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Topological indices of johnson graphs and their complements

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Abstract

The connectivity of the complements of the Johnson graphs $\mathcal{J}(\aleph, r)$ is completely characterized for every $\aleph \geq 2r$. A combinatorial identity as a variation of vandermonde's identity has been established to compute the Wiener index of Johnson graphs $\mathcal{J}(\aleph, r)$ and their compliments $\overline{\mathcal{J}(\aleph, r)}$ (whenever connected) by utilizing the vertex-transitivity of both the families of graphs. In addition, the Szeged index of $\mathcal{J}(\aleph, r)$ and $\overline{\mathcal{J}(\aleph, r)}$ is also obtained by

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using the edge transitivity and examining a connection between the topology of the vertices in both the graphs, respectively, because the complement of a symmetric graph need not to be edge-transitive, in particular, $\overline{\mathcal{J}(\aleph, r)}$ is not edge transitive for $r \neq 1, 2$ & $\aleph$-1.

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1. Introduction

The Johnson graph $\mathcal{J}(\aleph, r)$, where $\aleph, r$ positive integers and a set $\mathcal{X}$ having $\aleph$ elements, is a family of graphs with r-elements subsets of $\mathcal{X}$ correspond to its vertices, along with this, two of them are adjacent iff their corresponding subset has $r-1$ elements in common. Such a graph $\mathcal{J}(\aleph, r)$ has ${}^{\aleph}C_r$ vertices and $\frac{1}{2}r(\aleph - r)\,{}^{\aleph}C_r$ edges. It is not only $r(\aleph - r)$-regular graph but also symmetric. This means that the graph is vertex as well as edge transitive, i.e., $\mathcal{A}ut(\mathcal{J}(\aleph, r))$ acts transitively on both its vertices and edges. More precisely, for a given pair of vertices or edges in $\mathcal{J}(\aleph, r)$, there exists an automorphism of $\mathcal{J}(\aleph, r)$ which maps one of the vertices or edges to the other respectively.

In this article, any graph under consideration is a simple connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ having $\mathcal{V}$ & $\mathcal{E}$ as its edge and vertex set respectively. For more details on basic definitions and notations that have been used throughout this work can be found in [10].

Observe that the complement of a symmetric graph need not to be always symmetric, however it is always vertex-transitive i.e, the graph may not be edge-transitive. Thus, $\overline{\mathcal{J}(\aleph, r)}$, is vertex-transitive still it may or may not be edge-transitive. According to Lemma 3.3 in [18], it is pertinent to notice that $\overline{\mathcal{J}(\aleph, 2)} \cong \overline{\mathcal{J}(\aleph, \aleph - 2)}$ is the Kneser graph $\mathcal{KG}_{\aleph,2}$ which is a symmetric graph. Nevertheless, in Example 2.8, the orbits under the action of $\mathcal{A}ut(\overline{\mathcal{J}(\aleph, r)})$ on its edge set has been obtained, which infers that it is not edge-transitive. Moreover, in view of [5], the transitive action of $\mathcal{A}ut(\mathcal{G})$ is essential for computing topological indices of $\mathcal{G}$.

Topological indices describe numerical characteristics of a chemical graph by analysing topology, rather provide a quantitative description of its structure by emphasizing on the various parameters like degree of its vertices, the distance between the vertices and their sum, number of vertices at a given distance apart, the relative distance between the vertices and edges with respect to a given edge and so on. For the reason that the topological indices inherent applications in several disciplines including Chemical graph theory, Mathematical Chemistry, Combinatorics, Molecular Topology etc., a huge literature relevant to not only defining but also the computation and inter-relation of various different topological indices can be found in the last two decades (see [1, 4, 6, 11, 13, 15, 19]). A list of topological indices of a graph may include Wiener index, Szeged index, Randi c′ index, Padmakar-Ivan index, Zagreb index, Gutman index, Sombor index and many others. In addition, Afzal et al. [1] have used M-polynomial to derive closed formulas to compute several recently introduced topological indices for H-Naphtalenic nanotube. Furthermore, Gaur and Garg [9] associated a new degree known as deficiency degree to the vertices of a graph and derived five new topological indices based on deficiency degree of verticies for identified graphs $C(d,k)$ and $K(m,n)$. As an application point of view, topological indices have the ability of identifying the physical properties to analyze the physico-chemical features of chemical compounds that can be applied in biological activities such as drug delivery and pharmaceutical engineering [8, 17].

The Wiener index of a simple connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a numerical measure given by

$$\mathcal{W}(\mathcal{G}) = \sum_{u,v \in \mathcal{V}} \mathfrak{d}(u, v).$$

Equivalently,

$$\mathcal{W}(\mathcal{G}) = \frac{1}{2} \sum_{v \in \mathcal{V}} \mathfrak{d}(v),$$

where $\mathfrak{d}(u, v)$ denotes distance between $u, v \in \mathcal{V}$ and $\mathfrak{d}(v)$ represents the sum of distances of a vertex $v \in \mathcal{V}$ from each vertex in $\mathcal{V}$. The Wiener index was initially coined in 1947 by Wiener [21], while studying the effect of pure structural variation upon the boiling point of the chemical compound like paraffins, as in isomers the variations in physical properties are only due to the change in structural interrelationships or arrangements. In 1971, Hosoya [14] related it with distances in some chemical graphs. On the other hand, Szeged index is given by

$$\mathcal{S}z(\mathcal{G}) = \sum_{e=uv \in \mathcal{E}} n_u(e|\mathcal{G}) n_v(e|\mathcal{G}),$$

where $n_u(e|\mathcal{G}) = |\{w \in \mathcal{V}: \mathfrak{d}(w, u) < \mathfrak{d}(w, v)\}|$ and $n_v(e|\mathcal{G}) = |\{w \in \mathcal{V}: \mathfrak{d}(w, u) > \mathfrak{d}(w, v)\}|$ i.e., the count of members in $\mathcal{V}$ which are nearer to u than v and vice versa respectively. This topological index of a graph in some sense measures the distribution of the vertices in the graph with respect to the edges as it takes into account the number of vertices nearer or farther from the one or other ends of the edges. The Szeged index of F-sum of path graphs $P_{2n} +_F P_{n+1}$ (four sum operations of graphs on $P_{2n} \times P_{n+1}$) has been obtained and compared by Cancan et al. in [4]. For a deeper study regarding the Szeged index and its applications one can refer to [5, 12, 17, 18, 22].

The motivation behind the present study is the fact that Johnson graphs have applications in Chemistry, Chemical graph theory, Chemo-informatics etc. in studying the molecular structure of complex chemical compounds. The regularity of these graphs can be used to explore the connectivity of different atoms and functional groups in molecular complexes. This class of graphs can model a certain type of molecular structures as well as reactions involving specific types of combinatorial interactions between elements and functional groups. These models can be further used to know how different chemical entities or reaction steps are linked together, which helps in understanding reaction pathways and mechanisms.

In this article, we find topological indices of Johnson graphs and their complements whenever the graph is connected. To achieve this aim, we first resolve the problem concerning the connectivity of the complement of Johnson graph in Section 3. Finally, using the results regarding the transitivity property of Johnson graphs and their complement discussed in Section 2, the Wiener index of both the aforesaid graphs and the Szeged index of Johnson graphs are calculated. Additionally, in order to obtain the Szeged index of the complement of Johnson graphs as it may not be an edge transitive graph, a connection between the topology of the vertices of the graphs and their complement is also established.

2. Preliminaries

We begin with recalling next three results from [5] which can be applied for computing topological indices like Wiener index and Szeged index, whenever $\mathcal{A}ut(\mathcal{G})$ action is transitive on $\mathcal{V}$ & $\mathcal{E}$ or $\mathcal{E}$ splits into a finite number of orbits.

Theorem 2.1: [5] *If $\mathcal{A}ut(\mathcal{G})$ acts transitively on the vertex set of $\mathcal{G}$, then its Wiener index $\mathcal{G}$ is $\mathcal{W}(\mathcal{G}) = \frac{1}{2}|\mathcal{V}|\mathfrak{d}(v)$, where $\mathfrak{d}(v) = \sum_{u\in\mathcal{V}} \mathfrak{d}(u,v)$ for any vertex $v \in \mathcal{V}$.*

Theorem 2.2: [5] *If $\mathcal{A}ut(\mathcal{G})$ acts transitively on the edge set of $\mathcal{G}$, then $\mathcal{S}z(\mathcal{G}) = |\mathcal{E}|n_u(e|\mathcal{G})n_v(e|\mathcal{G})$.*

Theorem 2.3: [5] *If the action of $\mathcal{A}ut(\mathcal{G})$ on the edge set $\mathcal{E}$ has orbits $\mathcal{E}_t$ with representative e_t, $1 \le t \le k$, where e_t is an edge $u_t v_t \in \mathcal{E}$, then the Szeged index $\mathcal{S}z(\mathcal{G}) = \sum_{t=1}^{k} |\mathcal{E}_t| n_{u_t}(e_t|\mathcal{G}) n_{v_t}(e_t|\mathcal{G})$.*

Throughout this article, Γ denotes the group of automorphisms of both $\mathcal{J}(\aleph, r)$ and its complement. The set $\mathcal{V}$ of vertices coincides of both $\mathcal{J}(\aleph, r)$ and its complement, whereas $\mathcal{E}$ and $\overline{\mathcal{E}}$ are edge sets for $\mathcal{J}(\aleph, r)$ and its complement respectively.

Next, we study the action of Γ on the sets $\mathcal{V}$, $\mathcal{E}$ and $\overline{\mathcal{E}}$. Since $\mathcal{A}ut(\mathcal{G}) \cong \mathcal{A}ut(\overline{\mathcal{G}})$, therefore using [3, 16] we have

$$\Gamma \cong \mathcal{A}ut(\overline{\mathcal{J}(\aleph, r)}) \cong \mathcal{A}ut(\mathcal{J}(\aleph, r)) \cong \begin{cases} \mathcal{S}_\aleph & \text{when } \aleph \neq 2r \geq 4 \\ \mathcal{S}_\aleph \times \mathbb{Z}_2 & \text{when } \aleph = 2r \geq 4. \end{cases}$$

This leads us to the following result concerning the automorphism group of both Johnson graphs and their complements.

Theorem 2.4: *The automorphism group Γ contains a distance transitive subgroup of Γ isomorphic to the symmetric group $\mathcal{S}_\aleph$.*

Here, we recall the next result from [10] regarding the symmetry of $\mathcal{J}(\aleph, r)$.

Lemma 2.5: *The Johnson graph is vertex as well as edge transitive.*

Moreover, from [10] taking into consideration the invariance of vertex transitivity under taking complements, we collect the succeeding lemma.

Lemma 2.6: *The complement of the Johnson graph is vertex transitive.*

Next, we study the edge transitivity of the complement of the Johnson graphs. Since, the complement of $\mathcal{J}(\aleph, 2)$ is a Kneser graph $\mathcal{KG}_{\aleph,2}$ and Lemma 3.3 in [18] infers that $\mathcal{KG}_{\aleph,2}$ is edge transitive therefore, we deduce that $\overline{\mathcal{J}(\aleph, 2)}$ is edge transitive. Moreover, the graphs $\overline{\mathcal{J}(\aleph, 1)} \cong \overline{\mathcal{J}(\aleph, \aleph - 1)}$ are trivially edge transitive graphs.

Theorem 2.7: *The complement of Johnson graph* $\mathcal{J}(\aleph, r)$ *is edge transitive iff* $r = 1,2$ *or* $\aleph - 1$.

Proof: It follows in view of the fact that in the remaining possible cases i.e., when $r \neq 1,2, \aleph - 1$, Γ splits $\overline{\mathcal{E}}$ into $r - 1$ orbits, namely; $\mathcal{E}_t = \{(u, v): u, v \in \mathcal{V}, |u \cap v| = t\}$ for $t = 0,1,\ldots, r - 2$ corresponding to the reason that vertices constituting an edge $e = uv \in \overline{\mathcal{E}}$ can have $0,1,2,\ldots$, or $r - 2$ elements in their intersection.

The aforesaid orbits have been closely studied in Subsection 4.2 in the present article. Furthermore, an example to illustrate the above theorem is presented below. Also, the non-edge transitivity of $\overline{\mathcal{J}(6, 3)}$ has been verified using Sagemath [7].

Example 2.8: There are two orbits under the action of $\mathcal{A}ut(\overline{\mathcal{J}(6, 3)})$ on the set of edges in $\overline{\mathcal{J}(6,3)}$ namely; $\mathcal{E}_0$ and $\mathcal{E}_1$ having their orders 10 and 90, along with the edges $\{123\} \leftrightarrow \{456\}$ and $\{123\} \leftrightarrow \{345\}$ as representatives, respectively. In this case, the number of orbits exceeds 1, resulting in the non-edge transitivity of $\overline{\mathcal{J}(6,3)}$.

3. The Connectivity of the Complement of Johnson Graphs

In this section, the connectivity of the complement of Johnson's graph $\mathcal{J}(\aleph, r)$ is studied. First, we explore some easy cases in the next two propositions. Recall that $\mathcal{J}(\aleph, 1) \cong \mathcal{K}_\aleph$, while $\overline{\mathcal{K}_\aleph}$ is the empty graph having $\aleph$ vertices without an edge. So, we have

Proposition 3.1: The graph $\overline{\mathcal{J}(\aleph, 1)}$ is disconnected and has $\aleph$ components.

$\mathcal{J}(4, 2)$ $\overline{\mathcal{J}(4, 2)}$

Figure 1

Graphs: $\mathcal{J}(4, 2)$ and its complement

In addition, $\mathcal{J}(2,2)$ is the trivial graph which is an edgeless, self complementary graph with only one vertex and so its complement $\overline{\mathcal{J}(2,2)}$ is connected, whereas the graph $\mathcal{J}(3,2) \cong \mathcal{J}(3,1)$ is $\mathcal{K}_3$, whose complement is the empty graph with 3 vertices which is a disconnected graph. Now, let us view $\mathcal{J}(4,2)$ and its complement in Figure 1.

Clearly, the upcoming proposition follows from the above discussion.

Proposition 3.2: *The graph* $\overline{\mathcal{J}(\aleph, 2)}$ *is disconnected with three components, for* $\aleph = 3,4$.

Before proceeding further, observe that $\mathcal{J}(\aleph, r) \cong \mathcal{J}(\aleph, \aleph - r)$ (see Lemma 1.6.1 [10]), therefore in order to remove the repetitions one may consider $\aleph \geq 2r$. Thus, now onwards we fix $\aleph \geq 2r$ in $\mathcal{J}(\aleph, r)$.

Next, we recall a characterization in

Lemma 3.3: [2] *Let* $\mathcal{G}$ *be connected. Then* $\overline{\mathcal{G}}$ *is connected iff none of them has a spanning complete bipartite subgraph.*

With the discussion so far, we shall attempt to prove that the Johnson's graphs discussed in Proposition 3.1 and 3.2 will be the only graphs with disconnected complements in the upcoming results. Specifically; the only Johnson graphs with disconnected complements are $\mathcal{J}(\aleph, 1) \cong \mathcal{J}(\aleph, \aleph - 1)$ and $\mathcal{J}(4,2)$. Before proceeding, let us discuss an example given below.

Example 3.4: This example describes the connectedness of the complement of two graphs namely; $\mathcal{J}(5,2)$ and $\mathcal{J}(6,2)$, in view of Figure 2 & 3 as follows:

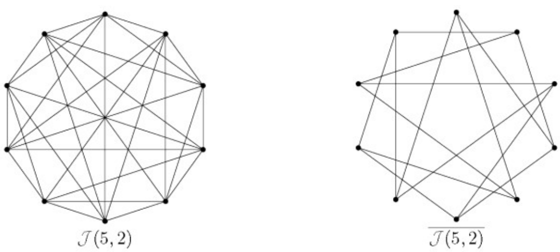


Figure 2

Graphs: $\mathcal{J}(5,2)$ and its complement

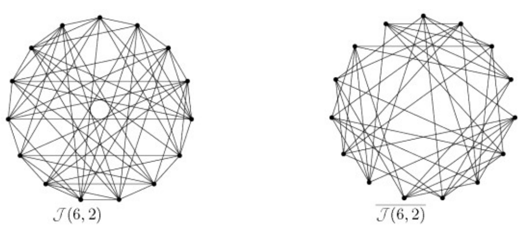


Figure 3

Graphs: $\mathcal{J}(6,2)$ and its complement

Theorem 3.5: *If the Johnson's graph* $\mathcal{J}(\aleph, r)$ *has a spanning complete bipartite subgraph, then*

(i) $\quad {}^{\aleph}C_{r} \leq 2r(\aleph - r)$.

(ii) $\mathcal{J}(\aleph, r)$ is either $\mathcal{K}_{\aleph}$ or $\mathcal{J}(4,2)$.

Proof: (i) Assume that the graph $\mathcal{J}(\aleph, r)$ contains a spanning complete bipartite subgraph $\mathcal{G}$. This infers that the vertex set $\mathcal{V}$ of $\mathcal{G}$ has $|\mathcal{V}| = {}^{\aleph}C_{r}$ vertices, along with this $\mathcal{V}$ has a partitioning $\mathcal{V} = \mathcal{A} \sqcup \mathcal{B}$ with each vertex of the set $\mathcal{A}$ is connected by an edge to every vertex in the set $\mathcal{B}$. Thus, we conclude that

$$ {}^{\aleph}C_{r} = |\mathcal{V}| = |\mathcal{A}| + |\mathcal{B}| \tag{1} $$

and for every vertex $a \in \mathcal{A}$, $b \in \mathcal{B}$,

$$ deg(a) = |\mathcal{B}| \text{ and } deg(b) = |\mathcal{A}|. \tag{2} $$

Further, according to the definition, a spanning subgraph is a graph obtained by removing either none or some of the edges from the original graph, we conclude that $deg(v)$ in $\mathcal{G}$ is less than or equal to the degree of v when regarded as a member of the vertex set of $\mathcal{J}(\aleph, r)$. Also, as $\mathcal{J}(\aleph, r)$ is $r(\aleph - r)$ -regular, therefore for every $v \in \mathcal{V}$, $deg(v) \leq r(\aleph - r)$. In fact, $deg(a) \leq r(\aleph - r)$ $\ \forall a \in \mathcal{A}$ and $deg(b) \leq r(\aleph - r)$ $\ \forall b \in \mathcal{B}$. Now, using Eqs. (1) and (2), we have ${}^{\aleph}C_{r} \leq 2r(\aleph - r)$.

(ii) Suppose that $\mathcal{J}(\aleph, r)$ with $\aleph \geq 2r$, has a spanning complete bipartite subgraph $\mathcal{G} = \mathcal{K}_{s,t}$. Notice that $\mathcal{G}$ is a connected subgraph with as many vertices as in $\mathcal{J}(\aleph, r)$. Therefore, using the definition of the diameter and the fact that a complete bipartite graph is connected, we conclude that

$$ diam(\mathcal{J}(\aleph, r)) \leq diam(\mathcal{G}). \tag{3} $$

But, if $\mathcal{G}$ is the complete bipartite graph $\mathcal{K}_{s,t}$, then $diam(\mathcal{G}) = \begin{cases} 1; & \text{if } s = t = 1 \\ 2; & \text{otherwise} \end{cases}$. On the other hand, the diameter of $\mathcal{J}(\aleph, r)$ is $min\{r, \aleph - r\}$. Thus, Eq. 3 infers that

$$ r \leq \begin{cases} 1; & \text{if } s = t = 1 \\ 2; & \text{otherwise} \end{cases}, \text{ as } \aleph - r \geq 1 $$

Now, if either $r = 1$ or $\aleph = r + 1$, then $\mathcal{J}(\aleph, r)$ is the complete graph $\mathcal{K}_{\aleph}$ with $\aleph$ vertices. On the other hand, we left with the following two possibilities:

Possibility 1: When $r = \aleph - r = 2$. In this case, $\mathcal{J}(\aleph, r)$ is the triangle graph $\mathcal{T}_4 \cong \mathcal{J}(4,2)$ which is already drawn in the above figure. Moreover, $\mathcal{K}_{2,4}$ is a spanning complete bipartite subgraph of $\mathcal{J}(4,2)$ with a bipartition which consists of one part as the vertices in any connected component of $\overline{\mathcal{J}(4,2)}$.

Possibility 2: When $2 = r < \aleph - r$. In this case, $r = 2$ and $\aleph \geq 5$. We claim that $\mathcal{J}(\aleph, 2)$ can not have a spanning complete bipartite subgraph, for $\aleph \geq 5$.

For this, consider Example 3.4 which concludes that $\overline{\mathcal{J}(5,2)}$ and $\overline{\mathcal{J}(6,2)}$, both are connected. Thus, by Lemma 3.3 $\mathcal{J}(5,2)$ and $\mathcal{J}(6,2)$ do not have any spanning complete bipartite subgraph and we are done in view of the given hypothesis.

Otherwise, let $\aleph = 2r + k$, where $k \geq 3$. Then

$$
\begin{aligned}
{}^{\aleph}C_r = {}^{2r+k}C_r = {}^{k+4}C_2 \\
&= \frac{(k+4)\,(k+3)}{2!} \\
&= \frac{(k+4)\,(k+3)}{2} \\
&= \frac{(k+4)\,(k+5)+2}{2!} \\
&> 4(k+2) = 2r(\aleph - r), \text{as } k \geq 3.
\end{aligned}
$$

This is a contradiction to (i). Therefore, $\mathcal{J}(\aleph, 2)$ for $\aleph \geq 5$, can not have a spanning complete bipartite subgraph.

Hence, $\mathcal{J}(\aleph, r)$ has a spanning complete bipartite subgraph, whenever $\mathcal{J}(\aleph, r)$ is either the complete graph $\mathcal{K}_{\aleph}$ or triangle graph with four vertices i.e., $\mathcal{J}(4, 2)$.

As an application of Lemma 3.3 and the conclusion obtained from the second possibility in the above theorem, we have

Theorem 3.6: *For $r = 2$, the graph $\overline{\mathcal{J}(\aleph, r)}$ is connected iff $\aleph \geq 5$.*

The next theorem provides the connectedness of the complement of $\mathcal{J}(\aleph, r)$ for $r > 2$.

Theorem 3.7: *The graph $\overline{\mathcal{J}(\aleph, r)}$ is connected, whenever $\frac{\aleph}{2} \geq r \geq 3$.*

Proof: Let $\mathcal{J}(\aleph, r)$ be the Johnson's graph with $\frac{\aleph}{2} \geq r \geq 3$. Next, we shall show that $\overline{\mathcal{J}(\aleph, r)}$ is not only connected, but also its diameter is 2. For this, it is enough to prove that any two vertices in $\overline{\mathcal{J}(\aleph, r)}$ are either adjacent, or if they are not, then one can find a vertex in $\overline{\mathcal{J}(\aleph, r)}$ adjacent to both those vertices. Now, in $\overline{\mathcal{J}(\aleph, r)}$, if vertices u, v are adjacent, then the distance between these vertices $\mathfrak{d}_{\overline{\mathcal{J}(\aleph,r)}}(u, v) = 1$ and we are done.

In case, if $e = uv \notin \overline{\mathcal{E}}$, then certainly $e = uv \in \mathcal{E}$. Therefore, u, v are two vertices in $\mathcal{J}(\aleph, r)$ i.e., u, v are two subsets of order r of an $\aleph$ ordered set say $\mathcal{X}$ satisfying $|u \cap v| = r - 1$. Next, we want to produce a vertex $x \in \mathcal{J}(\aleph, r)$ such that $xu, xv \notin \mathcal{E}$, equivalently, a subset x of order r of $\mathcal{X}$ satisfying $|x \cap u| < r - 1$ & $|x \cap v| < r - 1$. In fact, the number of subsets of order r fulfilling the above property is $[{}^{r-1}C_2\, {}^{\aleph-r}C_2 + {}^{r}C_2\, {}^{\aleph-r-1}C_2]$. Hence, there exist $[{}^{r-1}C_2\, {}^{\aleph-r}C_2 + {}^{r}C_2\, {}^{\aleph-r-1}C_2]$ number of vertices in $\overline{\mathcal{J}(\aleph, r)}$ with $\aleph \geq 2r \geq 3$, connected to both u, v in $\overline{\mathcal{J}(\aleph, r)}$. Most importantly, for $r \geq 3$ the quantities in the above count is meaningful and is at least one i.e., $[{}^{r-1}C_2\,{}^{\aleph-r}C_2 + {}^{r}C_2\,{}^{\aleph-r-1}C_2] \geq 1$. Thus, there indeed exists at least one vertex $x \in \overline{\mathcal{J}(\aleph,r)}$ such that $xu, xv \in \overline{\mathcal{E}}$. Also, in this case $\mathfrak{d}_{\overline{\mathcal{J}(\aleph,r)}}(u,v) = 2$.

Thus, we conclude that any two vertices in $\overline{\mathcal{J}(\aleph,r)}$ are either connected by an edge or if not, then they both are connected to another vertex in it. Hence, the graph $\overline{\mathcal{J}(\aleph,r)}$ is connected for $\frac{\aleph}{2} \geq r \geq 3$.

An immediate consequence of Theorem 1 in [20], for the case $r = 2$ and the above proof is contained in

Theorem 3.8: *The diameter of the complement of Johnson graph* $\overline{\mathcal{J}(\aleph,r)}$, *if it is connected, is* 2.

4. Computation of the Topological Indices

This section deals with computing Wiener and Szeged index of Johnson graphs and the complement of Johnson graphs, when the graph is connected. Throughout this section, graph $\overline{\mathcal{J}(\aleph,r)}$ under consideration is assumed to be connected.

4.1 *The Wiener Index*

We begin with the following proposition providing us with a formula to find the count of vertices in Johnson graph which are at a given distance t, $1 \leq t \leq r$, from a given vertex.

Proposition 4.1: If $u \in \mathcal{J}(\aleph,r)$, then $|\{v \in \mathcal{J}(\aleph,r) : \mathfrak{d}(u,v) = t\}| = {}^{r}C_{r-t}\,{}^{\aleph-r}C_t$.

Proof: Let $u \in \mathcal{J}(\aleph,r)$. Then, it is enough to count vertices in $\mathcal{J}(\aleph,r)$ which are at a distance t from the vertex u. Notice that such a vertex shares $r - t$ elements in common with u. Since, by definition any vertex in $\mathcal{J}(\aleph,r)$ is an r ordered subset, therefore $|\{v \in \mathcal{J}(\aleph,r) : \mathfrak{d}(u,v) = t\}| = {}^{r}C_{r-t}\,{}^{\aleph-r}C_t$.

Here, we recall an important combinatorial identity known as Vandermonde's identity:

$$\sum_{k=0}^{\gamma} {}^{\alpha}C_{k}\,{}^{\beta}C_{\gamma-k} = {}^{\alpha+\beta}C_{\gamma}, \tag{4}$$

it plays a notable role in the present paper. Moreover, we also need another Vandermonde-like identity proved in the upcoming lemma.

Lemma 4.2: *For non negative integers* α, β *and* γ, *the following identity holds:*

$$\sum_{k=0}^{\gamma} k\,{}^{\alpha}C_{k}\,{}^{\beta}C_{\gamma-k} = \alpha\left({}^{\alpha+\beta}C_{\gamma} - {}^{\alpha+\beta-1}C_{\gamma}\right). \tag{5}$$

Proof: From the binomial theorem, we have

$$(1 + x)^{\alpha} = \sum_{i=0}^{\alpha} {}^{\alpha}C_i\, x^i. \tag{6}$$

On differentiating the above equation with respect to x and then multiplied by x to get

$$\alpha x(1+x)^{\alpha-1} = \sum_{i=0}^{\alpha} i\, {}^{\alpha}C_i\, x^i.$$

On rearranging,

$$\alpha[(1+x)^{\alpha} - (1+x)^{\alpha-1}] = \sum_{i=0}^{\alpha} i\, {}^{\alpha}C_i\, x^i. \qquad (7)$$

Now, Eqs. (6) and (7) implies that

$$[\alpha(1+x)^{\alpha} - \alpha(1+x)^{\alpha-1}](1+x)^{\beta} = (\sum_{i=0}^{\alpha} i\, {}^{\alpha}C_i\, x^i)(\sum_{j=0}^{\beta}\, {}^{\beta}C_j\, x^j).$$

$$[\alpha(1+x)^{\alpha+\beta} - \alpha(1+x)^{\alpha+\beta-1}] = (\sum_{i=0}^{\alpha} i\, {}^{\alpha}C_i\, x^i)(\sum_{j=0}^{\beta}\, {}^{\beta}C_j\, x^j).$$

$$\alpha(\sum_{k=0}^{\alpha+\beta}\, {}^{\alpha+\beta}C_k\, x^k) - \alpha(\sum_{k=0}^{\alpha+\beta-1}\, {}^{\alpha+\beta-1}C_k\, x^k) = \sum_{\gamma=0}^{\alpha+\beta} (\sum_{k=0}^{\gamma} k\, {}^{\alpha}C_k\, {}^{\beta}C_{\gamma-k}) x^{\gamma}.$$

On comparing the coefficients of x^{γ}, for $\gamma = 0,1,\ldots,\alpha+\beta-1$, we have

$$\sum_{k=0}^{\gamma} k\, {}^{\alpha}C_k\, {}^{\beta}C_{\gamma-k} = \alpha({}^{\alpha+\beta}C_{\gamma}) - \alpha({}^{\alpha+\beta-1}C_{\gamma}),$$

while, for $\gamma = \alpha+\beta$,

$$\sum_{k=0}^{\gamma} k\, {}^{\alpha}C_k\, {}^{\beta}C_{\gamma-k} = \alpha({}^{\alpha+\beta}C_{\gamma}).$$

Since ${}^{\beta}C_{\gamma}$ is defined to be zero for $r > \beta$, therefore the above two equations can be combined for all values of $\gamma = 0,1,\ldots,\alpha+\beta$, to get the required result as follows:

$$\sum_{k=0}^{\gamma} k\, {}^{\alpha}C_k\, {}^{\beta}C_{\gamma-k} = \alpha({}^{\alpha+\beta}C_{\gamma} - {}^{\alpha+\beta-1}C_{\gamma}).$$

With this combinatorial identity in mind, we proceed to

Proposition 4.3: Let $u \in \mathcal{J}(\aleph, r)$. Then the summation of its distance from all the vertices in $\mathcal{J}(\aleph, r)$, i.e., $\sum_{v\in\mathcal{J}(\aleph,r)} \mathfrak{d}(u,v) = r\, {}^{\aleph-1}C_r$.

Proof: Assume $u \in \mathcal{J}(\aleph, r)$ is a fixed vertex. Since the diameter of $\mathcal{J}(\aleph, r)$ with $\aleph \geq 2r$ is $min\{r, \aleph - r\} = r$, therefore the distance of a vertex from u cannot exceed r. Further, Proposition 4.1 states that the count of vertices at a distance t satisfying $0 \leq t \leq r$ is ${}^{r}C_{r-t}\, {}^{\aleph-r}C_t$. Thus, the required sum in $\mathcal{J}(\aleph, r)$ is

$$\begin{aligned}
\sum_{v\in\mathcal{J}(\aleph,r)} \mathfrak{d}(u,v) &= 0\, {}^{r}C_r\, {}^{\aleph-r}C_0 + 1\, {}^{r}C_{r-1}\, {}^{\aleph-r}C_1 + \ldots + r\, {}^{r}C_0\, {}^{\aleph-r}C_r \\
&= \sum_{k=0}^{r} (r-k)\, {}^{r}C_k\, {}^{\aleph-r}C_{r-k} \\
&= r\sum_{k=0}^{r}\, {}^{r}C_k\, {}^{\aleph-r}C_{r-k} - \sum_{k=0}^{r} k\, {}^{r}C_k\, {}^{\aleph-r}C_{r-k}.
\end{aligned}$$

In view of Eqs. (4) and (5), this becomes

$$\begin{aligned}
&= r\, {}^{\aleph}C_r - r({}^{\aleph}C_r - {}^{\aleph-1}C_r) \\
&= r\, {}^{\aleph-1}C_r.
\end{aligned}$$

By virtue of Theorem 3.8, it is worth to note that in $\overline{\mathcal{J}(\aleph, r)}$, a pair of vertices is either at a distance one or two, because $diam(\overline{\mathcal{J}(\aleph, r)}) = 2$. Also, two vertices in $\overline{\mathcal{J}(\aleph,r)}$ are adjacent iff they are non-adjacent in $\mathcal{J}(\aleph,r)$ and vice versa. Therefore, the count of vertices in $\overline{\mathcal{J}(\aleph,r)}$ which are at a distance 0, 1 or 2 from a given fixed vertex is 1, ${}^{\aleph}C_r - r(\aleph - r) - 1$ or $r(\aleph - r)$ respectively. Thus, we have

Proposition 4.4: If $u \in \overline{\mathcal{J}(\aleph,r)}$, then $\sum_{v \in \overline{\mathcal{J}(\aleph,r)}} \mathfrak{d}(u,v) = {}^{\aleph}C_r + r(\aleph - r) - 1$.

Proof: Clearly, from the above discussion, the sum of the distances of the vertex $u \in \overline{\mathcal{J}(\aleph,r)}$ from all the other vertices in $\overline{\mathcal{J}(\aleph,r)}$ is given by

$$\begin{aligned}\sum_{v \in \overline{\mathcal{J}(\aleph,r)}} \mathfrak{d}(u,v) &= 0(1) + 1({}^{\aleph}C_r - r(\aleph - r) - 1) + 2(r(\aleph - r)) \\ &= {}^{\aleph}C_r + r(\aleph - r) - 1.\end{aligned}$$

Now, we are in a position to calculate Wiener index of both $\mathcal{J}(\aleph,r)$ and their complements. From Lemma 2.5 and 2.6, it follows that both these graphs are vertex transitive. Finally, using Theorem 2.1, the Weiner index is obtained in the next theorem.

Theorem 4.5: *Let* $\aleph > r > 0$.

(i) *The Wiener index of* $\mathcal{J}(\aleph,r)$ *is* $\mathcal{W}(\mathcal{J}(\aleph,r)) = \frac{1}{2} r\, {}^{\aleph}C_r\, {}^{\aleph-1}C_r$.

(ii) *The Wiener index of* $\overline{\mathcal{J}(\aleph,r)}$ *is* $\mathcal{W}(\overline{\mathcal{J}(\aleph,r)}) = \frac{1}{2}\, {}^{\aleph}C_r\, ({}^{\aleph}C_r + r(\aleph - r) - 1)$, *whenever* $\overline{\mathcal{J}(\aleph,r)}$ *is connected.*

4.2 *The Szeged Index*

Suppose that $e = uv \in \mathcal{J}(\aleph,r)$. Let $u \cap v = \{a_1, a_2, \ldots, a_{r-1}\}$, as u, v are adjacent, where $u = \{\alpha, a_1, a_2, \ldots, a_{r-1}\}$, $v = \{\beta, a_1, a_2, \ldots, a_{r-1}\}$. Now, a vertex u_1 in $\mathcal{J}(\aleph,r)$ with $\mathfrak{d}(u_1,u) = 1$ and $\mathfrak{d}(u_1,v) > 1$ is an r-ordered subset of a set $\mathcal{X}$ with $|\mathcal{X}| = \aleph$ satisfying:

(i) u_1 is adjacent to u,

(ii) u_1 has $r - 1$ elements common with the vertex u and

(iii) at most $r - 2$ elements common with the vertex v.

Observe that u_1 must contain the element α, otherwise, since $\mathfrak{d}(u_1,u) = 1$ so $\{a_1, a_2, \ldots, a_{r-1}\} \subset u_1$, but this will lead to $\mathfrak{d}(u_1,v) = 1$, a contradiction to $\mathfrak{d}(u_1,v) > 1$. Thus, the vertex u_1 as a subset of order r of the set $\mathcal{X}$, which along with α contains $r - 2$ elements from the set $u \cap v$ and an element θ_1 from $\mathcal{X} \backslash u \cup v$.

Next, a vertex u_2 in $\mathcal{J}(\aleph,r)$ satisfying $\mathfrak{d}(u_2,u) = 2$ and $\mathfrak{d}(u_2,v) > 2$ is an r-ordered subset of $\mathcal{X}$ satisfying:

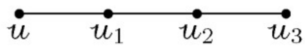
(i) there exists a vertex like u_1 adjacent to both the vertices u and u_2,

(ii) u_2 has exactly $r - 2$ elements common with the vertex u and

(iii) at most $r-3$ elements common with the vertex v.

Suppose that vertex $u_1=\{\alpha,\theta_1,a_1,a_2,\ldots,a_{r-2}\}$. Again, u_2 must contain both the elements from vertex u_1 other than the elements in $u\cap v=\{a_1,a_2,\ldots,a_{r-1}\}$ i.e., the elements α and θ_1, as u_2 is adjacent to u_1, so it must contain exactly $r-1$ elements of u_1 so at most only one element of the two can be left out. Further, if one of these two elements is omitted, then the elements $a_1,a_2,\ldots,a_{r-2}$ are members of u_2, implying $\eth(u_2,v)=2$, which is a contradiction. This concludes that α and θ_1 must be members of u_2. Next, $\eth(u_2,u)=2$ infers that u_2 can contain $r-3$ elements from the set $\{a_1,a_2,\ldots,a_{r-2}\}$. Therefore, we observe that the vertex u_2 is a subset of order r of $\mathcal{X}$ which along with α and θ_1 contains $r-3$ elements from the set $u\cap v$, while it contains an element θ_2 from $\mathcal{X}\setminus(u\cup v\cup\{\theta_1\})$.

Proceeding in this way, it follows that a vertex u_3 in $\mathcal{J}(\aleph,r)$ satisfying $\eth(u_3,u)=3$ and $\eth(u_3,v)>3$ is an r-ordered subset of $\mathcal{X}$ induces the existence of vertices like $u_1=\{\alpha,\theta_1,a_1,a_2,\ldots,a_{r-2}\}$, $u_2=\{\alpha,\theta_1,\theta_2,a_1,a_2,\ldots,a_{r-3}\}$ and a connected path



such that the vertex u_3 as a subset of order r of $\mathcal{X}$, along with α,θ_1,θ_2 contains $r-4$ elements from the set $u\cap v$, whereas another element θ_3 from $\mathcal{X}\setminus(u\cup v\cup\{\theta_1,\theta_2\})$. We summarize this in

Proposition 4.6: If $e=uv$ is a fixed edge in $\mathcal{J}(\aleph,r)$, then for some $0\le t\le r-1$, a vertex w in $\mathcal{J}(\aleph,r)$ satisfying $\eth(w,u)=t$ and $\eth(w,v)>t$, has exactly $r-t-1$ elements from the intersection of u and v.

Lemma 4.7: If $e=uv$ is a fixed edge in $\mathcal{J}(\aleph,r)$, then for some $0\le t\le r-1$, the count of vertices w in $\mathcal{J}(\aleph,r)$ satisfying $\eth(w,u)=t$ and $\eth(w,v)>t$ i.e., $|\{w\in\mathcal{J}(\aleph,r)|\eth(w,u)=t<\eth(w,v)\}| = {}^{r-1}C_{r-t-1}\,{}^{\aleph-r-1}C_t$.

Proof: Assume $uv=e$ is an edge in $\mathcal{J}(\aleph,r)$. Observe that a vertex w in $\mathcal{J}(\aleph,r)$ with $\eth(w,u)=t$ and $\eth(w,v)>t$ is a subset of order r of a set with $\aleph$ elements which shares $r-t$ elements with the vertex u and at most $r-t-1$ elements with v. Let $u=\{\alpha,a_1,a_2,\ldots,a_{r-1}\}$, $v=\{\beta,a_1,a_2,\ldots,a_{r-1}\}$ with $u\cap v=\{a_1,a_2,\ldots,a_{r-1}\}$, as u,v are adjacent. Now, notice from Proposition 4.6 that such a subset representing a vertex like w must contain exactly $r-t-1$ elements from the set $u\cap v=\{a_1,a_2,\ldots,a_{r-1}\}$, as if the number of elements exceeds $r-t-1$, then this contradict the fact that $\eth(w,v)>t$, while if the number of elements are less than $r-t-1$, then $\eth(w,u)$ can not be equal to t. Also, in light of the above discussion the only element in the set $u\setminus u\cap v$ is a member of the vertex w. Thus, number of vertices w in $\mathcal{J}(\aleph,r)$ satisfying $\eth(w,u)=t$ and $\eth(w,v)>t$ is given by ${}^{r-1}C_{r-t-1}\,{}^{\aleph-r-1}C_t$.

The upcoming proposition calculates the value of the parameters $\aleph_u(e|\mathcal{J}(\aleph,r))$ and $\aleph_v(e|\mathcal{J}(\aleph,r))$ required to find the Szeged index of $\mathcal{J}(\aleph,r)$.

Proposition 4.8: Let $e = uv \in \mathcal{J}(\aleph,r)$. Then $\aleph_u(e|\mathcal{J}(\aleph,r)) = \aleph_v(e|\mathcal{J}(\aleph,r)) = |\{w \in \mathcal{J}(\aleph,r) : \mathfrak{d}(w,u) < \mathfrak{d}(w,v)\}| = {}^{\aleph-2}C_{r-1}$.

Proof: Since, the count of w like vertices in $\mathcal{J}(\aleph,r)$ having $\mathfrak{d}(w,u) = t$ and $\mathfrak{d}(w,v) > t$ is ${}^{r-1}C_{r-t-1}\,{}^{\aleph-r-1}C_t$ and $diam(\mathcal{J}(\aleph,r)) = r$, therefore

$$\begin{aligned}
\aleph_u(e|\mathcal{J}(\aleph,r)) &= {}^{r-1}C_{r-1}\,{}^{\aleph-r-1}C_0 + {}^{r-1}C_{r-2}\,{}^{\aleph-r-1}C_1 + \ldots + {}^{r-1}C_0\,{}^{\aleph-r-1}C_{r-1} \\
&= \textstyle\sum_{k=0}^{r-1} {}^{r-1}C_{r-k-1}\,{}^{\aleph-r-1}C_k \\
&= \textstyle\sum_{k=0}^{r-1} {}^{\aleph-r-1}C_k\,{}^{r-1}C_{r-k-1} \\
&= {}^{\aleph-r-1+r-1}C_{k+r-1-k}, \text{ using Eq. (4)} \\
&= {}^{\aleph-2}C_{r-1}.
\end{aligned}$$

Moreover, the equality $\aleph_u(e|\mathcal{J}(\aleph,r)) = \aleph_v(e|\mathcal{J}(\aleph,r))$ holds, as $\mathcal{J}(\aleph,r)$ is symmetric.

Furthermore, for a non edge transitive graph $\mathcal{G}$, not only the value of the parameter $\aleph_u(e|\mathcal{G})$ may differ for different edges, but also $\aleph_u(e|\mathcal{G})$ may not be equal to $\aleph_v(e|\mathcal{G})$, where $e = uv$ is an edge. Therefore, computing Szeged index for $\overline{\mathcal{J}(\aleph,r)}$ is difficult, in general, as according to Theorem 2.7 the graph $\overline{\mathcal{J}(\aleph,r)}$ is not always an edge transitive graph. *To overcome this difficulty, we study the orbits of $\overline{\mathcal{E}}$ under the action of Γ and apply a different approach in which we get the value of $\aleph_u(e|\overline{\mathcal{J}(\aleph,r)})$ orbit-wise by translating it in the context of $\mathcal{J}(\aleph,r)$.* In this direction, the following lemma relates distances in $\mathcal{J}(\aleph,r)$ and its complement.

Lemma 4.9: *If $\mathfrak{d}$ and $\overline{\mathfrak{d}}$ are the distances between a pair of vertices u, v in $\mathcal{J}(\aleph,r)$ and $\overline{\mathcal{J}(\aleph,r)}$ respectively, then*

(i) $\overline{\mathfrak{d}} = 0$ iff $\mathfrak{d} = 0$.

(ii) $\overline{\mathfrak{d}} = 1$ iff $\mathfrak{d} > 1$.

(iii) $\overline{\mathfrak{d}} = 2$ iff $\mathfrak{d} = 1$.

Proof:

(i) Obviously, $\overline{\mathfrak{d}} = 0$ iff $u = v$ in $\overline{\mathcal{J}(\aleph,r)}$ and hence in $\mathcal{J}(\aleph,r)$ iff $\mathfrak{d} = 0$.

(ii) Assume that $\overline{\mathfrak{d}} = 1$. Therefore, there exists an edge e joining u, v in $\overline{\mathcal{J}(\aleph,r)}$ iff they are not adjacent in $\mathcal{J}(\aleph,r)$, i.e., $\mathfrak{d} > 1$.

(iii) Let $\overline{\mathfrak{d}} = 2$. Therefore, u and v are not adjacent in $\overline{\mathcal{J}(\aleph,r)}$. Equivalently, they are adjacent $\mathcal{J}(\aleph,r)$, so $\mathfrak{d} = 1$.

Recall that $e = uv \in \overline{\mathcal{E}}$ iff $e = uv \notin \mathcal{E}$ or equivalently the number of elements in $u \cap v \neq r - 1$. Therefore, we conclude that $e = uv \in \overline{\mathcal{E}}$ iff $|u \cap v|$ takes one of the values $0,1,2,\ldots,r-2$. This motivated us to realize $\overline{\mathcal{E}}$ as $\overline{\mathcal{E}} = \cup_{t=0}^{r-2} \mathcal{E}_t$, where $\mathcal{E}_t = \{(u,v): u, v \in \mathcal{V}, |u \cap v| = t\}$, for $t = 0,1,\ldots,r-2$ are nothing but the orbits under the action of Γ on $\overline{\mathcal{E}}$, the edge set of $\overline{\mathcal{J}(\aleph, r)}$.

Next, we establish the following properties of these orbits.

Proposition 4.10: Let $\mathcal{E}_t = \{(u,v): u, v \in \mathcal{V}, |u \cap v| = t\}$, for some $t = 0,1,\ldots,r-2$, be a subset of the edge set of $\overline{\mathcal{J}(\aleph, r)}$. Then

(i) $\Gamma = \mathcal{A}ut(\overline{\mathcal{J}(\aleph, r)})$ acts transitively on $\mathcal{E}_t$, for every t.

(ii) $|\mathcal{E}_t| = \frac{1}{2}\, {}^{\aleph}C_t\, {}^{\aleph-t}C_{r-t}\, {}^{\aleph-r}C_{r-t}$.

Proof: (i) Assume that $\Gamma = \mathcal{A}ut(\overline{\mathcal{J}(\aleph, r)})$. By Theorem 2.4, there exists a distance transitive subgroup of $\Gamma \cong \mathcal{S}_{\aleph}$. Now, it suffices to prove that for a pair $e_1, e_2 \in \mathcal{E}_t$ there exists a permutation $\sigma \in \mathcal{S}_{\aleph}$ such that ${e_1}^{\sigma} = e_2$. For this, observe that an edge $e \in \mathcal{E}_t$ is an ordered pair of vertices or equivalently an ordered pair of two r ordered subsets of an $\aleph$ ordered set having t elements in their intersection. Next, suppose that $e_1 = u_1 v_1, e_2 = u_2 v_2 \in \mathcal{E}_t$, where u_1, u_2, v_1 and v_2 are vertices in $\mathcal{J}(\aleph, r)$ such that $|u_1 \cap v_1| = t = |u_2 \cap v_2|$. Let $u_1 = \{1,2,\ldots,t,t+1,\ldots,r\}$, $v_1 = \{1,2,\ldots,t,r+1,r+2,\ldots,2r-t\}$, $u_2 = \{1',2',\ldots,t',(t+1)',\ldots,r'\}$ and $v_2 = \{1',2',\ldots,t', (r+1)',(r+2)',\ldots,(2r-t)'\}$, where $i, i' \in \{1,2,..,\aleph\}$, for all $i = 1,2,\ldots,2r-t$. Finally, consider the map $\sigma: \{1,2,\ldots,\aleph\} \to \{1,2,\ldots,\aleph\}$ with $\sigma(i) = i'$ for all $i = 1,2,\ldots,2r-t$, while $\sigma(k) = k$ for all $k \in \{1,2,\ldots,\aleph\} \backslash \{1,2,\ldots,2r-t\}$. Clearly, under the action of $\sigma \in \mathcal{S}_{\aleph} \subset \Gamma$, we have $u_1 \mapsto u_2$ and $v_1 \mapsto v_2$. Thus, we have proved that for any pair of edges $e_1, e_2 \in \mathcal{E}_t$, there exists a permutation $\sigma \in \Gamma$ such that ${e_1}^{\sigma} = e_2$. Hence, $\mathcal{A}ut(\overline{\mathcal{J}(\aleph, r)})$ acts transitively on the set $\mathcal{E}_t$, for all t.

(ii) Recall that for some $t = 0,1,\ldots,r-2$, the set $\mathcal{E}_t = \{(u,v): |u \cap v| = t\}$ contains all those edges or pairs of vertices in $\mathcal{J}(\aleph, r)$ which have t elements in their intersection when the vertices of $\mathcal{J}(\aleph, r)$ are viewed as the subsets of order r. Now, the number of such vertices can be counted as follows: Since the intersection between the pair of vertices is t, therefore the possible ways to select the t elements in the intersection is ${}^{\aleph}C_t$. Further, the remaining $r-t$ elements of one of the two vertices can be selected by choosing $r-t$ elements from the remaining $\aleph - t$ elements in ${}^{\aleph-t}C_{r-t}$ ways, although for the other vertex it can be done by picking $r-t$ elements from the remaining $\aleph - r$ elements in ${}^{\aleph-r}C_{r-t}$ possible ways. In addition, the pairs of vertices in $\mathcal{E}_t$ have been counted twice, as the pairs of two distinct vertices (u,v) or

(v,u) correspond to a single edge in $\mathcal{E}_t$. Thus, the number of edges in $\mathcal{J}(\aleph,r)$, whose corresponding pair of vertices have t elements in their intersection, i.e., $|\mathcal{E}_t|$ is $\frac{1}{2}\ {}^{\aleph}C_t\ {}^{\aleph-t}C_{r-t}\ {}^{\aleph-r}C_{r-t}$.

The next result establishes the equality between $n_u(e|\overline{\mathcal{J}(\aleph,r)})$ and $n_v(e|\overline{\mathcal{J}(\aleph,r)})$ where $e\in\mathcal{E}_t$ for some t, even though $\overline{\mathcal{J}(\aleph,r)}$ is not edge transitive.

Theorem 4.11: *If* $e=uv\in\mathcal{E}_t$ *be an edge in* $\overline{\mathcal{J}(\aleph,r)}$, *then*

(i) $\quad n_v(e|\overline{\mathcal{J}(\aleph,r)})=\begin{cases}1+\ {}^{r}C_{r-1}\ {}^{\aleph-r}C_1, & \text{if } t<r-2\\ 1+\ {}^{r}C_{r-1}\ {}^{\aleph-r-2}C_1, & \text{if } t=r-2.\end{cases}$

(ii) $\quad n_u(e|\overline{\mathcal{J}(\aleph,r)})=n_v(e|\overline{\mathcal{J}(\aleph,r)})$.

Proof: (i) Let $e=uv\in\mathcal{E}_t$ is an edge in $\overline{\mathcal{J}(\aleph,r)}$. We shall calculate $n_u(e|\overline{\mathcal{J}(\aleph,r)})=|\{w\in\overline{\mathcal{J}(\aleph,r)}:\ \ \bar{\mathfrak{d}}(w,u)<\bar{\mathfrak{d}}(w,v)\}|$ by translating the right hand side of it into the context of the graph $\mathcal{J}(\aleph,r)$ as:

By Theorem 3.8, the diameter, when the graph $\mathcal{J}(\aleph,r)$ has a connected complement is 2, therefore for $u,v\in\overline{\mathcal{J}(\aleph,r)}$, $\mathfrak{d}(u,v)$ can take the values 0,1 or 2. Suppose that in $\mathcal{J}(\aleph,r)$, $\mathfrak{d}_1=\mathfrak{d}(w,u)$ & $\mathfrak{d}_2=\mathfrak{d}(w,u)$, while $\bar{\mathfrak{d}}_1$ & $\bar{\mathfrak{d}}_2$ are the corresponding distances between the respective vertices in $\overline{\mathcal{J}(\aleph,r)}$. So, for an edge $e=uv\in\overline{\mathcal{J}(\aleph,r)}$, we have

$$\begin{aligned}\{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1<\bar{\mathfrak{d}}_2\}= \quad & \{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1=0,\bar{\mathfrak{d}}_2=1\}\\ & \sqcup\{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1=0,\bar{\mathfrak{d}}_2=2\}\\ & \sqcup\{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1=1,\bar{\mathfrak{d}}_2=2\}.\end{aligned}$$

In addition, using Lemma 4.9 the set $\{w\in\overline{\mathcal{J}(\aleph,r)}:\ \bar{\mathfrak{d}}_1<\bar{\mathfrak{d}}_2\}$ can be rewritten in the framework of $\mathcal{J}(\aleph,r)$ as

$$\begin{aligned}\{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1<\bar{\mathfrak{d}}_2\}= \quad & \{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1=0,\mathfrak{d}_2>1\}\\ & \sqcup\{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1=0,\mathfrak{d}_2=1\}\\ & \sqcup\{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1>1,\mathfrak{d}_2=1\},\end{aligned}$$

where $e=uv\in\mathcal{E}_t$ is an edge in $\overline{\mathcal{J}(\aleph,r)}$. This infers that

$$\begin{aligned}n_u(e|\overline{\mathcal{J}(\aleph,r)})= \quad & |\{w\in\overline{\mathcal{J}(\aleph,r)}:\bar{\mathfrak{d}}_1<\bar{\mathfrak{d}}_2\}|\\ = \quad & |\{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1=0,\mathfrak{d}_2>1\}|\\ & +|\{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1=0,\mathfrak{d}_2=1\}|\\ & +|\{w\in\mathcal{J}(\aleph,r):\mathfrak{d}_1>1,\mathfrak{d}_2=1\}|.\end{aligned} \tag{8}$$

Next, we claim that the cardinality of the sets in the first and second addend in the above equation is 1 and 0 respectively. In order to verify this, notice that only the vertex u may or may not be the member of each of these sets, as any vertex w which is a member of any of the two sets must satisfy $\mathfrak{d}_1 = 0$. Additionally, the distance $\mathfrak{d}(u,v) \neq 1$. This implies that in Eq. (8), the set in the first addend is the singleton set $\{u\}$, while the set in the second addend is empty and this completes the verification.

On the other hand, the remaining addend in Eq. (8) counts those adjacent vertices of v in $\mathcal{J}(\aleph, r)$ which are at a distance atleast 2 from the vertex u, provided the vertices u, v as subsets which satisfy $|u \cap v| = t$ i.e., u and v are at a distance $r - t$ from each other. Since $e = uv \in \mathcal{E}_t$ is an edge in $\overline{\mathcal{J}(\aleph, r)}$ and $|u \cap v| = t$, therefore t can take the values $0,1,2,\ldots,r-3, r-2$ and correspondingly, $\mathfrak{d}(u,v)$ can be $r, r-1,\ldots,3,2$ respectively. Now, if $\mathfrak{d}(u,v) \geq 3$, then the vertices adjoining v must be at a distance atleast 2 from u. So, all of them will be the members of the set $\{w \in \mathcal{J}(\aleph, r) : \mathfrak{d}_1 > 1, \mathfrak{d}_2 = 1\}$ and hence $|\{w \in \mathcal{J}(\aleph, r) : \mathfrak{d}_1 > 1, \mathfrak{d}_2 = 1\}| = {}^{r}C_{r-1}\, {}^{\aleph-r}C_1$. Otherwise, if $\mathfrak{d}(u,v) = 2$, then the vertices adjacent to v but not u are the only members of the set $\{w \in \mathcal{J}(\aleph, r) : \mathfrak{d}_1 > 1, \mathfrak{d}_2 = 1\}$. Thus, the number of such vertices is ${}^{r}C_{r-1}\, {}^{\aleph-r-2}C_1$, as $\mathfrak{d}(u,v) = 2$, the count of vertices adjoining u, v is ${}^{r}C_{r-1}\, {}^{2}C_1 = 2r$. Therefore, if $\mathfrak{d}(u,v) = 2$, then $|\{w \in \mathcal{J}(\aleph, r) : \mathfrak{d}_1 > 1, \mathfrak{d}_2 = 1\}| = {}^{r}C_{r-1}\, {}^{\aleph-r-2}C_1$. Hence, using Eq. (8) we conclude that $n_v(e|\overline{\mathcal{J}(\aleph, r)}) = \begin{cases} 1 + {}^{r}C_{r-1}\, {}^{\aleph-r}C_1, & \text{if } t < r-2 \\ 1 + {}^{r}C_{r-1}\, {}^{\aleph-r-2}C_1, & \text{if } t = r-2. \end{cases}$

(ii) Observe that the expression for $n_v(e|\overline{\mathcal{J}(\aleph, r)})$ is given by Eq. (8). However, using the same reasoning as above, the expression for $n_u(e|\overline{\mathcal{J}(\aleph, r)})$ will be similar to $n_v(e|\overline{\mathcal{J}(\aleph, r)})$ which can be obtained from Eq. (8) by interchanging u with v. Hence, the result follows from in view of the observation that the $\overline{\mathcal{J}(\aleph, r)}$ is only vertex transitive, while $\mathcal{J}(\aleph, r)$ is both vertex and edge transitive.

Finally, in view of Theorem 2.6 and 2.7, we have

Theorem 4.12: *Let* $\aleph > r$ *be positive integers.*

(i) The Szeged index of $\mathcal{J}(\aleph, r)$ is $\mathcal{S}_z(\mathcal{J}(\aleph, r)) = \frac{1}{2} r(\aleph - r)\, {}^{\aleph}C_r\, {}^{\aleph-2}C_{r-1}\, {}^{\aleph-2}C_{r-1}$.

(ii) The Szeged index of $\overline{\mathcal{J}(\aleph, r)}$, whenever $\overline{\mathcal{J}(\aleph, r)}$ is connected, is

$$\mathcal{S}_z(\overline{\mathcal{J}(\aleph, r)}) = \sum_{t=0}^{r-2} \frac{1}{2}\, {}^{\aleph}C_t\, {}^{\aleph-t}C_{r-t}\, {}^{\aleph-r}C_{r-t} (\phi_t)^2,$$

where $\phi_t = \begin{cases} 1 + {}^{r}C_{r-1}\, {}^{\aleph-r}C_1, & \text{if } t < r-2 \\ 1 + {}^{r}C_{r-1}\, {}^{\aleph-r-2}C_1, & \text{if } t = r-2 \end{cases}.$

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