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Geometric meaning and variation of parameter method of modified α-fractional derivative

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Abstract

In this study, we address a frequently asked topic by scholars, "What is the geometrical meaning of the modified α-derivative", by using the notion of fractional cords. There is also discussion of fractional orthogonal trajectories. We give several examples to explain the notions of fractional cords and fractional orthogonal trajectories. Moreover, we also discuss the variation of the parameter method and we examine how to apply this technique to discover a particular solution to a set of nonhomogeneous linear fractional differential

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equations. A formula for modified α-fractional differential equations is established, which is identical to that for ordinary differential equations.

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1. Introduction

Fractional derivatives are defined in a variety of ways in the literature. The most important are the Riemann Liouville and Caputo definitions, which can be found in [12, 15], as well as [8,6,13], for some applications. The Riemann - Liouville definition and the Caputo definition of fractional derivative, is as follows.

(*i*) Riemann-Liouville definition[15]: For $\alpha \in [p-1, p)$ the α - fractional derivative of Φ is defined as

$$D_a^{\alpha}\Phi(\xi) = \frac{1}{\Gamma(p-\alpha)} \frac{d^p}{d\zeta^p} \int_a^{\alpha} \frac{\Phi(\xi)}{(\zeta-\xi)^{\alpha-p+1}} d\xi.$$

(*ii*) Caputo definition[15]: For $\alpha \in [p-1, p)$ the α - fractional derivative of Φ is defined as

$$D_a^{\alpha}\Phi(\xi) = \frac{1}{\Gamma(p-\alpha)} \int_a^{\alpha} \frac{\Phi^{(p)}(\xi)}{(\zeta-\xi)^{\alpha-p+1}} d\xi.$$

Some features, including the Product Rule, Quotient Rule, Chain Rule, Rolle's Theorem and Mean Value Theorem, are not provided by these fractional derivatives in this situation. Khalil et al. (cite:8) developed an innovative concept that expands the well-known limit definition of the derivative of a function to address some of these and other issues. Using his novel definition of fractional derivative, he obtained some results. In the works cited in [7, 8, 9, 10] and [1], Khalil et al. established a new definition of the term "fractional derivative" and used it to demonstrate various findings.

In [2], Almeida et al., proposed a new limit definition for the local fractional derivative, which is defined as follows.

Definition 1.1 : [2] Let $k:[a,b] \to R$ be a continous non-negative function with $k(\xi) \neq 0$, if $\xi > a$. If $\alpha \in (0,1)$ a real, then $\phi:[a,b] \to R$ is called as α -differentiable at $\xi > a$, in relation to kernel k,, if the limit.

$$\phi^{(\alpha)}(\xi) := \lim_{\varepsilon\to 0} \frac{\phi(\xi+\varepsilon k(\xi)^{1-\alpha}) - \phi(\xi)}{\varepsilon}$$

exist.

They get some known cases for certain relevant kernel options. Also, they discovered a connection between this new notion & conventional ordinary differentiation. Furthermore, most of the essential features of the fractional derivative may be determined directly using there formula and results. Recently, Katugampola[5], introduced a new definition of fractional derivative and utilising his revised definition of fractional derivative to explain the concept of fractional derivative. The fundamental purpose of this paper is to present a limit definition of the derivative of a function that obeys classical features such as linearity, product rule, quotient rule and chain rule. For undefined & unexplained concepts and terms the readers can see[11, 12].

2. Modified α-Fractional Derivative

We developed the notion of α-fractional derivative and integral in [14] by modifying the conventional definition of an ordinary derivative, which is described as follows.

Definition 2.1 : As $\Phi : [0,\infty) \to \mathbb{R}$ and $\alpha \in [0,1)$ then α-fractional derivative of Φ of order α is defined as

$$T_\alpha(\Phi(\zeta)) = \lim_{\mu\to 0} \frac{\Phi(\zeta e^{\mu\zeta^{1-\alpha}}) - \Phi(\zeta)}{\mu}, \tag{2.1}$$

for all $\zeta > 0$, and if $\alpha = 1$ then $T_\alpha(\Phi(\zeta)) = \Phi'(\zeta)$.

Theorem 2.2 : [14] *If Φ is a α-differential and $\alpha \in [0,1)$ then*

$$T_\alpha \Phi(\zeta) = \zeta^{2-\alpha} \frac{d\Phi(\zeta)}{dx}$$

Theorem 2.3 : [14] *If Φ and Ψ are α-differentiable functions at ϑ then*

1. If Φ is a α-differentiable at ϑ then Φ is continuous at ϑ.
2. $T_\alpha(\beta\Phi) = \beta T_\alpha(\Phi), \quad \forall \quad \beta \in \mathbb{R}$
3. $T_\alpha(\Phi + \Psi)(\vartheta) = T_\alpha(\Phi)(\vartheta) + T_\alpha(\Psi)(\vartheta)$.
4. $T_\alpha(\Phi\Psi)(\vartheta) = \Phi T_\alpha(\Psi)(\vartheta) + \Psi T_\alpha(\Phi)(\vartheta)$.
5. $T_\alpha\left(\frac{\Phi}{\Psi}\right)(\vartheta) = \left(\frac{\Psi(\vartheta)T_\alpha\Phi(\vartheta) - \Phi(\vartheta)T_\alpha\Psi(\vartheta)}{\Psi(\vartheta)^2}\right)$.

6. $T_\alpha\left(\frac{1}{\Psi}\right) = -\frac{T_\alpha \Psi}{\Psi^2}.$

7. $T_\alpha(\Phi \circ \Psi)(\vartheta) = T_\alpha \Phi(\Psi(\vartheta)) T_\alpha \Psi(\vartheta).$

3. α-Fractional cords

Let $\phi(\zeta, \xi) = \kappa$ be some curve in the $\zeta\xi$-plane. For a point (ζ_0, ξ_0) on the curve, the equation $\frac{\xi - \xi_0}{\zeta - \zeta_0} = \xi'(\zeta_0)$ denotes the formula of the tangent line to curve $\phi(\zeta, \xi) = \kappa$ at the point (ζ_0, ξ_0).

Definition 3.1 : If $\phi(\zeta, \xi) = \kappa$ be a some curve in the $\zeta\xi$-plane then equation

$$\frac{\xi^\alpha - \xi_0^\alpha}{\zeta^\alpha - \zeta_0^\alpha} = \frac{\xi_0^{\alpha-2}}{\zeta_0^{\alpha-2}} \xi^{(\alpha)}(\zeta_0) \tag{3.1}$$

is called as fractional cord of the curve $\phi(\zeta, \xi) = \kappa$ at the point (ζ_0, ξ_0).

Here Fractional cords represent deviation curves from the tangent line, in the sense.

$$\frac{\xi^\alpha - \xi_0^\alpha}{\zeta^\alpha - \zeta_0^\alpha} = \frac{\xi_0^{\alpha-2}}{\zeta_0^{\alpha-2}} \xi^{(\alpha)}(\zeta_0)$$

And by definition if $\alpha = 1$ then

$$\frac{\xi - \xi_0}{\zeta - \zeta_0} = \xi'(\zeta_0).$$

Theorem 3.2 : *The α-fractional derivative $\xi^{(\alpha)}(\zeta_0)$ of the function $\xi(\zeta)$ in the equation $\phi(\zeta, \xi) = \kappa$, is the tangent line's slope to the fractional cords connected with the curve $\phi(\zeta, \xi) = \kappa$ at (ζ_0, ξ_0).*

Thus fractional cords indicate deviation curves from the tangent line, in the following sense

$$\frac{\xi^\alpha - \xi_0^\alpha}{\zeta^\alpha - \zeta_0^\alpha} = \frac{\xi_0^{\alpha-2}}{\zeta_0^{\alpha-2}} \xi^{(\alpha)}(\zeta_0)$$

with respect to ζ_0 at (ζ_0, ξ_0) and, by definition if $\alpha = 1$, we get $\xi'(\zeta_0) = \frac{\xi - \xi_0}{\zeta - \zeta_0}$.

Example 3.3 : Consider $\phi(\zeta, \xi) = \xi - \zeta^2 = 0$.

If $\alpha = 3/2$ then the fractional cord at $(2, 4)$ is

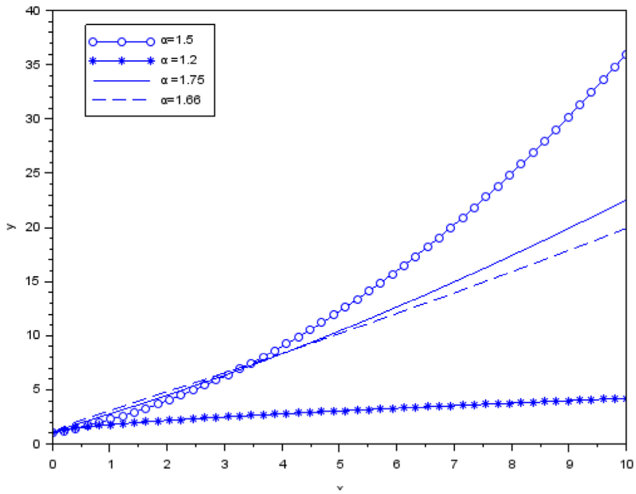

Figure 1

The fractional cord to the curve $\xi = \zeta^2$ at (2.4), for different values of α

$$\frac{\sqrt{\xi}-2}{\sqrt{\zeta}-\sqrt{2}} = \frac{\sqrt{2}}{2}\xi^{(3/2)}(2) = \frac{\sqrt{2}}{2}(4\sqrt{2}) = 4$$

$$\Rightarrow \qquad \sqrt{\xi} - 2 = 4(\sqrt{\zeta} - \sqrt{2}). \tag{3.2}$$

The slope of the tangent to the fractional cord (3.2) is $\xi'(2) = 4\sqrt{2}$. This is $\xi^{(3/2)}(2)$, for the curve $\xi = \zeta^2$, the fractional cord is shown in Figure 1.

4. α-Fractional orthogonal curves

If $\phi(\zeta, \xi) = \lambda$, $\psi(\zeta, \xi) = \eta$ are two families of curves then the curve $\phi(\zeta, \xi) = \lambda$, is said to be α-fractionally orthogonal to $\psi(\zeta, \xi) = \eta$ at (ζ, ξ), if $\xi^{(\alpha)}(\zeta)$ for $\phi(\zeta, \xi) = \lambda$ is equal to $\frac{-1}{\xi^{(\alpha)}(\zeta)}$ for $\psi(\zeta, \xi) = \eta$.

Let

$$\frac{\xi^{\alpha} - \xi_0^{\alpha}}{\zeta^{\alpha} - \zeta_0^{\alpha}} = \frac{\xi_0^{\alpha-2}}{\zeta_0^{\alpha-2}}\xi^{(\alpha)}(\zeta_0) \tag{4.1}$$

be the fractional cord at (ζ_0, ξ_0) for the curve $\phi(\zeta, \xi) = 0$. Then

$$\frac{\xi^{\alpha} - \xi_0^{\alpha}}{\zeta^{\alpha} - \zeta_0^{\alpha}} = \frac{\xi_0^{\alpha-2}}{\zeta_0^{\alpha-2}}\left(\frac{-1}{\xi^{(\alpha)}(\zeta)}\right) \tag{4.2}$$

is the orthogonal α-fractional cord at (ζ_0, ξ_0) for the curve $\phi(\zeta, \xi) = 0$. Indeed, the slope of the tangent to (4.1) at (ζ_0, ξ_0) is $\xi^{\alpha}(\zeta_0)$ while the slope of the tangent to (4.2) is $\frac{-1}{\xi^{(\alpha)}(\zeta)}$

Example 4.1 : If $\xi = \zeta + c$ be the family of curves, then $\xi^{(3/2)} = \sqrt{\zeta}$. Put $\frac{-1}{\xi^{(3/2)}} = \sqrt{\zeta}$ then $\zeta\xi' = -1$. Hence $\xi = -\ln\zeta + C, \zeta > 0$ is the $\frac{3}{2}$-orthogonal trajectories to the family $\xi = \zeta + c$ at (ζ, ξ).

Example 4.2 : Consider the curve Ω given by $\xi = \sin\zeta$. Now $(\frac{\pi}{4}, \frac{1}{\sqrt{2}}) \in \Omega$. Take $0 < \alpha \leq 1$. Then

$$\xi^{(\alpha)}(\frac{\pi}{4}) = (\frac{\pi}{4})^{2-\alpha}\frac{1}{\sqrt{2}}.$$

As the fractional cord of Ω at $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ is given by

$$\frac{\xi^{\alpha} - (\frac{1}{\sqrt{2}})^{\alpha}}{\zeta^{\alpha} - (\frac{\pi}{4})^{\alpha}} = \frac{1}{\sqrt{2}}(\frac{\pi}{4})^{2-\alpha}\left(\frac{4}{\sqrt{2}\pi}\right)^{2-\alpha}$$

the orthogonal fractional cord at $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ is

$$\frac{\xi^{\alpha} - (\frac{1}{\sqrt{2}})^{\alpha}}{\zeta^{\alpha} - (\frac{\pi}{4})^{\alpha}} = \sqrt{2}(\frac{\pi}{4})^{2-\alpha}\left(\frac{4}{\sqrt{2}\pi}\right)^{2-\alpha}$$

5. α-Fractional linear differential equations

Definition 5.1 : If $\alpha \in [0, 1)$ and $n \in N$. Then we write

$${}^{n}T_{\alpha}f = T_{\alpha}T_{\alpha}\ldots T_{\alpha}\phi.$$

We use writing for simplicity $\phi^{(n\alpha)}$ for ${}^{n}T_{\alpha}$. So $\gamma^{(2\alpha)}$ stand for $\frac{d^{\alpha}}{d\nu^{\alpha}}(\frac{d^{\alpha}\gamma}{d\nu^{\alpha}})$

Definition 5.2 : An equation

$${}^{n}T_{\alpha}\gamma + a_{n-1}{}^{n-1}T_{\alpha}\gamma + \cdots + a_1T_{\alpha}\gamma + a_0\gamma = \phi(\nu) \tag{5.1}$$

is called a linear FDE of order n. The coefficients $a_0, a_1, \ldots, a_{n-1}$ may be constants or variables.

As $\alpha \in [0,1)$ and γ is n-times differentiable, then $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be the set of n-independent solutions for the homogeneous differential equation.

$$^{n}T_{\alpha}\gamma + \cdots + a_0\gamma = 0 \tag{5.2}$$

To determine a particular solution of (5.1), one can use either undetermined coefficients or the variation of parameters method. When $n = 2$, we actually discover a comprehensive formula for γ_p.

Definition 5.3 : If γ_1 and γ_2 are two independent functions then function

$$W_{\alpha}[\gamma_1, \gamma_2] = \begin{vmatrix} \gamma_1 & \gamma_2 \\ T_{\alpha}\gamma_1 & T_{\alpha}\gamma_2 \end{vmatrix}$$

will be called the α-Wronskian of γ_1 and γ_2

More generally, if $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be the set of n linearly independent functions, then

$$W_{\alpha}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \begin{vmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ T_{\alpha}\gamma_1 & T_{\alpha}\gamma_2 & \cdots & T_{\alpha}\gamma_n \\ \vdots & \vdots & \vdots & \vdots \\ ^{n-1}T_{\alpha}\gamma_1 & ^{n-1}T_{\alpha}\gamma_2 & \cdots & ^{n-1}T_{\alpha}\gamma_n \end{vmatrix}$$

6. The Main Results

Let γ_1 and γ_2 be two independent solutions for

$$T_{\alpha}(T_{\alpha}\gamma) + a_1 T_{\alpha}\gamma + a_0\gamma = 0 \tag{6.1}$$

Our purpose is to locate γ_p for

$$T_{\alpha}(T_{\alpha}\gamma) + a_1 T_{\alpha}\gamma + a_0\gamma = \varphi(\nu) \tag{6.2}$$

Procedure to is to find γ_p is as fallows:

Step (I) : Let

$$\gamma_p = \sum_{i=1}^{2} c_i \gamma_i$$

where $c_1 = c_1(\nu)$ and $c_2 = c_2(\nu)$ and hence, we have

$$\gamma_p(\nu) = \sum_{i=1}^{2} c_i(\nu)\gamma_i(\nu)$$

$$\Rightarrow T_\alpha \gamma_p(\nu) = T_\alpha c_1(\nu)\gamma_1(\nu) + c_1(\nu) T_\alpha \gamma_1(\nu) + T_\alpha c_2(\nu)\gamma_2(\nu) + c_2(\nu) T_\alpha \gamma_2(\nu)$$

Step (II) : Put

$$T_\alpha c_1(\nu)\gamma_1(\nu) + T_\alpha c_2(\nu)\gamma_2(\nu) = 0 \tag{6.3}$$

$$\begin{aligned} T_\alpha T_\alpha \gamma_p &= (c_1 T_\alpha \gamma_1 + c_2 T_\alpha \gamma_2) \\ &= T_\alpha c_1 T_\alpha \gamma_1 + c_1{}^2 T_\alpha \gamma_1 + T_\alpha c_2 T_\alpha \gamma_2 + c_2{}^2 T_\alpha \gamma_2 \end{aligned}$$

Step (III). Substitute $T_\alpha T_\alpha \gamma_p$ in the equation (6.2) to get

$$\begin{aligned} T_\alpha c_1 T_\alpha \gamma_1 + c_1{}^2 T_\alpha \gamma_1 + T_\alpha c_2 T_\alpha \gamma_2 + c_2{}^2 T_\alpha \gamma_2 + a_1(c_1 T_\alpha \gamma_1 + c_2 T_\alpha \gamma_2) \\ + a_0(c_1\gamma_1 + c_2\gamma_2) = \varphi(\nu) \end{aligned}$$

$$\begin{aligned} \Rightarrow c_1({}^2 T_\alpha \gamma_1 + a_1 T_\alpha \gamma_1 + a_0\gamma_1) + c_2({}^2 T_\alpha \gamma_2 + a_1 T_\alpha \gamma_2 + a_0\gamma_2) + \\ T_\alpha c_1 T_\alpha \gamma_1 + c_2 T_\alpha \gamma_2 = \varphi(\nu) \end{aligned}$$

Step (IV) : Since γ_1 and γ_2 are solutions for (6.2), we get

$$T_\alpha c_1 T_\alpha \gamma_1 + c_2 T_\alpha \gamma_2 = \varphi(\nu). \tag{6.4}$$

After solving the equations (6.3) and (6.4), we get

$$T_\alpha c_1 = -\frac{\varphi(\nu)\gamma_2(\nu)}{W_\alpha[\gamma_1, \gamma_2]}$$

$$T_\alpha c_2 = \frac{\varphi(\nu)\gamma_1(\nu)}{W_\alpha[\gamma_1, \gamma_2]}$$

$$\gamma_p(\nu) = \int_a^\nu \frac{\begin{vmatrix} \gamma_1(\xi) & \gamma_2(\xi) \\ \gamma_1(\nu) & \gamma_2(\nu) \end{vmatrix}}{W_\alpha[\gamma_1, \gamma_2]} \frac{\varphi(\xi)}{\xi^{2-\alpha}} d\xi$$

$$\begin{aligned} \gamma_p(\nu) &= I_\alpha^a \left[\frac{\begin{vmatrix} \gamma_1(\xi) & \gamma_2(\xi) \\ \gamma_1(\nu) & \gamma_2(\nu) \end{vmatrix}}{W_\alpha[\gamma_1, \gamma_2]} \varphi(\xi) \right] \\ &= I_\alpha^a (K(\nu, \xi)\varphi(\xi)), \end{aligned}$$

$$K(\nu, \xi) = \frac{\begin{vmatrix} \gamma_1(\xi) & \gamma_2(\xi) \\ \gamma_1(\nu) & \gamma_2(\nu) \end{vmatrix}}{W_\alpha[\gamma_1, \gamma_2]}$$

Higher order linear fractional differential equations can be taken into consideration in a similar manner. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ solutions of (5.2) be linearly independent. As was previously shown, it is possible to find a particular solution to the nonhomogeneous equation (5.1) of the given form.

$$\gamma_p(\nu) = \sum_{i=1}^{n} c_i(\nu)\gamma_i(\nu). \tag{6.5}$$

The following nonhomogeneous algebraic linear system of n equations for $T_\alpha c_1, T_\alpha c_2, \ldots, T_\alpha c_n$ may be produced by using the same methods as in the previous procedure $T_\alpha c_1, T_\alpha c_2, \ldots, T_\alpha c_n$

$$\gamma_1 T_\alpha c_1 + \gamma_2 T_\alpha c_2 + \cdots + \gamma_n T_\alpha c_n = 0$$

$$T_\alpha \gamma_1 T_\alpha c_1 + T_\alpha \gamma_2 T_\alpha c_2 + \cdots + T_\alpha \gamma_n T_\alpha c_n = 0$$

$${}^2T_\alpha \gamma_1 T_\alpha c_1 +^2 T_\alpha \gamma_2 T_\alpha c_2 + \cdots +^2 T_\alpha \gamma_n T_\alpha c_n = 0$$

$$\vdots$$

$${}^{(n-1)}T_\alpha \gamma_1 T_\alpha c_1 +^{(n-1)} T_\alpha \gamma_2 T_\alpha c_2 + \cdots +^{(n-1)} T_\alpha \gamma_n T_\alpha c_n = 0$$

Cramer's rule is used to determine

$$T_\alpha c_m(\nu) = \frac{\varphi(\nu) W_\alpha^k(\nu)}{W_\alpha(\nu)} \tag{6.6}$$

where $W_\alpha(\nu) = W_\alpha(\gamma_1, \gamma_2, \ldots, \gamma_n)(\nu)$ and W_α^k, is the determinant derived from W_α by substituting the m^{th} column by e_m. Thus

$$\gamma_p(\nu) = \sum_{k=1}^{n} \gamma_k \int_a^\xi \frac{\varphi(\xi) W_\alpha^k(\xi)}{W_\alpha(\xi)\xi} d\xi \tag{6.7}$$

where a is any positive constant that you choose.

7. Applications

Example 7.1 : Consider the following fractional differential equation

$$\nu T_{\frac{3}{2}}(T_{\frac{3}{2}}\gamma) - \frac{1}{2}\gamma = \nu^{3/2} + \nu, \quad \nu > 0 \tag{7.1}$$

One can easily show that $\gamma_1 = v$ and $\gamma_2 = 1/\sqrt{v}$ are two solutions to the equivalent homogeneous equation that are linearly independent.

$$vT_{\frac{3}{2}}(T_{\frac{3}{2}}\gamma) - \frac{1}{2}\gamma = 0. \tag{7.2}$$

Let us use the formula to obtain the particularÂ solution $\gamma_p(\xi)$ of equation (7.1),

$$\gamma_p(v) = -\gamma_1 \int \frac{\varphi(v)\gamma_2}{W_\alpha[\gamma_1,\gamma_2]v^{2-\alpha}}dv + \gamma_2 \int \frac{\varphi(v)\gamma_1}{W_\alpha[\gamma_1,\gamma_2]v^{2-\alpha}}dv$$

$$= -\gamma_1 \int \frac{\varphi(v)\gamma_2}{W[\gamma_1,\gamma_2]v^{4-2\alpha}}dv + \gamma_2 \int \frac{\varphi(v)\gamma_1}{W[\gamma_1,\gamma_2]v^{4-2\alpha}}dv.$$

Substitute $\gamma_1 = v,\ \gamma_2 = 1/\sqrt{v}$ $\varphi(v) = v^{1/2} + 1$ and $W[\gamma_1, \gamma_2] = -3/2$ $v^{-1/2}$ in the above equation, we get

$$\gamma_p(v) = -v\int \frac{\xi^{-1/2}(v^{1/2}+1)}{-3/2\,v^{-1/2}v}dv + v^{-1/2}\int \frac{v(v^{1/2}+1))}{-3/2\,v^{-1/2}v}dv$$

$$= v^{3/2} + \frac{2}{3}v\ln v - \frac{4}{9}v$$

As a result, the general solution of equation (7.1) is as follows:

$$\gamma(v) = \phi_1 v + \phi_2 v^{-1/2} + v^{3/2}\,\frac{2}{3}v\ln v - \frac{4}{9}v$$

Example 7.2 : Let's try to solve the fractional differential equation below.

$$T_{\frac{3}{2}}(T_{\frac{3}{2}}\gamma) = \ln v, \quad v > 0. \tag{7.3}$$

It is possible to verify that $\gamma_1 = 1$ and $\gamma_2 = \sqrt{(v)}$ are the linearly independent solutions of the homogeneous equation.

$$T_\alpha(T_\alpha\gamma) = 0.$$

Let us use the method of parameter variation to find the specific solution $\gamma_p(v)$ of equation (7.3). More specifically, we want to use the formula.

$$\gamma_p(v) = -\gamma_1 \int \frac{\varphi(v)\gamma_2}{W_\alpha[\gamma_1,\gamma_2]v^{2-\alpha}}dv + \gamma_2 \int \frac{\varphi(v)\gamma_1}{W_\alpha[\gamma_1,\gamma_2]\xi^{2-\alpha}}dv$$

$$= -\gamma_1 \int \frac{\varphi(v)\gamma_2}{W[\gamma_1,\gamma_2]v^{4-2\alpha}}dv + \gamma_2 \int \frac{\varphi(v)\gamma_1}{W[\gamma_1,\gamma_2]v^{4-2\alpha}}dv$$

Since $\gamma_1 = 1$ and $\gamma_2 = v, \varphi(v) = \ln v$ and $W[\gamma_1, \gamma_2] = \frac{1}{2\sqrt{v}}$, , we obtain

$$\gamma_p(v) = -\int \frac{\sqrt{v}\ln v}{v} dv + \sqrt{v}\int \frac{\ln v}{\frac{1}{2\sqrt{v}} v} dv$$

$$= -2\int \ln v dv + 2\sqrt{v}\int \frac{\ln v}{\sqrt{v}} dv$$

$$= 2v\ln v - 6v.$$

Thus, the general solution $\gamma(v)$ can be written as

$$\gamma(v) = \phi_1 + \phi_2\sqrt{v} + 2v\ln v - 6v.$$

Example 7.3 : Let's try to solve the fractional differential equation below.

$$T_{\frac{3}{2}}(T_{\frac{3}{2}}(T_{\frac{3}{2}}\gamma)) = v^{5/2} \tag{7.4}$$

The linearly independent solutions of the homogeneous equation are $\gamma_1 = 1$, $\gamma_2 = v$ and $\gamma_3 = v^{1/2}$, which can be verified.

$$T_{\frac{3}{2}}(T_{\frac{3}{2}}(T_{\frac{3}{2}}\gamma)) = 0$$

$$W_{\frac{3}{2}} = -\frac{-1}{4}, W^1_{\frac{3}{2}} = -\frac{-1}{2}v, W^2_{\frac{3}{2}} = -\frac{-1}{2} \text{ and } W^3_{\frac{3}{2}} = v^{3/2}$$

Now, see (6.7),

$$\gamma_p(v) = \gamma_1\int \frac{W^1_\alpha f(\xi)}{W_\alpha v^{1/2}} dv + \gamma_2\int \frac{W^2_\alpha f(\xi)}{W_\alpha v^{1/2}} dv + \gamma_3\int \frac{W^3_\alpha f(\xi)}{W_\alpha v^{1/2}} dv$$

$$= \int \frac{-\frac{-1}{2} v v^{5/2}}{-\frac{-1}{4} v^{1/2}} dv + v\int \frac{-\frac{-1}{2} v^{5/2}}{-\frac{-1}{4} v^{1/2}} dv + v^{1/2}\int \frac{v^{5/2} v^{1/2}}{-\frac{-1}{4} v^{1/2}} dv$$

$$= 2\int x^3 dv + 2v\int v^2 dv - 4v^{1/2}\int v^{5/2} dv$$

$$= \frac{1}{42} v^4$$

As a result, the general solution of equation (7.4) is

$$\gamma(\xi) = \phi_1 + \phi_2 v + \phi_3 v^{1/2} + \frac{1}{42} v^4$$

Conclusion

In this work, we have discussed the geometrical meaning of the modified α-fractional derivative by using the concept of fractional cords.

Later, the concept of fractional orthogonal trajectories was also covered. SomeÂ illustrations are used to explain the notions of fractional cords and fractional orthogonal trajectories. Moreover, we also discussed the variation of the parameter method and we looked at how to use this method to discover a specific solution to a set of nonhomogeneous linear fractional differential equations. A formula for modified α-derivative differential equations is established and it is found that this formula is identical to that formula of ordinary differential equations.

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