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The neighborhood version of the Harmonic index of trees

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Abstract

In a graph G, the neighborhood version of the harmonic index is calculated by summing the values $\frac{2}{\Lambda_G(x)+\Lambda_G(y)}$ for all pairs of adjacent vertices x and y. Here, $\Lambda_G(x)$ represents the total degree of all vertices that are neighbors of vertex x. In this study, we present a lower bound on this kind of index in trees and determine the extremal trees achieve our bound.

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1. Introduction

Topological indices are parameters in graphs calculated from molecular structures that reflect the physical and chemical characteristics of the molecules. In recent years, there has been considerable interest from researchers in topological indices because of their relevance to numerous fields, including chemistry, physics, and electrical engineering. The first and second Zagreb indices rank among the oldest and most frequently utilized topological indices, which underscores their significance and the interest they have garnered from numerous researchers [12, 13]. Comprehensive discussions on their mathematical foundations and chemical applications are available in the literature [2, 11, 28]. Recently, several innovative variations of the Zagreb indices have been proposed, such as Zagreb coindices [10, 18, 30], reformulated Zagreb indices [14, 19], multiplicative Zagreb indices [3, 31, 32], Lanzhou index [6, 33], entire Zagreb indices [1, 16], connection degree index [7], harmonic index [15], etc.

Let G as graph withvertex set $V(G)$ and edge set $E(G)$ which is simple and connected. Suppose $\grave{a}$ be a vertex of G, the open neighborhood of $\grave{a}$ is the set $N_G(\grave{a}) = N(\grave{a}) = \{\grave{b} \in V(G) | \grave{a}\grave{b} \in E(G)\}$. $|N_G(\grave{a})| = \deg_G(\grave{a})$ is degree of a vertex $\grave{a}$ in G. We use the notations $\rho(G) = \rho$, $\Psi(G) = \Psi$ and $d(\grave{a}, \grave{b})$ for the maximum degree, minimum degree and the distance between $\grave{a}, \grave{b} \in V(G)$, respectively.

The Harmonic Index is a graph invariant that is used in the graph theory to quantify the complexity of a graph G, It is defined as [15]

$$H(G) = \sum_{\grave{a}\grave{b} \in E(G)} \frac{2}{\deg_G(\grave{a}) + \deg_G(\grave{b})}.$$

The applications of harmonic index can be seen in various fields of science such as chemistry, computer science, and social network analysis. In chemistry, the harmonic index has been used in the study of molecular graphs. Molecular graphs are used to represent the structure of molecules, with atoms as vertices and chemical bonds as edges. By applying graph theoretical concepts such as the harmonic index, researchers can analyze the structural properties of molecules and predict certain chemical and physical properties. The harmonic index of a molecular graph can provide information about the stability or reactivity of a molecule, as well as its molecular energy. The harmonic index has been used in quantitative structure-activity relationship (QSAR) studies, where it helps in predicting the biological activity or other properties of chemical compounds based on their molecular structure. Overall, the harmonic index serves as a powerful tool in both graph theory and chemistry, allowing researchers to analyze the structural complexity of graphs and molecules and derive valuable insights from them.

The study of harmonic index in graphs has been an active area of research with applications in determining the connectivity and properties of networks. Several research papers have explored different aspects of the harmonic index in graphs, providing insights into its behavior and significance [8, 9, 17].

Recently, Mondal et al. introduced some new indices [20, 21] based on neighborhood degree sum of vertices. Here, we consider a varient of these indices named neighborhood version of the harmonic index. The neighborhood version of the harmonic index defined in [27] as:

$$HN(G) = \sum_{\grave{a}\grave{b}\in E(G)} \frac{2}{\Lambda_G(\grave{a})+\Lambda_G(\grave{b})},$$

where $\Lambda_G(\grave{a}) = \sum_{\grave{c}\in N_G(\grave{a})} \deg_G(\grave{c})$. For more information see [4, 5, 22, 23, 24, 25, 26, 29]. In this paper, We give a lower bound on the neighborhood version of the harmonic index in trees and we determine the extremal trees achieve this bound.

2. Major Results

In a tree $\mathcal{T}$, we consider a particular vertex as a root, this tree is called *rooted tree*. Suppose $a \in V(\mathcal{T})$, if $\deg_{\mathcal{T}}(a) = 1$, then a is called a *leaf* and if $\deg_{\mathcal{T}}(a) > 2$, then a is said a *branching vertex*.

A tree which has only one branching vertex ia called a *spider*. The *center* of a spider is it's branching vertex. There is a path from center and any leaf that is called *leg* of a spider. A spider graph that all legs of it are of length one actually is a *star*.

In this section, we first give a lower bound on the neighborhood harmonic index of trees. Then We prove that our bound is sharp for spiders.

Let $\mathcal{T}$ is a tree that is rooted at vertex a and $\deg_{\mathcal{T}}(a) = \Psi$ and $N_{\mathcal{T}}(a) = \{a_1, a_2, \dots, a_\Psi\}$. Suppose that $\Im(n,\Psi)$ for $n, \Psi \in \mathbb{Z}^+$, be the collection of all trees that have n vertex and maximum degree Ψ. At first we approve some Lemmas and Propositions to obtain the major result.

Lemma 1: *Suppose $\mathcal{T} \in \Im(n,\Psi)$ and is a branching vertex of $\mathcal{T}$ that has maximum distance form a. If adjacent to at least two leaves, then there is a tree $\mathcal{T}_1 \in \Im(n,\Psi)$ that $HN(\mathcal{T}_1) > HN(\mathcal{T})$.*

Proof: Assume that $\deg_{\mathcal{T}}(\hbar) = \rho$ and $N_{\mathcal{T}}(\hbar) = \{1,2,\dots,\rho\}$, where $\hbar_\rho$ lies on the path from to x. By our supposition, for $1 \le \kappa \le \rho - 1$, $\deg_{\mathcal{T}}(\hbar_\kappa) = 1$ or $\deg_{\mathcal{T}}(\hbar_\kappa) = 2$. Since adjacent to at least two leaves, we let $\hbar_1$ and $\hbar_2$ be leaves. Suppose in tree $\mathcal{T}$ we remove the edge h_1 and add the edge $\hbar_{1_2}$, we say the new tree has been obtained $\mathcal{T}_1$. It is easy to see that $\Lambda_{\mathcal{T}}(\hbar) \ge 4$. Then

$$\begin{aligned}
\alpha_1 &= \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)}\right) \\
&= \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)-1}\right) < 0,
\end{aligned}$$

$$\begin{aligned}
\alpha_2 &= \sum_{i=3}^{\rho} \sum_{u\in N_{\mathcal{T}}(\hbar_i)-\{\hbar\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(u)+\Lambda_{\mathcal{T}_1}(\hbar_i)}\right) \\
&= \sum_{i=3}^{\rho} \sum_{u\in N_{\mathcal{T}}(\hbar_i)-\{\hbar\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)-1}\right) < 0,
\end{aligned}$$

and

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_1)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar_1)+\Lambda_{\mathcal{T}_1}(\hbar_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_2)}$$

$$= \frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho} - \frac{2}{\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} - \frac{2}{\rho+2} \leq \frac{2}{\rho+4} - \frac{2}{\rho+2} < 0.$$

Hence $HN(\mathcal{T}) - HN(\mathcal{T}_1) = \alpha_1 + \alpha_2 + \alpha_3 < 0$.

Lemma 2: Suppose $\mathcal{T} \in \mathfrak{I}(n, \Psi)$ and $\hbar$ be a branching vertex of $\mathcal{T}$ with maximum distance to a. If adjacent to exactly one leaf, then there is a tree $\mathcal{T}_1 \in \mathfrak{I}(n, \Psi)$ that $HN(\mathcal{T}_1) > HN(\mathcal{T})$.

Proof: Assume that $\deg_{\mathcal{T}}(\hbar) = \rho$ and $N_{\mathcal{T}}(\hbar) = \{\hbar_1, \hbar_2, \dots, \hbar_\rho\}$, where $\hbar_\rho$ lies on the path from $\hbar$ to a. By our supposition, for $1 \leq i \leq \rho - 1$, $\deg_{\mathcal{T}}(\hbar_i) = 1$ or $\deg_{\mathcal{T}}(\hbar_i) = 2$. Since adjacent to exactly one leaf, we let $\hbar_1$ be a leaf and by Lemma 1, $\deg_{\mathcal{T}}(\hbar_i) = 2$ for $2 \leq i \leq \rho - 1$. Let $z_1 z_2 \dots z_l$ be a path in $\mathcal{T}$ with $\hbar_2 = z_1$ and $l \geq 2$. Suppose in tree $\mathcal{T}$ we remove the edge $\hbar_1$ and add the edge $\hbar_{1_2}$, we say the new tree has been obtained $\mathcal{T}_1$. It is easy to see that $\Lambda_{\mathcal{T}}(\hbar) \geq 5$. Also

$$\alpha_1 = \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)} \right)$$
$$= \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)-2} \right) < 0,$$

and

$$\alpha_2 = \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{h\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(u)+\Lambda_{\mathcal{T}_1}(\hbar_i)} \right)$$
$$= \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{h\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)-1} \right) < 0.$$

- If $l = 2$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_1)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)} + \frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar_1)+\Lambda_{\mathcal{T}_1}(z_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(\hbar)}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{2}{\rho+3} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} - \frac{2}{\rho+4} - \frac{2}{5}$$
$$= \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\rho+4} \right) + \left(\frac{2}{\rho+3} - \frac{2}{5} \right)$$
$$\leq \left(\frac{2}{\rho+6} - \frac{2}{\rho+4} \right) + \left(\frac{2}{6} - \frac{2}{5} \right) < 0.$$

Hence $HN(\mathcal{T}) - HN(\mathcal{T}_1) = \alpha_1 + \alpha_2 + \alpha_3 < 0$.

- If $l = 3$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_1)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)} + \frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)} + \frac{2}{\Lambda_{\mathcal{T}}(z_2)+\Lambda_{\mathcal{T}}(z_3)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar_1)+\Lambda_{\mathcal{T}_1}(z_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_3)+\Lambda_{\mathcal{T}_1}(z_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(\hbar)}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} + \frac{2}{\rho+5} + \frac{2}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} - \frac{2}{\rho+5} - \frac{2}{7} - \frac{2}{5}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{7} \leq \frac{2}{10} - \frac{2}{7} < 0.$$

Hence $HN(\mathcal{T}) - HN(\mathcal{T}_1) = \alpha_1 + \alpha_2 + \alpha_3 < 0$.

- Now if $l > 3$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_1)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)} + \frac{2}{\Lambda_{\mathcal{T}}(z_l)+\Lambda_{\mathcal{T}}(z_{l-1})} + \frac{2}{\Lambda_{\mathcal{T}}(z_{l-1})+\Lambda_{\mathcal{T}}(z_{l-2})}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar_1)+\Lambda_{\mathcal{T}_1}(z_l)} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_l)+\Lambda_{\mathcal{T}_1}(z_{l-1})} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_{l-1})+\Lambda_{\mathcal{T}_1}(z_{l-2})} - \frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(\hbar)}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} + \frac{2}{\Lambda_{\mathcal{T}}()+\rho+2} + \frac{2}{7} + \frac{2}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho} - \frac{2}{8} - \frac{2}{7} - \frac{2}{5}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{8} \leq \frac{2}{10} - \frac{2}{8} < 0.$$

Hence $HN(\mathcal{T}) - HN(\mathcal{T}_1) \leq \alpha_1 + \alpha_2 + \alpha_3 < 0$. This complete the proof.

Lemma 3: *Suppose $\mathcal{T} \in \Im(n, \Psi)$ and let $\hbar$ be a branching vertex of $\mathcal{T}$ with maximum distance to a. Therefore there is a tree $\mathcal{T}_1 \in \Im(n, \Psi)$ that $HN(\mathcal{T}_1) > HN(\mathcal{T})$.*

Proof: Assume that $\deg_{\mathcal{T}}(\hbar) = \rho$ and $N_{\mathcal{T}}(\hbar) = \{\hbar_1, \hbar_2, \dots, \hbar_\rho\}$, where $\hbar_\rho$ lies on the path from to a. By Lemmas 1 and 2, $\deg_{\mathcal{T}}(\hbar_i) = 2$ for $1 \leq i \leq \rho - 1$. Let $z_1 z_2 \dots z_l$ and $t_1 t_2 \dots t_s$, $l, s \geq 2$, be two paths in $\mathcal{T}$ with $\hbar_1 = z_1$ and $\hbar_2 = t_1$. Let $\mathcal{T}_1$ be the tree derived from $\mathcal{T} - \{\hbar z_1\}$ by attaching the edge $t_s z_1$. It is easy to see that $\Lambda_{\mathcal{T}}(\hbar) \geq 2\rho$. Also

$$\alpha_1 = \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)}\right)$$
$$= \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)-2}\right) < 0,$$

and

$$\alpha_2 = \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{h\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(u)+\Lambda_{\mathcal{T}_1}(\hbar_i)}\right)$$
$$= \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{h\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)-1}\right) < 0.$$

- If $l = s = 2$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)} + \frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)} + \frac{2}{\Lambda_{\mathcal{T}}(h)+\Lambda_{\mathcal{T}}(t_1)} + \frac{2}{\Lambda_{\mathcal{T}}(t_1)+\Lambda_{\mathcal{T}}(t_2)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)} - - \frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(t_2)+\Lambda_{\mathcal{T}_1}(t_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(\hbar)}$$
$$= \frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{4}{\rho+3} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} - \frac{2}{\rho+5} - \frac{2}{7} - \frac{2}{5}$$
$$= \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\rho+5}\right) + \left(\frac{4}{\rho+3} - \frac{24}{35}\right)$$
$$\leq \left(\frac{2}{\rho+7} - \frac{2}{\rho+5}\right) + \left(\frac{4}{6} - \frac{24}{35}\right) < 0.$$

Hence $HN(\mathcal{T}) - HN(\mathcal{T}_1) = \alpha_1 + \alpha_2 + \alpha_3 < 0$.

- If $l=2$ and $s=3$, then

$$\begin{aligned}
\alpha_3 = &\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)}+\frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(t_1)}\\
&+\frac{2}{\Lambda_{\mathcal{T}}(t_1)+\Lambda_{\mathcal{T}}(t_2)}+\frac{2}{\Lambda_{\mathcal{T}}(t_2)+\Lambda_{\mathcal{T}}(t_3)}\\
&-\frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)}--\frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_3)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_3)+\Lambda_{\mathcal{T}_1}(t_2)}\\
&-\frac{2}{\Lambda_{\mathcal{T}_1}(t_2)+\Lambda_{\mathcal{T}_1}(t_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(\hbar)}\\
=&\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}+\frac{2}{\rho+5}+\frac{2}{\rho+3}+\frac{2}{5}\\
&-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{\rho+5}-\frac{2}{8}-\frac{2}{7}-\frac{2}{5}\\
=&\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+3}-\frac{15}{28}\\
\leq&\frac{2}{3\rho+2}+\frac{2}{\rho+3}-\frac{15}{28}\leq\frac{17}{33}-\frac{15}{28}=-\frac{19}{924}<0.
\end{aligned}$$

Hence $HN(\mathcal{T})-HN(\mathcal{T}_1)=\alpha_1+\alpha_2+\alpha_3<0$.

- If $l=2$ and $s>3$, then

$$\begin{aligned}
\alpha_3 = &\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)}+\frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(t_1)}\\
&+\frac{2}{\Lambda_{\mathcal{T}}(t_s)+\Lambda_{\mathcal{T}}(t_{s-1})}+\frac{2}{\Lambda_{\mathcal{T}}(t_{s-1})+\Lambda_{\mathcal{T}}(t_{s-2})}\\
&-\frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_s)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_s)+\Lambda_{\mathcal{T}_1}(t_{s-1})}\\
&-\frac{2}{\Lambda_{\mathcal{T}_1}(t_{s-1})+\Lambda_{\mathcal{T}_1}(t_{s-2})}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(\hbar)}\\
=&\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}+\frac{2}{\rho+3}+\frac{2}{7}+\frac{2}{5}\\
&-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{\rho+5}-\frac{4}{8}-\frac{2}{7}-\frac{2}{5}.
\end{aligned}$$

If $\rho\geq 4$, then

$$\alpha_3<\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+3}-\frac{1}{2}\leq\frac{3}{7}-\frac{1}{2}=-\frac{1}{14}<0,$$

and if $\rho=3$, then

$$\alpha_3\leq\frac{2}{11}+\frac{2}{10}+\frac{2}{6}-\frac{3}{4}=\frac{118}{165}-\frac{3}{4}=-\frac{23}{660}<0.$$

Hence $HN(\mathcal{T})-HN(\mathcal{T}_1)=\alpha_1+\alpha_2+\alpha_3<0$.

- If $l=s=3$, then

$$\begin{aligned}
\alpha_3 = &\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)}+\frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}+\frac{2}{\Lambda_{\mathcal{T}}(z_2)+\Lambda_{\mathcal{T}}(z_3)}\\
&+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(t_1)}+\frac{2}{\Lambda_{\mathcal{T}}(t_1)+\Lambda_{\mathcal{T}}(t_2)}+\frac{2}{\Lambda_{\mathcal{T}}(t_2)+\Lambda_{\mathcal{T}}(t_3)}\\
&-\frac{2}{\Lambda_{\mathcal{T}_1}(z_3)+\Lambda_{\mathcal{T}_1}(z_2)}-\frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)}--\frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_3)}
\end{aligned}$$

$$-\frac{2}{\Lambda_{\mathcal{T}_1}(t_3)+\Lambda_{\mathcal{T}_1}(t_2)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_2)+\Lambda_{\mathcal{T}_1}(t_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(\hbar)}$$
$$=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{4}{\rho+5}+\frac{4}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{\rho+5}-\frac{4}{8}-\frac{2}{7}-\frac{2}{5}$$
$$=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+5}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{4}{8}-\frac{2}{7}.$$

If $\rho \geq 4$, then

$$\alpha_3<\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+5}+\frac{2}{5}-\frac{2}{7}-\frac{1}{2}\leq\frac{28}{45}-\frac{9}{14}=-\frac{13}{630}<0.$$

Now let $\rho=3$. Then

$$\alpha_3=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+5}+\frac{2}{8}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+2}-\frac{2}{7}-\frac{1}{2}.$$

If $\xi(x)=\frac{4}{x+5}-\frac{2}{x+2}$, then $\xi'(x)=\frac{-2x^2+4x+34}{(x^2+7x+10)^2}$ and $\xi(x)$ is a decreasing function, when $x\geq 6$. Therefore

$$\alpha_3\leq\frac{4}{11}+\frac{2}{5}-\frac{2}{7}-\frac{1}{2}=\frac{42}{55}-\frac{11}{14}=-\frac{17}{770}<0.$$

Hence $HN(\mathcal{T})-HN(\mathcal{T}_1)=\alpha_1+\alpha_2+\alpha_3<0$.

- If $l=3$ and $s>3$, then

$$\alpha_3\leq\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)}+\frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}+\frac{2}{\Lambda_{\mathcal{T}}(z_2)+\Lambda_{\mathcal{T}}(z_3)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(t_1)}$$
$$+\frac{2}{\Lambda_{\mathcal{T}}(t_1)+\Lambda_{\mathcal{T}}(t_2)}+\frac{2}{\Lambda_{\mathcal{T}}(t_s)+\Lambda_{\mathcal{T}}(t_{s-1})}+\frac{2}{\Lambda_{\mathcal{T}}(t_{s-1})+\Lambda_{\mathcal{T}}(t_{s-2})}$$
$$-\frac{2}{\Lambda_{\mathcal{T}_1}(z_3)+\Lambda_{\mathcal{T}_1}(z_2)}-\frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_s)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_s)+\Lambda_{\mathcal{T}_1}(t_{s-1})}$$
$$-\frac{2}{\Lambda_{\mathcal{T}_1}(t_{s-1})+\Lambda_{\mathcal{T}_1}(t_{s-2})}-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(t_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(t_2)}$$
$$=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+5}+\frac{2}{\rho+6}+\frac{2}{7}+\frac{4}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{\rho+5}-\frac{3}{4}-\frac{2}{7}-\frac{2}{5}$$
$$=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+6}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{3}{4}.$$

First let $\rho\geq 4$. Then $\Lambda_{\mathcal{T}}(\hbar)\geq 8$ and

$$\alpha_3\leq\frac{2}{14}+\frac{2}{10}+\frac{2}{5}-\frac{3}{4}=\frac{26}{35}-\frac{3}{4}=-\frac{1}{140}<0.$$

Now let $\rho=3$. Then

$$\alpha_3=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+5}+\frac{2}{9}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+2}-\frac{3}{4}.$$

Since $\xi(x)=\frac{4}{x+5}-\frac{2}{x+2}$ is a decreasing function, when $x\geq 6$, then

$$\alpha_3\leq\frac{4}{11}+\frac{2}{9}+\frac{2}{5}-1=\frac{488}{495}-1<0.$$

Hence $HN(\mathcal{T})-HN(\mathcal{T}_1)=\alpha_1+\alpha_2+\alpha_3<0$.

- Finally if $l,s>3$, then

$$\alpha_3\leq\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(z_1)}+\frac{2}{\Lambda_{\mathcal{T}}(z_1)+\Lambda_{\mathcal{T}}(z_2)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(t_1)}$$

$$+\frac{2}{\Lambda_{\mathcal{T}}(t_1)+\Lambda_{\mathcal{T}}(t_2)}+\frac{2}{\Lambda_{\mathcal{T}}(t_s)+\Lambda_{\mathcal{T}}(t_{s-1})}+\frac{2}{\Lambda_{\mathcal{T}}(t_{s-1})+\Lambda_{\mathcal{T}}(t_{s-2})}$$
$$-\frac{2}{\Lambda_{\mathcal{T}_1}(z_2)+\Lambda_{\mathcal{T}_1}(z_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(z_1)+\Lambda_{\mathcal{T}_1}(t_s)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_s)+\Lambda_{\mathcal{T}_1}(t_{s-1})}$$
$$-\frac{2}{\Lambda_{\mathcal{T}_1}(t_{s-1})+\Lambda_{\mathcal{T}_1}(t_{s-2})}-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(t_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(t_1)+\Lambda_{\mathcal{T}_1}(t_2)}$$
$$=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{4}{\rho+6}+\frac{2}{7}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{\rho+5}-1.$$

If $\rho \geq 4$, then $\Lambda_{\mathcal{T}}(\hbar) \geq 8$. First let $\Lambda_{\mathcal{T}}(\hbar) \geq 9$. Since $\xi(x)=\frac{4}{x+6}-\frac{2}{x+5}$ is a decreasing function, when $x \geq 4$, then

$$\alpha_3 \leq \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{4}{5}+\frac{2}{7}-\frac{2}{9}-1 \leq \frac{2}{15}+\frac{4}{5}+\frac{2}{7}-\frac{2}{9}-1=\frac{128}{105}-\frac{11}{9}=-\frac{1}{315}.$$

If $\Lambda_{\mathcal{T}}(\hbar)=8$, then $\rho=4$, and

$$\alpha_3 \leq \frac{4}{7}+\frac{4}{5}-\frac{2}{11}-\frac{2}{9}-1=\frac{48}{35}-\frac{139}{99}=-\frac{113}{3465}.$$

Now let $\rho=3$. Then

$$\alpha_3=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+5}+\frac{4}{9}+\frac{2}{9}+\frac{2}{5}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+2}-\frac{1}{4}-1.$$

Since $\xi(x)=\frac{4}{x+5}-\frac{2}{x+2}$ is a decreasing function, when $x \geq 6$, then

$$\alpha_3 \leq \frac{4}{11}+\frac{4}{9}+\frac{2}{7}+\frac{2}{5}-\frac{3}{2}=\frac{5176}{3465}-\frac{3}{2}=-\frac{43}{6930}<0.$$

Hence $HN(\mathcal{T})-HN(\mathcal{T}_1)=\alpha_1+\alpha_2+\alpha_3<0$ and this complete the proof.

Theorem A: *Let $\xi(e,d)$ and $g(e,d)$ be functions that are differentiable. If there is a local extreme point (either a minimum or maximum) of $\xi(e,d)$ constrained by the curve defined by $g(e,d)=0$ at the point $P=(a,b)$, and given that $\nabla g_p \neq 0$, there exists a scalar λ satisfying the equation*

$$\nabla\xi_p=\lambda\nabla g_p.$$

Proposition 4: The function $\xi_1(e,d)=\frac{20}{2e+d+2}-\frac{18}{2e+d+1}-\frac{2}{2e+d-1}+\frac{4}{e+5}-\frac{2}{e+6}-\frac{19}{140}$ for every $(e,d) \in A$ that $A=\{(e,d)|3 \leq e \leq 10, 2 \leq d \leq e\}$ is negative.

Proof: Let $\xi_1(e,d)=\frac{20}{2e+d+2}-\frac{18}{2e+d+1}-\frac{2}{2e+d-1}+\frac{4}{e+5}-\frac{2}{e+6}-\frac{19}{140}$ and $g(e)=d-e$. By Theorem A,

$$\xi_{1e}(e,d)=\lambda g_e(e,d) \quad \text{and} \quad \xi_{1d}(e,d)=\lambda g_d(e,d),$$

then

$$\frac{-40}{(2e+d+2)^2}+\frac{36}{(2e+d+1)^2}+\frac{4}{(2e+d-1)^2}-\frac{4}{(e+5)^2}+\frac{2}{(e+6)^2}=-\lambda, \tag{1}$$
$$\frac{-20}{(2e+d+2)^2}+\frac{18}{(2e+d+1)^2}+\frac{2}{(2e+d-1)^2}=\lambda. \tag{2}$$

MATLAB solves the equations (1), (2) and gives complex roots and then $\xi_1(e,d)$ has no real local maximum or minimum point. By the Figure 1, $\xi_1(e,d)$

is a monotonic function, then it has the maximum or minimum values in the corners of A. The corners of A are $(10,10)$, $(10,2)$, $(3,2)$, $(3,3)$ and

$$\xi_1(10,10) = -0.0187, \qquad \xi_1(10,2) = -0.0386,$$
$$\xi_1(3,2) = -0.143, \qquad \xi_1(3,3) = -0.0898.$$

Then $\max \xi_1(e,d)|_A = -0.0187 < 0$.

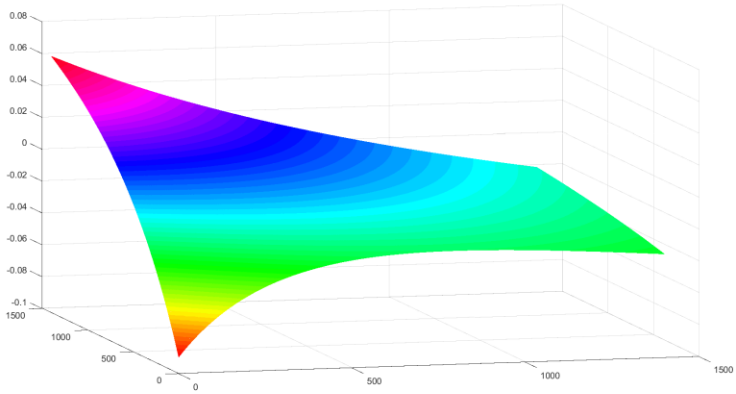

Figure 1
Figure of the function $\xi_1(e,d)$.

Proposition 5: The function $\xi_2(e,d) = \frac{30}{2e+d+2} - \frac{28}{2e+d+1} - \frac{2}{2e+d-1} + \frac{2}{e+5} - \frac{1}{10}$ for every $(e,d) \in B$ that $B = \{(e,d) | 3 \le e \le 15, 2 \le d \le e\}$ is negative.

Proof: Let $\xi_2(e,d) = \frac{30}{2e+d+2} - \frac{28}{2e+d+1} - \frac{2}{2e+d-1} + \frac{2}{e+5} - \frac{1}{10}$ and $g(e) = d - e$. By Theorem A,

$$\xi_{2e}(e,d) = \lambda g_e(e,d) \quad and \quad \xi_{2d}(e,d) = \lambda g_d(e,d),$$

then

$$\frac{-60}{(2e+d+2)^2} + \frac{56}{(2e+d+1)^2} + \frac{4}{(2e+d-1)^2} - \frac{2}{(e+5)^2} = -\lambda, \tag{3}$$
$$\frac{-30}{(2e+d+2)^2} + \frac{28}{(2e+d+1)^2} + \frac{1}{(2e+d-1)^2} = \lambda. \tag{4}$$

MATLAB solves the equations (3), (4) and gives complex roots and then $\xi_2(e,d)$ has no real local maximum or minimum point. By the Figure 2, $\xi_2(e,d)$ is a monotonic function, then it has the maximum or minimum values in the corners of B. The corners of B are $(15,15)$, $(15,2)$, $(3,2)$, $(3,3)$ and

$$\xi_2(15,15) = -0.0159, \qquad \xi_2(15,2) = -0.03,$$
$$\xi_2(3,2) = -0.246, \qquad \xi_2(3,3) = -0.172.$$

Then $\max \xi_2(e,d)|_B = -0.0159 < 0$.

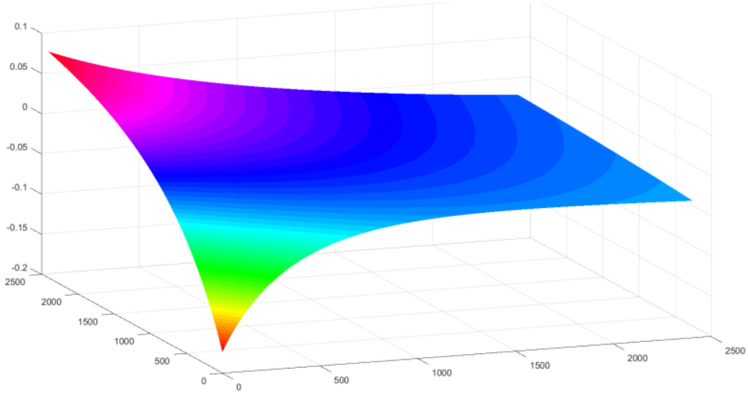

Figure 2
Figure of the function $\xi_2(e,d)$.

Proposition 6: The function $\xi_3(e,d)=\frac{28}{2e+d+2}-\frac{28}{2e+d+1}-\frac{2}{2e+d-1}+\frac{2}{e+6}-\frac{9}{140}$ for every $(e,d)\in C$ that $C=\{(e,d)|3\le e\le 25, 2\le d\le e\}$ is negative.

Proof: Let $\xi_3(e,d)=\frac{28}{2e+d+2}-\frac{28}{2e+d+1}-\frac{2}{2e+d-1}+\frac{2}{e+6}-\frac{9}{140}$ and $g(e)=d-e$. By Theorem A,

$$\xi_{3e}(e,d)=\lambda g_e(e,d) \quad and \quad \xi_{3d}(e,d)=\lambda g_d(e,d),$$

then

$$\frac{-56}{(2e+d+2)^2}+\frac{56}{(2e+d+1)^2}+\frac{4}{(2e+d-1)^2}-\frac{2}{(e+6)^2}=-\lambda, \tag{5}$$

$$\frac{-28}{(2e+d+2)^2}+\frac{28}{(2e+d+1)^2}+\frac{2}{(2e+d-1)^2}=\lambda. \tag{6}$$

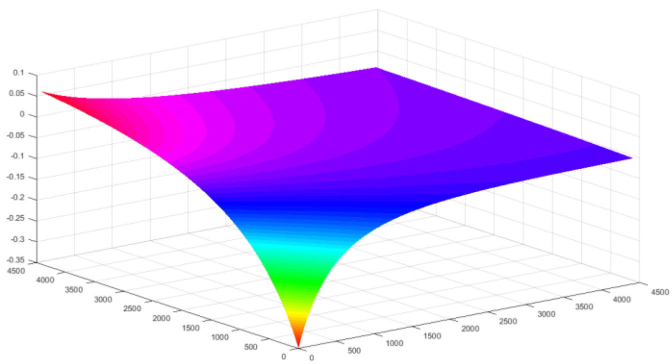

Figure 3
Figure of the function $\xi_3(e,d)$.

MATLAB solves the equations (5), (6) and gives complex roots and then $\xi_3(e,d)$ has no real local maximum or minimum point. By the Figure 3, $\xi_3(e,d)$ is a monotonic function, then it has the maximum or minimum values in the corners of B. The corners of C are $(25,25)$, $(25,2)$, $(3,2)$, $(3,3)$ and

$$\xi_3(25,25) = -0.034, \qquad \xi_3(25,2) = -0.055,$$
$$\xi_3(3,2) = -0.66, \qquad \xi_3(3,3) = -0.528.$$

Then $\max \xi_3(e,d)|_B = -0.034 < 0$.

Lemma 7: *Let $\mathcal{T} \in \Im(n,\Psi)$ be a spider with $\Psi \geq 3$ such that $\mathcal{T}$ has two legs of length more than one. Then there exists a spider $\mathcal{T}_1 \in \Im(n,\Psi)$ with $HN(\mathcal{T}_1) > HN(\mathcal{T})$.*

Proof: Assume that with $deg_{\mathcal{T}}(\hbar) = \rho$ be the center of $\mathcal{T}$ and $\hbar b_1 b_2 \dots b_l$, $\hbar c_1 c_2 \dots c_s$ be two legs of length more than one in $\mathcal{T}$. Let $\mathcal{T}_1$ be the tree deduced from $\mathcal{T} - \{b_1 b_2\}$ by attaching the edge $c_s b_2$.

$$\alpha_1 = \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)} \right)$$
$$= \sum_{i=3}^{\rho} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)-2} \right) < 0,$$

and

$$\alpha_2 = \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{\hbar\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}_1}(u)+\Lambda_{\mathcal{T}_1}(\hbar_i)} \right)$$
$$= \sum_{i=3}^{\rho} \sum_{u \in N_{\mathcal{T}}(\hbar_i)-\{\hbar\}} \left(\frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)} - \frac{2}{\Lambda_{\mathcal{T}}(u)+\Lambda_{\mathcal{T}}(\hbar_i)-1} \right) < 0.$$

- If $l = s = 2$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)} + \frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)} + \frac{2}{\Lambda_{\mathcal{T}}(c_1)+\Lambda_{\mathcal{T}}(c_2)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_1)+\Lambda_{\mathcal{T}_1}(c_2)}$$
$$= \frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{4}{\rho+3} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} - \frac{2}{\rho+5} - \frac{2}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}$$
$$= \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} \right) + \left(\frac{4}{\rho+3} - \frac{2}{5} - \frac{2}{\rho+5} \right).$$

It is clear that $\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} < 0$. If $\xi(e) = \frac{4}{e+3} - \frac{2}{e+5}$, then $\xi(e)$ is decreasing function for $e \geq 4$. Therefore

$$\alpha_3 \leq \frac{4}{7} - \frac{2}{5} - \frac{2}{9} < 0.$$

If $\Psi = 3$ and $\Lambda_{\mathcal{T}}(\hbar) = 5$, then

$$\alpha_3 \leq \frac{2}{3} + \frac{2}{9} - \frac{2}{5} - \frac{1}{4} - \frac{2}{7} < 0,$$

and if $\Psi = 3$ and $\Lambda_{\mathcal{T}}(\hbar) = 6$, then

$$\alpha_3 \leq \frac{2}{3} - \frac{1}{5} - \frac{1}{2} < 0.$$

- If $l = 2$ and $s = 3$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)} + \frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)} + \frac{2}{\Lambda_{\mathcal{T}}(c_1)+\Lambda_{\mathcal{T}}(c_2)}$$
$$+ \frac{2}{\Lambda_{\mathcal{T}}(c_2)+\Lambda_{\mathcal{T}}(c_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(c_1)+\Lambda_{\mathcal{T}_1}(c_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_2)+\Lambda_{\mathcal{T}_1}(c_3)} = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{2}{\rho+3} + \frac{2}{\rho+5}$$
$$+ \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} + \frac{2}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} - \frac{2}{5} - \frac{2}{7} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\rho+6}$$
$$= \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) + \left(\frac{2}{\rho+3} + \frac{2}{\rho+5} - \frac{2}{7} - \frac{2}{\rho+6}\right).$$

For $\rho \geq 4$,

$$\alpha_3 = \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) + \left(\frac{2}{\rho+3} + \frac{2}{\rho+5} - \frac{2}{7} - \frac{2}{\rho+6}\right)$$
$$\leq \frac{2}{14} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} + \frac{2}{7} + \frac{2}{9} - \frac{2}{7} - \frac{2}{\rho+6}$$
$$\leq \frac{16}{63} - \frac{2}{3\rho-1} - \frac{2}{\rho+6}.$$

The function $d = \frac{-2}{3e-1} - \frac{2}{e+6}$ is always increasing, then

$$\alpha_3 \leq \frac{16}{63} - \frac{2}{11} - \frac{2}{10} \simeq -0.12 < 0.$$

For $\rho = 3$,

$$HN(\mathcal{T}) - HN(\mathcal{T}_1) = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_3)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)} + \frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)}$$
$$+ \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)} + \frac{2}{\Lambda_{\mathcal{T}}(c_1)+\Lambda_{\mathcal{T}}(c_2)} + \frac{2}{\Lambda_{\mathcal{T}}(c_2)+\Lambda_{\mathcal{T}}(c_3)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_3)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_1)+\Lambda_{\mathcal{T}_1}(c_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_2)+\Lambda_{\mathcal{T}_1}(c_3)}$$
$$= \frac{4}{11} + \frac{2}{10} + \frac{2}{6} + \frac{2}{8} + \frac{2}{5} - \frac{2}{10} - \frac{2}{8} - \frac{2}{5} - \frac{2}{10} - \frac{2}{9} - \frac{2}{7}$$
$$= -0.01 < 0$$

- If $l = 2$ and $s > 3$, then

$$\alpha_3 = \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)} + \frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)} + \frac{2}{\Lambda_{\mathcal{T}}(c_{l-1})+\Lambda_{\mathcal{T}}(c_l)}$$
$$- \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_l)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_{l-1})+\Lambda_{\mathcal{T}_1}(c_l)}$$
$$= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{2}{\rho+3} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} + \frac{2}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} - \frac{2}{5} - \frac{2}{7}$$
$$- \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}.$$

If $\rho = 3$ and $\Lambda_{\mathcal{T}}(\hbar) = 5$ then

$$\alpha_3 = \frac{2}{10} - \frac{4}{7} + \frac{2}{6} = -\frac{4}{105} < 0.$$

If $\rho = 3$ and $\Lambda_{\mathcal{T}}(\hbar) = 6$ then

$$\alpha_3 = \frac{2}{11} - \frac{1}{4} + \frac{2}{6} - \frac{2}{7} = -\frac{38}{1848} < 0.$$

If $\rho \geq 4$, then

$$\alpha_3 \leq \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) + \left(\frac{2}{\rho+3} - \frac{2}{7}\right) < 0.$$

- If $l = s = 3$, then

$$\begin{aligned}
\alpha_3 = & \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)} + \frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)} + \frac{2}{\Lambda_{\mathcal{T}}(b_2)+\Lambda_{\mathcal{T}}(b_3)} + \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)} \\
& + \frac{2}{\Lambda_{\mathcal{T}}(c_1)+\Lambda_{\mathcal{T}}(c_2)} + \frac{2}{\Lambda_{\mathcal{T}}(c_2)+\Lambda_{\mathcal{T}}(c_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_3)} \\
& - \frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(b_3)} - \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_1)+\Lambda_{\mathcal{T}_1}(c_2)} - \frac{2}{\Lambda_{\mathcal{T}_1}(c_2)+\Lambda_{\mathcal{T}_1}(c_3)} \\
= & \frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} + \frac{4}{\rho+5} + \frac{4}{5} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} - \frac{2}{5} - \frac{2}{7} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\rho+6} - \frac{2}{8} \\
= & \left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) + \left(\frac{4}{\rho+5} - \frac{2}{\rho+6} - \frac{19}{140}\right).
\end{aligned}$$

It is clear that $\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} < 0$. For $e > 0$, the function $d = \frac{4}{e+5} - \frac{2}{e+6}$ is decreasing and then for $\rho \geq 11$, $\frac{4}{\rho+5} - \frac{2}{\rho+6} - \frac{19}{140} < 0$ and therefor $\alpha_3 < 0$.

Now let $3 \leq \rho \leq 10$.

$$\begin{aligned}
\alpha_1 + \alpha_2 &= \sum_{i=3}^{\rho} \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)} - \sum_{i=3}^{\rho} \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)} \\
&= \sum_{i=3}^{9} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right) \\
&= (\rho-2)\left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right) \\
&\leq 8\left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right),
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 \leq & \left(\frac{16}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{16}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right) + \left(\frac{4}{\rho+5} - \frac{2}{\rho+6} - \frac{19}{140}\right) \\
& + \left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) \\
= & \frac{20}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{18}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1} + \frac{4}{\rho+5} - \frac{2}{\rho+6} - \frac{19}{140}.
\end{aligned}$$

Let $\rho = e$ and $\Lambda_{\mathcal{T}}(\hbar) = \rho + d = e + d$, then $3 \leq e \leq 10, 2 \leq d \leq e$, therefor for proving $\alpha_1 + \alpha_2 + \alpha_3 < 0$, we must prove the function $\xi_1(e,d) = \frac{20}{2e+d+2} - \frac{18}{2e+d+1} - \frac{2}{2e+d-1} + \frac{4}{e+5} - \frac{2}{e+6} - \frac{19}{140}$ in the region

$$A = \{(e,d) | 3 \leq e \leq 10, 2 \leq d \leq e\}$$

is negative. According to the Proposition 4, we have $\alpha_1 + \alpha_2 + \alpha_3 < 0$. Therefore $HN(\mathcal{T}) - HN(\mathcal{T}_1) < 0$.

- If $l = 3$ and $s > 3$, then

$$\begin{aligned}\alpha_3 &= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)}+\frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)}\\ &+\frac{2}{\Lambda_{\mathcal{T}}(c_{s-1})+\Lambda_{\mathcal{T}}(c_s)}+\frac{2}{\Lambda_{\mathcal{T}}(c_{s-2})+\Lambda_{\mathcal{T}}(c_{s-1})}-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_s)}\\ &-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(c_{s-1})+\Lambda_{\mathcal{T}_1}(c_s)}-\frac{2}{\Lambda_{\mathcal{T}_1}(c_{s-2})+\Lambda_{\mathcal{T}_1}(c_{s-1})}\\ &=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+5}+\frac{2}{5}+\frac{2}{7}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{2}{7}-\frac{1}{2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}=\\ &\left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right)+\left(\frac{2}{\rho+5}-\frac{1}{10}\right).\end{aligned}$$

For $\rho \geq 16$, $\frac{2}{\rho+5}-\frac{1}{10}<0$ and it is clear that $\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}<0$, then $\alpha_3<0$.

For $3 \leq \rho \leq 15$,

$$\begin{aligned}\alpha_1+\alpha_2 &= \sum_{i=3}^{\rho}\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(\hbar_i)}-\sum_{i=3}^{\rho}\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(\hbar_i)}\\ &=\sum_{i=3}^{9}\left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right)\\ &=(\rho-2)\left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right)\\ &\leq 13\left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(a)+\rho+1}\right),\end{aligned}$$

and

$$\begin{aligned}\alpha_1+\alpha_2+\alpha_3 &\leq \left(\frac{26}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{26}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right)+\left(\frac{2}{\rho+5}-\frac{1}{10}\right)\\ &+\left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right)\\ &=\frac{30}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{28}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}+\frac{2}{\rho+5}-\frac{1}{10}.\end{aligned}$$

Let $\rho=e, \Lambda_{\mathcal{T}}(\hbar)=\rho+d=e+d$ and then $3 \leq e \leq 15, 2 \leq d \leq e$, therefor for proving $\alpha_1+\alpha_2+\alpha_3<0$, we must prove the function $\xi_2(e,d)=\frac{30}{2e+d+2}-\frac{28}{2e+d+1}-\frac{2}{2e+d-1}+\frac{2}{e+5}-\frac{1}{10}$ in region the

$$B=\{(e,d)|3 \leq e \leq 15, 2 \leq d \leq e\}.$$

is negative. According to the Proposition 5, we have $\alpha_1+\alpha_2+\alpha_3<0$. Therefore $HN(\mathcal{T})-HN(\mathcal{T}_1)<0$.

- If $l,s>3$, then

$$\begin{aligned}\alpha_3 &= \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(b_1)}+\frac{2}{\Lambda_{\mathcal{T}}(b_1)+\Lambda_{\mathcal{T}}(b_2)}+\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(c_1)}\\ &+\frac{2}{\Lambda_{\mathcal{T}}(c_{s-1})+\Lambda_{\mathcal{T}}(c_s)}+\frac{2}{\Lambda_{\mathcal{T}}(c_{s-2})+\Lambda_{\mathcal{T}}(c_{s-1})}-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(b_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(b_2)+\Lambda_{\mathcal{T}_1}(c_s)}\\ &-\frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(c_1)}-\frac{2}{\Lambda_{\mathcal{T}_1}(c_{s-1})+\Lambda_{\mathcal{T}_1}(c_s)}-\frac{2}{\Lambda_{\mathcal{T}_1}(c_{s-2})+\Lambda_{\mathcal{T}_1}(c_{s-1})}\\ &=\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}+\frac{2}{\rho+6}+\frac{2}{5}+\frac{2}{7}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}-\frac{3}{4}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}=\\ &\left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2}-\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}-\frac{2}{\Lambda_{\mathcal{T}}(a)+\rho-1}\right)+\left(\frac{2}{\rho+6}-\frac{9}{140}\right).\end{aligned}$$

If $\rho > 25$, then $\alpha_3 < 0$. For $3 \leq \rho \leq 25$

$$\begin{aligned}\alpha_1 + \alpha_2 &= \sum_{i=3}^{\rho} \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\Lambda_{\mathcal{T}}(i)} - \sum_{i=3}^{\rho} \frac{2}{\Lambda_{\mathcal{T}_1}(\hbar)+\Lambda_{\mathcal{T}_1}(i)} \\ &= \sum_{i=3}^{9} \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right) \\ &= \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right)(\rho-2) \\ &\leq \left(\frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right)23,\end{aligned}$$

and

$$\begin{aligned}\alpha_1 + \alpha_2 + \alpha_3 &\leq \left(\frac{46}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{46}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1}\right) + \left(\frac{2}{\rho+6} - \frac{9}{140}\right) \\ &+\left(\frac{4}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho-1}\right) \\ &= \frac{28}{\Lambda_{\mathcal{T}}(\hbar)+\rho+2} - \frac{28}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} - \frac{2}{\Lambda_{\mathcal{T}}(\hbar)+\rho+1} + \frac{2}{\rho+6} - \frac{9}{140}.\end{aligned}$$

Let $\rho = e, \Lambda_{\mathcal{T}}(\hbar) = \rho + d = e + d$ and then $3 \leq e \leq 25, 2 \leq d \leq e$, therefor for proving $\alpha_1 + \alpha_2 + \alpha_3 < 0$, we must prove the function $\xi_3(e,d) = \frac{28}{2e+d+2} - \frac{28}{2e+d+1} - \frac{2}{2e+d-1} + \frac{2}{e+6} - \frac{9}{140}$ in the region

$$C = \{(e,d) | 3 \leq e \leq 25, 2 \leq d \leq e\}.$$

is negative. According to the Proposition 6, we have $\alpha_1 + \alpha_2 + \alpha_3 < 0$. Therefore $HN(\mathcal{T}) - HN(\mathcal{T}_1) < 0$.

Theorem 8: *Let $\mathcal{T} \in \mathfrak{I}(n, \Psi)$ be a tree with $n \geq 4$. Then*

$$HN(\mathcal{T}) \leq \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+6} + \frac{n-\Psi-4}{4} + \frac{24}{35},$$

when $\Psi < n-3$,

$$HN(\mathcal{T}) \leq \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+5} + \frac{2}{5},$$

when $\Psi = n-3$,

$$HN(\mathcal{T}) = \frac{2(\Psi-1)}{2\Psi+1} + \frac{1}{\Psi+1} + \frac{2}{\Psi+3},$$

when $\Psi = n-2$ and $HN(\mathcal{T}) = 1$, when $\Psi = n-1$. The equality is achieved if and only if $\mathcal{T}$ is a spider having at most one leg of length greater or equal to 2.

Proof: If $\Psi = n-1$, then $\mathcal{T}$ is a star, $HN(\mathcal{T}) = 1$. Now if $\Psi = n-2$, then

$$HN(\mathcal{T}) = \frac{2(\Psi-1)}{2\Psi+1} + \frac{1}{\Psi+1} + \frac{2}{\Psi+3},$$

and $\Psi = n-3$, then

$$HN(\mathcal{T}) \leq \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+5} + \frac{2}{5}.$$

If $\Psi = 2$, then $T = P_n$ and

$$\begin{aligned}HN(P_n) &= \frac{4}{5} + \frac{4}{7} + \frac{n-5}{4} \\ &= \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+6} + \frac{n-\Psi-4}{4} + \frac{24}{35}.\end{aligned}$$

Therefore, let $3 \leq \Psi \leq n-4$. Assume that $\mathcal{T}' \in \mathfrak{J}(n,\Psi)$ with $HN(\mathcal{T}') \leq HN(\mathcal{T})$ for every tree $\mathcal{T} \in \mathfrak{J}(n,\Psi)$. Let v be a vertex with $\deg_{\mathcal{T}'}(v) = \Psi$ and root $\mathcal{T}'$ at v. By the choice of $\mathcal{T}'$ and Lemmas 1,2 and 3, we conclude that $\mathcal{T}'$ is a spider with center v. Then by Lemma 7, $\mathcal{T}'$ is a spider with at most one leg of length at least two. It is easy to see that

$$HN(\mathcal{T}') = \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+6} + \frac{n-\Psi-4}{4} + \frac{24}{35},$$

and the proof is complete.

Observation 9: For a graph G and for any $e \notin E(G)$, the inequality $HN(G+e) < HN(G)$, holds.

The Observation 9 and Theorem 8 yield the following:

Corollary 10: *Let G be a graph of order n and maximum degree Ψ. Then*

$$HN(G) \leqslant \begin{cases} \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+6} + \frac{n-\Psi-4}{4} + \frac{24}{35} & \text{if } \Psi < n-3 \\ \frac{2(\Psi-1)}{2\Psi+1} + \frac{2}{2\Psi+3} + \frac{2}{\Psi+5} + \frac{2}{5} & \text{if } \Psi = n-3 \\ \frac{2(\Psi-1)}{2\Psi+1} + \frac{1}{\Psi+1} + \frac{2}{\Psi+3} & \text{if } \Psi = n-2 \\ 1 & \text{if } \Psi = n-1, \end{cases}$$

The equality is achieved if and only if G is a spider having at most one leg of length greater or equal to 2.

Theorem 11: *For every tree $\mathcal{T}$ of order $n \geq 6$*

$$HN(\mathcal{T}) \leq \frac{48}{35} + \frac{n-5}{4}.$$

The equality is achieved if and only if $T = P_n$.

Proof: Let $\xi(e) = \frac{2(e-1)}{2e+1} + \frac{2}{2e+3} + \frac{2}{e+6} + \frac{n-e-4}{4} + \frac{24}{35}$. Then

$$\xi'(e) = \frac{6}{(2e+1)^2} - \frac{4}{(2e+3)^2} - \frac{2}{(e+6)^2} - \frac{1}{4}.$$

If $2 \leq e \leq n-4$, then $\frac{6}{(2e+1)^2} < \frac{1}{4}$ and $\xi(e)$ is a decreasing function. Therefore this function has the maximum value for $e = 2$. Then

$$HN(\mathcal{T}) \leq \frac{48}{35} + \frac{n-5}{4}.$$

The Observation 9 and Theorem 11 yield the following:

Theorem 12: *For every graph G of order $n \geq 6$*

$$\frac{n}{2(n-1)} \leq HN(G) \leq \frac{48}{35} + \frac{n-5}{4}.$$

The equalities are achieved if and only if $G = K_n$ and $G = P_n$.

Author contribution statement

Hamideh Aram: Visualization; Investigation; Validation; Methodology; Writing review and editing. Maryam Atapour: Visualization; Investigation; Validation; Resources; Writing review and editing. Nasrin Dehgardi: Conceptualization; Visualization; Investigation; Methodology; Validation; Writing original draft.

Availability of data and materials

The paper does not include or use any datasets.

Conflicts of interest

The authors declare that there is no competing interest related to this paper.

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