

The transport of combinatorial structures via braid group actions : A species-theoretic approach

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Abstract

This paper introduces a novel framework for the transport of combinatorial structures using the braid group $\mathfrak{B}_m$, generalizing the classical theory of species of structures developed by Joyal and Bergeron. While traditional species theory relies on symmetric group actions for label permutations, our approach leverages the richer algebraic structure of braid groups to encode topological and combinatorial relabelings of t-structures-objects constructed over finite sets with m -strand elements. We define t -structures as pairs $(\gamma, [U]_m)$, where γ is a construction on $[U]_m$, and establish their transport along braid group bijections $\omega_i : [U]_m \rightarrow [V]_m$ as functorial operations. Key contributions include:

- A rigorous formulation of braid group actions on species of t-structures, with explicit examples (e.g. rooted trees).
- Proofs of isomorphism invariance under transport, demonstrating that t-structures preserve their combinatorial type under relabeling by braid words.
- Connections to geometric braid diagrams and algebraic presentations of $\mathfrak{B}_m$, highlighting the interplay between combinatorics and low-dimensional topology.

Our results extend the scope of enumerative combinatorics by incorporating braid-theoretic symmetries, paving the way for future work on generating series and cycle index techniques for t-structures. This work bridges gaps between species theory, braid groups, and labeled/unlabeled enumeration, offering tools for recursive structural analysis in mathematics and theoretical computer science.

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1. Introduction

In 1981, Joyal [2] pioneered a groundbreaking framework where discrete structures are modeled as functors mapping finite sets and bijections to combinatorial configurations. This approach unifies enumeration and structural analysis under a categorical lens. Central to this theory is the notion of transport of structures: given a species of structures $\mathfrak{F}$, each bijection $\omega: U \rightarrow V$ induces a relabelling operation $\mathfrak{F}[\omega] : \mathfrak{F}[U] \rightarrow \mathfrak{F}[V]$, preserving the underlying combinatorial form. This formalism elegantly reconciles labeled enumeration (via generating series) with unlabeled enumeration (via Pólya theory), while enabling recursive specifications through functional equations on series.

However, classical species theory restricts transport to actions of the symmetric group $\mathfrak{S}_m$, where relabelings are mere permutations of atomic labels. This paper generalizes this paradigm by introducing species of t -structures, whose transport is governed by the richer algebraic and topological structure of the braid group $\mathfrak{B}_m$. Here, a t -structure $s_t = (\gamma_t, [U]_m)$ is a combinatorial object constructed over a finite set $[U]_m$ whose relabelings are encoded by braid group actions.

Motivation and Contributions:

1. **Braid Groups as Combinatorial Tools:** Braid groups $\mathfrak{B}_m$ (Artin [1], Garside [11]) are fundamental in low-dimensional topology and algebra, with generators σ_i representing crossings of strands. We leverage their non-commutativity and isotopy invariance to enrich the transport of structures: a bijection $\omega_t: [U]_m \rightarrow [V]_m$ is now a braid word $\omega_t = \prod_i \sigma_i^{f_i}$, acting on labels via strand crossings (Definition 0.2). This extends symmetric group actions by incorporating topological equivalences.
2. **Functorial Transport of t -Structures:** We define species of t -structures as functors that associate:
 - To each $[U]_m$, a set $\mathfrak{F}[U]_m$ of t -structures (e.g., rooted trees with braided labels; Example 0.4).
 - To each braid bijection ω_t , a transport map $\mathfrak{F}[\omega_t]$ satisfying functoriality (Definition 0.3). Crucially, isomorphic t -structures (related by $\mathfrak{F}[\omega_t]$) preserve their combinatorial type while allowing strand reconfigurations (Theorem 0.1).
3. **Applications and Future Directions:**
 - **Enumeration:** Braid-relabeled structures enable refined counting of isomorphism types, with potential links to the cycle index series of $\mathfrak{B}_m$ (Remark 0.1).
 - **Topological Combinatorics:** Geometric braids provide diagrammatic calculus for t -structures, bridging combinatorics and knot theory.
 - **Recursive Structures:** The transport's functoriality (Proposition 0.1) suggests new functional equations for series of t -structures, generalizing Joyal's implicit species theorem.

While Bergeron, Labelle, and Leroux [5] systematized species theory via $\mathfrak{S}_m$ -actions, and Dehornoy [4], Birman [7] studied braid group algebraically, our work is the first to merge these threads.

Recent studies have emphasized the role of combinatorial methods in the analysis of integer sequences and matrix-based identities, including combinatorial identities derived from powers of square matrices, structural properties of Fibonacci-related objects, and combinatorial interpretations of sequences such as the Perrin sequence [16, 17, 18].

Prior efforts (e.g., [12] on generating series) lacked the functorial braid-group perspective we develop here.

Section 2 formalizes t -structures and braid group presentations. Section 3 proves transport invariance and gives examples (rooted trees, graphs). Section 4 discusses implications for enumeration and topology.

2. Formal Framework for Braided Species

2.1. Notion of t -structures

The Möbius species construction [14] provides inspiration for our definition of t -structures, particularly in handling braid-invariant properties.

Definition 0.1: A t -structure s_t is a pair $(\gamma_t, [U]_m)$, such that:

- $[U]_m$ is a finite set of labeled elements equipped with m strands (e.g., $U = \{1, \dots, n\}$, strands labeled $1, \dots, m$ per element).
- γ_t is a combinatorial construction (e.g., graph, tree, partition) on $[U]_m$ whose edges or relations may encode braid-like interactions between strands.

Example 0.1: (Braided Graph).

Let $[U]_3$ be set with 2 elements, each with 3 strands. A t -structure $s_t = (\gamma_t, [U]_3)$ could be a graph where:

- Vertices: Elements of $[U]_3$ (e.g., (u_1, u_2)).
- Edges: Pairs (u_i, u_j) with braid constraints (e.g., edge (u_1, u_2) requires σ_i -crossing of strands 1 and 2).

While Bergeron, Labelle, and Leroux [5] employs symmetric group actions $\mathfrak{S}_m$ to describe the transport of species on $\mathfrak{F}[U]$, our approach extends this framework by introducing braid group actions $\mathfrak{B}_m$. This generalization motivates the term "species of t -structures".

2.2. Braid Group $\mathfrak{B}_m$ and Combinatorial Relabeling

The braid group $\mathfrak{B}_m$, as introduced in [3, 4], parametrizes isotopy classes of m strands geometric braids. Joyal, in [2], provided its algebraic presentation. Thus, we define $\mathfrak{B}_m$ as the group with the following presentation:

$$\mathfrak{B}_m = \langle \sigma_1, \dots, \sigma_{m-1} \mid \begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{for } |i - j| \geq 2; \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & \text{for } |i - j| = 1. \end{cases} \rangle \quad (1)$$

Thus a braid with m strands is an equivalence class of words in the letters $\sigma_i^{f_i}$, where the map $f_i: [m] \rightarrow \{-1, 1\}$ is defined by:

$$f_i = \begin{cases} 1, & \text{if strand } i \text{ crosses under strand } i+1; \\ -1, & \text{if strand } i \text{ crosses over strand } i+1. \end{cases}$$

A braid word $\omega_t = \prod_i \sigma_i^{f_i}, (f_i \in \{-1, 1\})$ encodes strand crossing.

Example 0.2: (Braid-Action on Labels).

For $[U]_3 = \{u_1, u_2, u_3\}$, the braid word $\omega_t = \sigma_1 \sigma_2^{-1}$ relabels:

- σ_1 : Swaps $u_1 \leftrightarrow u_2$ with strand 1 under strand 2.
- σ_2^{-1} : Swaps $u_2 \leftrightarrow u_3$ with strand 2 under strand 3.

This labeling convention adheres to Larue's irreflexivity principle [10] for braid word construction.

Notation 0.1: Let $\omega_t: [U]_m \rightarrow [V]_m$ denote the bijection induced by a braid word, where each $u \in [U]_m$ is mapped to $\omega_t(u) \in [V]_m$ via strand reconfiguration.

Definition 0.2: (Elementary Braid Words)

An elementary braid word $\omega_t^{(i)}$ is a formal composition of Artin generators σ_i with crossing directions, constructed as:

$$\omega_t^{(i)} = \prod_{i \in \mathcal{I}} \sigma_i^{f_i}$$

Where

1. $\mathcal{I} \subseteq \{1, \dots, m-1\}$ marks participating strands.
2. Crossing Signs: $f_i \in \{-1, 1\}$ encodes:
 - $f_i = 1$: Strand k passes under strand $k+1$.
 - $f_i = -1$: Strand k passes over strand $k+1$.
3. Local Action: Each $\sigma_i^{f_i}$ represents a single crossing event between adjacent strands.

Classification:

- Positive Braid Word: $f_i = 1, \forall i \in \mathcal{I}$;
- Negative Braid Word: $f_i = -1, \forall i \in \mathcal{I}$.

3. Transport of Species of t -Structures

3.1. Functorial Transport via Braid Group Actions

Proposition 0.1: Let s_t be a t -structure on $[U]_m$. The transport of s_t along ω_t induces a t -structure on $[V]_m$.

Proof 0.1: Consider a construction γ_t on $[U]_m$ (denoted $s_t = (\gamma_t, [U]_m)$) and a braid f_i -word concatenation $\omega_t^{(i)}, i \in [U]_m$. The transport along ω_t maps each

element $u \in [U]_m$ to a braid f_u -word $\omega_t^{(u)} \in [V]_m$, thereby relabeling all elements of $u \in [U]_m$. This process preserves the t -structure but changes the vertex labels.

Remark 0.1: Note that $\text{Card}([U]_m) = \text{Card}([V]_m)$. The construction of the elements of $[V]_m$ along ω_t is non-unique, with each vertex having $2^{m-1}(m-1)!$ possible braid f_i -words.

Example 0.3: Consider the species $\mathfrak{A}$ of rooted trees with braid-decorated vertices. For any finite set $[U]_m$ where each element carries m strands:

- $\mathfrak{A}[U]_m$ denote all rooted tree t -structures $(\gamma_t, [U]_m)$.
- $\mathfrak{G}[U]_m$ represent simple graphs with braid-constrained edges.

That means,

$$\mathfrak{G}[U]_m = \{g_t | g_t = (\gamma_t, [U]_m), \gamma_t \in \mathcal{P}^{[2]}[U]_m\}$$

$\mathcal{P}^{[2]}[U]_m$ contains all possible braid-compatible edge pairs of $[U]_m$. So,

$$\mathfrak{A}[U]_m = \mathfrak{G}^\circ[U]_m.$$

The transport mechanism for rooted tree t -structures under braid group action demonstrates the following categorical behavior:

1. Transport Mapping:

For any t -structure $\alpha_t = (\gamma_t, [U]_m) \in \mathfrak{A}[U]_m$, the braid-induced transport:

$$\mathfrak{A}[\omega_t]: \mathfrak{A}[U]_m \rightarrow \mathfrak{A}[V]_m$$

effects a strand-preserving vertex transformation where:

- Each vertex $u \in [U]_m$ maps to $\gamma_t(u) \in [V]_m$.
- Edge relations $\{u, v\}$ transform to $\{\gamma_t(u), \gamma_t(v)\}$.
- The rooted tree hierarchy remains invariant.

2. Functorial Properties:

The transport operation [6] satisfies:

- Compositional Integrity: $\forall \omega_t: [U]_m \rightarrow [V]_m; \varpi_t: [V]_m \rightarrow [W]_m$:

$$\mathfrak{A}[\omega_t \bullet \varpi_t] = \mathfrak{A}[\omega_t] \bullet \mathfrak{A}[\varpi_t].$$

- Identity Preservation:

$$\mathfrak{A}[Id_{[U]_m}] = Id_{\mathfrak{A}[U]_m}.$$

The functoriality of the transport operation for t -structures $\mathfrak{A}[\omega_t]$ is captured by these two equalities. This property is what the definition of species of t -structures aims to abstract.

Definition 0.3: (Braid-Compatible Species).

A t -structure species $\mathfrak{F}$ consists of:

1. Object assignment: $[U]_m \mapsto \mathfrak{F}[U]_m$ (finite sets of t -structures).
2. Morphism action: $\omega_t \mapsto \mathfrak{F}[\omega_t]$ (transport maps between t -structure sets).

Satisfying braided functoriality:

- Composition:

$$\mathfrak{F}[\omega_t \bullet \varpi_t] = \mathfrak{F}[\omega_t] \bullet \mathfrak{F}[\varpi_t]$$

for composite braids.

- Identity: $\mathfrak{F}$ preserves trivial braid actions.

Where $\bullet$ denotes braid concatenation and function composition.

Remark 0.2: A t -structure s_t belonging to $\mathfrak{F}[U]_m$ represents an instance of the species $\mathfrak{F}$ defined over a finite set $[U]_m$, where each element is equipped with m strands. The mapping $\mathfrak{F}[\omega_t]$ describes how t -structures are systematically relocated according to the braid bijection ω_t . This framework offers flexibility: any contribution rule $\mathfrak{F}$ generating $\mathfrak{F}[U]_m$ and its associated transport maps is admissible, provided it satisfies functorial coherence.

Example 0.4: Let $\alpha_t = (\gamma_t, [U]_m)$ be a rooted tree species with $U = \{1, 2, 3, 4\}$. Under the braid-induced bijection ω_t , each label $u \in [U]_m$ is mapped to a new vertex $\omega_t^{(u)}$ in $[V]_m$, constructing an isomorphic tree $\alpha' = (\omega_t \cdot \gamma_t, [V]_m)$

Formally, α' is the image of α under the transport map ω_t .

Through the lens of set theory, this transformation systematically substitutes every instance of $u \in [U]_m$ within γ_t with its mapped counterpart $\omega_t(u) \in [V]_m$, effectively implementing the braid-induced relabeling prescribed by ω_t while maintaining structural integrity.

3.2. Isomorphism Invariance and Combinatorial Types

Definition 0.4: Let $s_t \in \mathfrak{F}[U]_m$ and $\tau_t \in \mathfrak{F}[V]_m$ be two $\mathfrak{F}$ - t -structures. An isomorphism from s_t to τ_t is a bijection $\omega_t: [U]_m \rightarrow [V]_m$ such that $\tau_t = \mathfrak{F}[\omega_t](s_t)$. When such an isomorphism exists, s_t and τ_t are said to be of the same isomorphism type.

Remark 0.3: Self-isomorphisms of a t -structure s_t are termed automorphisms. The framework's power lies in its flexibility: any construction rule $\mathfrak{F}$ producing both the set $\mathfrak{F}[U]_m$ and their associated transport maps $\mathfrak{F}[\omega_t]$ remains valid provided it satisfies the fundamental functoriality axioms.

The algebraic topology of braid configurations [8] guarantees that isomorphism t -structures maintain equivalent strand linkage patterns.

Theorem 0.1: (Structural Equivalence Principle).

For t -structures $s_t \in \mathfrak{F}[U]_m$ and $\tau_t \in \mathfrak{F}[V]_m$, combinatorial isomorphism occurs precisely when there exists $\omega_t \in \mathfrak{B}_m$ effecting

$$\tau_t = \mathfrak{F}[\omega_t](s_t).$$

Isomorphism t -structures share the same combinatorial type, but may differ in strand configurations.

Proof: Consider isomorphic $\mathfrak{F}$ - t -structures $s_t \in \mathfrak{F}[U]_m$ and $\tau_t \in \mathfrak{F}[V]_m$. The transport operation from s_t to τ_t consists of replacing the labels of s_t to yield the labels of τ_t on $[V]_m$. Similarly, the transport from t -structure τ_t to t -structure s_t cause the labels of t -structure τ_t to be replaced to obtain those of t -structure s_t . This reciprocal transport $\tau_t = \mathfrak{F}[\omega_t](s_t)$ and $s_t = \mathfrak{F}[\omega_t](\tau_t)$, from t -structure s_t to t -structure τ_t is only possible when the two t -structures have the same labels. That means they cannot be distinguished, they have the same type.

Example 0.5: (Rooted Trees).

Let $s_t, \tau_t \in \mathfrak{a}[U]_3$ be rooted trees with braided labels:

- s_t : Edges $\{u_1, u_2\}, \{u_2, u_3\}$ with σ_1 -crossing.
- τ_t : Same edges but with $\sigma_2^{-1}\sigma_1$ -crossing.

Then $\tau_t = \mathfrak{a}[\omega_t](s_t)$ for $\omega_t = \sigma_2^{-1}$.

3.3. Applications to Enumeration

Corollary 0.1: (Counting Principle).

The enumeration of braid-equivariant t -structures is governed by:

$$G_{\mathfrak{F}}(x) = \sum_{n \geq 0} |\mathfrak{F}[n]_m| \frac{x^n}{n!}$$

where $|\mathfrak{F}[n]_m|$ counts t -structures on n vertices, weighted by braid-equivalence.

4. Applications and Future Directions

The theory of species of t -structures opens new avenues in enumerative combinatorics, topological graph theory, and beyond. This section explores two key directions: refined enumeration via braid-group actions, diagrammatic calculus for t -structures.

4.1. Enumeration of t -Structures via Braid-Group Symmetries

Classical species theory enumerates labeled and unlabeled structures by exploiting symmetric group actions (via generating series and cycle index series [2, 5]). For t -structures, the braid group $\mathfrak{B}_m$ replaces $\mathfrak{S}_m$, requiring new tools:

Definition 0.5: (Braid-Group Cycle Index).

The symmetry series for $\mathfrak{F}$ under $\mathfrak{B}_m$ actions is:

$$Z_{\mathfrak{B}_m}(\mathfrak{F}) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\omega_t \in \mathfrak{B}_m} \text{Fix}(\mathfrak{F}[\omega_t]) \cdot x_t^\omega,$$

where $\text{Fix}(\mathfrak{F}[\omega_t])$ counts t -structures fixed under the braid relabeling ω_t , and x_t^ω encodes the braid's Artin generators.

Building on the generalized series framework developed by Méndez and Yang [13], our cycle index techniques for $\mathfrak{B}_m$ extend these enumerative methods to braid-equivariant structures.

Example 0.6: (Rooted Trees with Braided Leaves).

For the species $\mathcal{T}_t$ of rooted trees whose leaves carry m -strand braids, the cycle index $Z_{\mathfrak{B}_2}(\mathcal{T}_t)$ enumerates trees up to braid isotopy. For $m = 2$, contributions arise from braid words $\omega_t = \sigma_1^k$ (twists), with $\text{Fix}(\mathcal{T}_t[\sigma_1^k])$ counting trees invariant under k half-twists.

4.2. Topological Combinatorics of t -Structures

The geometric nature of braids enables a diagrammatic calculus for t -structures:

Proposition 0.2: (Isotopy Invariance).

Two t -structures s_t, s'_t are isomorphic iff their braid-labeled diagrams are ambient isotopic in $\mathbb{R}^3$.

Following Fromentin's normalization procedure [6], we ensure topological consistency when applying braid relations to t -structure diagrams.

Proof 0.3: Functoriality of $\mathfrak{F}$ ensures that transport along a braid ω_t preserves combinatorial type, while Artin's braid relations [1] guarantee isotopy invariance.

Example 0.7: (Braided Graphs).

Let $\mathfrak{G}_t$ be the species of graphs whose vertices are labeled by braids. A planar embedding of $\mathfrak{G}_t$ defines a link diagram, with crossings induced by braid generators. The Jones-Kauffman polynomial of this link may distinguish non-isomorphic t -graphs.

Application: (Topological Quantum Computation).

t -structures with $\mathfrak{B}_3$ -actions model non-abelian anyons in topological quantum fields [12]. Enumeration of t -structures could quantify degeneracies in anyonic fusion spaces.

5. Conclusion

As observed in the relationship between Burnside rings and species [15], our work reveals deeper connections between braid group representations and combinatorial enumeration. Our framework extends Cartier's program [9] for applying braid theory to algebraic combinatorics through species-theoretic methods. By defining t -structures as functors from the category of braid-labeled finite sets to combinatorial structures, we unified algebraic, topological, and enumerative perspectives in a single formalism. Our key contributions include: Braid-Group Transport, Enumeration and Symmetry, Topological Combinatorics. This work lays the foundation for several interdisciplinary directions: Physics, Category Theory, Algorithmics.

In summary, species of t -structures enrich classical combinatorics with the geometric flexibility of braids, offering new tools to explore the interplay between discrete mathematics, topology, and physics. We anticipate this framework will inspire further cross-pollination between these fields.

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