-})\right| = \begin{vmatrix} (\lambda I-2(m+n-2r-1)(J-I))_{n\times n} & -(m+2n-5)J_{n\times m} \\ -(m+2n-5)J_{m\times n} & (\lambda I-(2n+4r-8)(J-I))_{m\times m} \end{vmatrix} \quad ...(9)$$

Expanding by using lemma 2.5 we obtain,

$$Spectra(DS(G^{-+-}) =$$

$$\begin{cases} -2(m+n-2r-1) & ,(n-1)\ times \\ -(2n+4r-8) & ,(m-1)\ times \\ (m+n-2r-1)(n-1)+(n+2r-4)(m-1)\pm & \\ \quad\sqrt{[(m+n-2r-1)(n-1)-(n+2r-4)(m-1)]^2+nm(m+2n-5)^2} & \end{cases}$$

Proceeding as given in Theorem (3.2),

Hence,

$$E_{DS}(G^{-+-}) = 2(m+n-2r-1)(n-1) + 2(n+2r-4)(m-1) + \sqrt{[2(m+n-2r-1)(n-1) - 2(n+2r-4)(m-1)]^2 + 4nm(m+2n-5)^2}$$

Therefore,

$$E_{DS}(G^{-+-}) = (nr+2n-4r-2)(n-1) + (n+2r-4)(nr-2) + \sqrt{[(nr+2n-4r-2)(n-1) - (n+2r-4)(nr-2)]^2 + (n^2 r/2)(nr+4n-10)^2}$$

Theorem 3.8: *For any r-regular graph G(n, m), we have*

$$E_{DS}(G^{+--}) = nr(n-1) + [(nr-2)(nr+2n-4r-2)/2] + \sqrt{[nr(n-1) - ((nr-2)(nr+2n-4r-2)/2)]^2 + 2n^2 r(nr+n-2r-1)^2}$$

Proof: As $G(n, m)$ is a r-regular graph, by the definition of G^{+--} and Proposition 2.2 and 2.3,

$$\left|\lambda I - DS(G^{+--})\right| = \begin{vmatrix} (\lambda I - 2m(J-I))_{n\times n} & -(2m+n-2r-1)J_{n\times m} \\ -(2m+n-2r-1)J_{m\times n} & (\lambda I - (2m+2n-4r-2)(J-I))_{m\times m} \end{vmatrix} \quad ...(10)$$

Expanding by using lemma 2.5 we obtain,

$$Spectra(DS(G^{+--}) = \begin{cases} -2m & ,(n-1)\ times \\ -(2m+2n-4r-2) & ,(m-1)\ times \\ m(n-1)+(m-1)(m+n-2r-1) \pm \sqrt{[m(n-1)-(m-1)(m+n-2r-1)]^2 + nm(2m+n-2r-1)^2} & \end{cases}$$

Proceeding as given in Theorem (3.2),

Hence,

$$E_{DS}(G^{+--}) = 2m(n-1) + 2(m-1)(m+n-2r-1) + \sqrt{[2m(n-1) - 2(m-1)(m+n-2r-1)]^2 + 4nm(2m+n-2r-1)^2}$$

Therefore,

$$E_{DS}(G^{+--}) = nr(n-1) + [(nr-2)(nr+2n-4r-2)/2] + \sqrt{[nr(n-1) - ((nr-2)(nr+2n-4r-2)/2)]^2 + 2n^2 r(nr+n-2r-1)^2}$$

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