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# Parametric linear programming : The case of parameterized right-hand sides of the constraints and the general parametric linear programming problem

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**Abstract**

In this paper, the parametric linear programming problems are considered, in particular, the case of parameterized right-hand sides of constraints and the general case, with parameterized both the objective function and the right-hand sides of the constraints. Methods for solving such problems are presented, illustrative numerical examples are included, and postoptimal and sensitivity analysis is made. This paper is continuation of paper [Stefanov 14].

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## 1. Introduction

Recall ([14]) the statement of parametric linear programming problems.

Linear programming problem, whose coefficients of the objective function depend linearly on the parameter $t$, consists in solving problem (1)-(3) for each value of the parameter $t \in [\alpha, \beta]$:

$$\max\{L(x) = \sum_{j=1}^{n} (c_j' + c_j''t)x_j\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1, \dots, m \tag{2}$$

$$x_j \geq 0, j = 1, \dots, n, \tag{3}$$

where $c_j', c_j'', a_{ij}$ and $b_i$ are given constants.

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If the right-hand sides of the constraints of a linear program depend linearly on the parameter $t$, then this problem consists in solving problem (4)-(6) for each value of the parameter $t \in [\alpha, \beta]$:

$$\max\{L(x) = \sum_{j=1}^{n} c_j x_j\} \tag{4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i' + b_i'' t, \quad i = 1, \dots, m \tag{5}$$

$$x_j \geq 0, j = 1, \dots, n, \tag{6}$$

where $c_j, a_{ij}, b_i', b_i''$ are given constants.

In the case when both coefficients of the objective function and the right-hand sides of constraints of a linear program depend linearly on the parameter $t$, then this problem consists in solving problem (7)-(9) for each value of the parameter $t \in [\alpha, \beta]$:

$$\max\{L(x) = \sum_{j=1}^{n} (c_j' + c_j'' t) x_j\} \tag{7}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i' + b_i'' t, i = 1, \dots, m \tag{8}$$

$$x_j \geq 0, \; j = 1, \dots, n. \tag{9}$$

Problem (7)-(9) is called the *general parametric linear program*.

Generalization of the three problems (1)-(3), (4)-(6), and (7)-(9), considered above, is the so-called *general problem of the parametric linear optimization*, in which coefficients of the objective function, coefficients of variables $x_j$ in the left-hand sides of constraints, and the right-hand sides of constraints depend linearly on the parameter $t$. Statement of this problem is as follows: for each feasible value of the parameter $t \in [\alpha, \beta]$ solve the problem

$$\max\{L(x) = \sum_{j=1}^{n} (c_j' + c_j'' t) x_j\} \tag{10}$$

subject to

$$\sum_{j=1}^{n} (a_{ij}' + a_{ij}'' t) x_j = b_i' + b_i'' t, \; i = 1, \dots, m \tag{11}$$

$$x_j \geq 0, \; j = 1, \dots, n. \tag{12}$$

Parametric linear programming problems, sensitivity analysis for these problems, and related topics are studied in [1-16], etc.

The paper is organized as follows. In Section 2, the parametric linear programming problem with parameterized right-hand sides of the constraints is considered and the solution of an illustrative example is presented. In Section 3, the general parametric linear programming problem is studied and the solution of an illustrative example is included. In Section 4, an open problem for future research is formulated.

## 2. The parametric linear programming problem with parameterized right-hand sides of the constraints

The algorithm for solving problem (4)-(6) is similar to the algorithm for solving problem (1)-(3) ([14]).

Assign the parameter $t$ some value $t_0$ and solve the linear programming problem, obtained from problem (4)-(6) with $t = t_0$, that is, either find an optimal solution of this problem or establish that this problem is unsolvable. Let in the first case the obtained solution be optimal for an arbitrary value of $t$, $\underline{t} \le t \le \overline{t}$, where

$$\underline{t} = \begin{cases} \max\left(-\frac{q_i}{p_i}\right), & \text{if there is } p_i > 0, \\ -\infty, & \text{if all } p_i \le 0, \end{cases}$$

$$\overline{t} = \begin{cases} \min\left(-\frac{q_i}{p_i}\right), & \text{if there is } p_i < 0, \\ \infty, & \text{if all } p_i \ge 0, \end{cases}$$

and the numbers $p_i$ and $q_i$ are determined from components of the optimal solution and they depend on $t_0$: $x_i^* = q_i + t_0 p_i$.

If for $t = t_0$ problem (4)-(6) is unsolvable, then either the objective function $L(x)$, defined by (4), is unbounded in the feasible set (5)-(6) or the system of equations (5) has no nonnegative solution(s), as the constraints (6) require. In the first case, the considered problem is unsolvable for all $t \in [\alpha, \beta]$, and in the second case, determine all values of parameter $t \in [\alpha, \beta]$, for which the system of equations (5) is inconsistent, and exclude these values of $t$ from further consideration.

After determining the interval for the parameter $t$, in which the problem (4)-(6) has the same optimal solution or in which this problem is unsolvable, choose a new value of the parameter $t$, which does not belong to the interval determined above, and solve the linear programming problem, obtained from problem (4)-(6) for the chosen value of $t$, using the dual simplex method ([15]). Continuing in the same manner, after finite number of steps (iterations) we obtain an optimal solution of problem (4)-(6).

**Algorithm for solving problem (4)-(6).**

1. Assign the parameter $t$ a value, equal to some number $t_0 \in [\alpha, \beta]$, and find an optimal solution $x^*$ of problem (4)-(6) with $t = t_0$ or establish that this linear programming problem is unsolvable. Go to 2.
2. Determine the values of the parameter $t \in [\alpha, \beta]$, for which the optimal solution of problem (4)-(6) is the same or for which the problem is unsolvable. These values of the parameter are excluded from further consideration. Go to 3.
3. Choose a value of the parameter $t$ in the remaining part of interval $[\alpha, \beta]$, and using the dual simplex, we find an optimal solution (if there is any) of the linear program, obtained from (4)-(6) with the chosen value of $t$. Go to 4.
4. Determine the set of all values of the parameter $t$, for which the given problem has the same optimal solution or for which this problem is unsolvable. The calculations continue for all values of the parameter $t \in [\alpha, \beta]$. Go to 5.

5. End.

**Example 1:** For all values of parameter $t \in (-\infty, \infty)$ solve the problem

$$\max\{L(x) = 3x_1 - 2x_2 + 5x_3 - 4x_5\} \tag{13}$$

subject to

$$\begin{array}{rrrrrcl} x_1 & +x_2 & +x_3 & & & = & 12+t \\ 2x_1 & -x_2 & & +x_4 & & = & 8+4t \\ -2x_1 & +2x_2 & & & +x_5 & = & 10-6t \end{array} \tag{14}$$

$$x_j \geq 0, j = 1, \dots, 5. \tag{15}$$

**Solution:** Set, for example, $t = 0$ in the system (14) of equality constraints and solve the obtained canonical linear programming problem; note that the system of equality constraints of this problem is in basic form with respect to variables $x_3$, $x_4$, and $x_5$:

**Table 1**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|---|
| | | | 3 | $-2$ | 5 | 0 | $-4$ |
| $x_3$ | 5 | $t+12$ | 1 | 1 | 1 | 0 | 0 |
| $x_4$ | 0 | $4t+8$ | 2 | $-1$ | 0 | 1 | 0 |
| $x_5$ | $-4$ | $-6t+10$ | $-2$ | **2** | 0 | 0 | 1 |
| $\Delta$ | | $-29t-20$ | $-10$ | 1 | 0 | 0 | 0 |
| $x_3$ | 5 | $4t+7$ | 2 | 0 | 1 | 0 | $-\frac{1}{2}$ |
| $x_4$ | 0 | $t+13$ | 1 | 0 | 0 | 1 | $\frac{1}{2}$ |
| $x_2$ | $-2$ | $-3t+5$ | $-\mathbf{1}$ | 1 | 0 | 0 | $\frac{1}{2}$ |
| $\Delta$ | | $-26t-25$ | $-9$ | 0 | 0 | 0 | $-\frac{1}{2}$ |

It is clear from Table 1 that $\mathbf{x}_0^* = (0, -3t+5, 4t+7, t+13, 0)$ is an optimal solution for $t = 0$. Obviously $\mathbf{x}_0^*$ is also an optimal solution if and only if $\mathbf{x}_0^*$ has no negative components, that is, if and only if $x_2^* \equiv -3t+5 \geq 0$, $x_3^* \equiv 4t+7 \geq 0$, $x_4^* \equiv t+13 \geq 0$, and these three conditions are satisfied for $-\frac{7}{4} \leq t \leq \frac{5}{3}$. Therefore, if $t \in \left[-\frac{7}{4}, \frac{5}{3}\right]$, then $\mathbf{x}_0^* = (0, -3t+5, 4t+7, t+13, 0)$ is an optimal solution of problem (13)-(15), and $L_{\max} = L(\mathrm{x}_0^*) = 26t + 25$.

Study whether the considered problem has optimal solution(s) for $t > \frac{5}{3}$. If $t > \frac{5}{3}$, then $\mathbf{x}_0^*$ is an infeasible solution of the problem because $x_2^* \equiv -3t + 5 < 0$ in $\mathbf{x}_0^*$ in this case. That is why, for $t > \frac{5}{3}$ we have to find a new feasible solution, which is an optimal solution as well. This can be done if in the row of $x_2$ in the second simplex table of Table 1 there are negative elements (recall that the problem can be solved by the *dual* simplex method). In this case, this condition is satisfied (see second table of Table 1, the element $-1$ in the row of $x_2$ and the column of $x_1$). In the new pivot transformation, the basic variable $x_2$ leaves the basis, and the nonbasic variable $x_1$ becomes a basic variable in the next basis:

**Table 2**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|---|
| | | | 3 | $-2$ | 5 | 0 | $-4$ |
| $x_3$ | 5 | $-2t+17$ | 0 | 2 | 1 | 0 | $\frac{1}{2}$ |
| $x_4$ | 0 | $-2t+18$ | 0 | 1 | 0 | 1 | 1 |
| $x_1$ | 3 | $3t-5$ | 1 | $-1$ | 0 | 0 | $-\frac{1}{2}$ |
| $\Delta$ | | $t-70$ | 0 | $-9$ | 0 | 0 | $-5$ |

From Table 2 it follows that $\mathbf{x}_1^* = (3t-5, 0, -2t+17, -2t+18, 0)$ turns out to be an optimal solution of the considered problem for all $t$, which satisfy $\mathrm{x}_1^* \equiv 3t-5 \geq 0$, $x_3^* \equiv -2t+17 \geq 0$, $x_4^* \equiv -2t+18 \geq 0$, that is, for all $\frac{5}{3} \leq t \leq \frac{17}{2}$. Therefore $\mathbf{x}_1^* = (3t-5, 0, -2t+17, -2t+18, 0)$ is an optimal solution for the problem for all $t \in \left[\frac{5}{3}, \frac{17}{2}\right]$, and $L_{\max} = L(\mathrm{x}_1^*) = 70 - t$.

If $t > \frac{17}{2}$, then $\mathbf{x}_1^* = (3t-5, 0, -2t+17, -2t+18, 0)$ is a nonoptimal solution of the considered problem for these values of $t$ because the third component $x_3^* = 17 - 2t$ of $\mathbf{x}_1^*$ is negative for $t > \frac{17}{2}$. Since in the row of $x_3$ in Table 2 there are no negative elements, then for $t > \frac{17}{2}$ the given problem is unsolvable.

Study the solvability of the given problem for $t < -\frac{7}{4}$. In this case, the vector $\mathbf{x}_0^* = (0, -3t+5, 4t+7, t+13, 0)$, obtained above, is an infeasible solution of the problem because its third component $x_3^* = 4t+7$ is negative for $t < -\frac{7}{4}$. In order to find an optimal solution for $t < -\frac{7}{4}$ (this can be done because in the row of $x_3$ in the second simplex table of Table 1 there is a negative element $-\frac{1}{2}$), the basic variable $x_3$ of $x_0^*$ leaves the basis, and the nonbasic variable $x_5$ becomes a basic variable in the next basis:

**Table 3**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|---|
| | | | 3 | $-2$ | 5 | 0 | $-4$ |
| $x_5$ | $-4$ | $-8t-14$ | $-4$ | 0 | $-2$ | 0 | 1 |
| $x_4$ | 0 | $5t+20$ | 3 | 0 | 1 | 1 | 0 |
| $x_2$ | $-2$ | $t+12$ | 1 | 1 | 1 | 0 | 0 |
| $\Delta$ | | $-30t-32$ | $-11$ | 0 | $-1$ | 0 | 0 |

From Table 3 it is clear that $\mathbf{x}_2^* = (0, t+12, 0, 5t+20, -8t-14)$ is an optimal solution of the given problem for all values of the parameter $t$, for which

$x_2^* \equiv t + 12 \geq 0,\ x_4^* \equiv 5t + 20 \geq 0,\ x_5^* \equiv -8t - 14 \geq 0$, that is, for all values $-4 \leq t \leq -\frac{7}{4}$. Therefore for $t \in \left[-4, -\frac{7}{4}\right]$ the given problem has an optimal solution $\mathbf{x}_2^* = (0, t + 12, 0, 5t + 20, -8t - 14)$, and $L_{\max} = L(\mathbf{x}_2^*) = 30t + 32$.

From Table 3 it is also clear that for $t < -4$ the given problem is unsolvable because in this case, for the fourth component of $\mathbf{x}_2^*$ we have $x_4^* \equiv 5t + 20 < 0$ for $t < -4$ and in the row of $x_4$ there are no negative elements (see Table 3).

Summary of the obtained results for problem (13)-(15):
for $t \in (-\infty, -4)$: problem (13)-(15) is unsolvable,
for $t \in \left[-4, -\frac{7}{4}\right]$: $\mathbf{x}_2^* = (0, t + 12, 0, 5t + 20, -8t - 14), L_{\max} = 30t + 32$,
for $t \in \left[-\frac{7}{4}, \frac{5}{3}\right]$: $\mathbf{x}_0^* = (0, -3t + 5, 4t + 7, t + 13, 0), L_{\max} = 26t + 25$,
for $t \in \left[\frac{5}{3}, \frac{17}{2}\right]$: $\mathbf{x}_1^* = (3t - 5, 0, -2t + 17, -2t + 18, 0), L_{\max} = 70 - t$,
for $t \in \left(\frac{17}{2}, \infty\right)$: problem (13)-(15) is unsolvable.

## 3. The general parametric linear programming problem

Using the algorithms for solving some parametric linear programs, described in ([14], Section 2) and in previous Section 2 of this paper, we can also solve the parametric linear programming problems, in which both coefficients of the objective function and the right-hand sides of the constraints depend linearly on the parameter $t$.

**Example 2:** For all values of parameter $t \in (-\infty, \infty)$ solve the problem

$$\max\{L(x) = (-5t + 8)x_1 + (-3t + 9)x_2 + (5t - 3)x_3 - (4t + 2)x_4\} \tag{16}$$

subject to

$$\begin{array}{rrrrcl} x_1 & -x_2 & +x_3 & & = & -12t + 24 \\ -x_1 & +2x_2 & & +x_4 & = & 10t - 18 \end{array} \tag{17}$$

$$x_1 \geq 0, x_2 \geq 0. \tag{18}$$

**Solution:** Set, for example, $t = 2$ (this value is arbitrarily chosen) and using simplex method, solve the obtained linear programming problem for $t = 2$:

**Table 4**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|---|
| | | | $-5t+8$ | $-3t+9$ | $5t-3$ | $-4t-2$ |
| $x_3$ | $5t-3$ | $-12t+24$ | $1$ | $-1$ | $1$ | $0$ |
| $x_4$ | $-4t-2$ | $10t-18$ | $-1$ | $\mathbf{2}$ | $0$ | $1$ |
| $\Delta$ | | $100t^2-208t+36$ | $-14t+9$ | $10t+10$ | $0$ | $0$ |
| $x_3$ | $5t-3$ | $-7t+15$ | $\frac{1}{2}$ | $0$ | $1$ | $\frac{1}{2}$ |
| $x_2$ | $-3t+9$ | $5t-9$ | $-\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ |
| $\Delta$ | | $50t^2-168t+126$ | $-9t+14$ | $0$ | $0$ | $-5t-5$ |

Table 4 shows that $\mathbf{x}_0^* = (0, -9 + 5t, 15 - 7t, 0)$ is an optimal solution of the given problem for $t = 2$. This optimal solution also remains optimal for all values of the parameter $t$, such that $\Delta_1 \equiv -9t + 14 \leq 0$ and $\Delta_4 \equiv -5t - 5 \leq 0$, that is, for $t \geq \frac{14}{9}$, and there are no negative components of $\mathrm{x}_0^*$. At the beginning we will consider namely such values of the parameter $t$.

Then $\mathrm{x}_0^*$ is an optimal solution of the considered problem for $x_2^* \equiv 5t - 9 \geq 0$ and $x_3^* \equiv -7t + 15 \geq 0$, that is, for $\frac{9}{5} \leq t \leq \frac{15}{7}$. Therefore, if $t \in \left[\frac{9}{5}, \frac{15}{7}\right]$ (note that $\frac{9}{5} > \frac{14}{9}$), then $\mathrm{x}_0^* = (0{,}5t - 9, -7t + 15{,}0)$ is an optimal solution of the considered problem and $L_{\max} = L(\mathbf{x}_0^*) = -50t^2 + 168t - 126$.

If $t < \frac{9}{5}$, then $x_2^* \equiv 5t - 9 < 0$ and $\mathbf{x}_0^* = (0{,}5t - 9, -7t + 15{,}0)$ is not a feasible solution of the considered problem. That is why for $t < \frac{9}{5}$ we have to find a new simplex Table and a new feasible corner point, corresponding to this simplex Table. This can be done because in the row of $x_2$ in the second simplex table of Table 4 there is a negative element: $-\frac{1}{2}$. In the new pivot transformation with pivot element $-\frac{1}{2}$, the basic variable $x_2$ leaves the basis, and the nonbasic variable $x_1$ becomes a basic variable in the new basis:

**Table 5**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|---|
| | | | $-5t+8$ | $-3t+9$ | $5t-3$ | $-4t-2$ |
| $x_3$ | $5t-3$ | $-2t+6$ | 0 | **1** | 1 | 1 |
| $x_1$ | $-5t+8$ | $-10t+18$ | 1 | $-2$ | 0 | $-1$ |
| $\Delta$ | | $-40t^2+134t-126$ | 0 | $-18t+28$ | 0 | $-14t+9$ |

Table 5 shows that $\mathrm{x}_1^* = (-10t + 18{,}0, -2t + 6{,}0)$ is an optimal solution of the considered problem for all values of $t \geq \frac{14}{9}$, such that $x_1^* \equiv -10t + 18 \geq 0$ and $x_3^* \equiv -2t + 6 \geq 0$, that is, for $t \leq \frac{9}{5}$. Therefore for $t \in \left[\frac{14}{9}, \frac{9}{5}\right]$, the optimal solution of the given problem is $\mathbf{x}_1^* = (-10t + 18{,}0, -2t + 6{,}0)$ and $L_{\max} = L(\mathbf{x}_1^*) = 40t^2 - 134t + 126$.

Let now $t > \frac{15}{7}$. For these values of $t$, the vector $\mathbf{x}_0^* = (0{,}5t - 9, -7t + 15{,}0)$ is not a feasible solution of the problem because $x_3^* \equiv -7t + 15 < 0$ in $\mathrm{x}_0^*$ in this case. Since in the row of $x_3$ in Table 5 there are no negative elements, then for $t > \frac{15}{7}$ the given problem is unsolvable.

Up to now we have found optimal solutions of the considered problem for $t \in \left[\frac{14}{9}, \infty\right)$. Consider the interval $t \in \left(-\infty, \frac{14}{9}\right)$. If $t < \frac{14}{9}$, then $\Delta_2 = -18t + 28 > 0$ (Table 5). That is why we have to find a new adjacent feasible corner point. In the new pivot transformation, the basic variable $x_3$ leaves the basis, and the nonbasic variable $x_2$ becomes a basic variable in the next basis:

**Table 6**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|---|
| | | | $-5t+8$ | $-3t+9$ | $5t-3$ | $-4t-2$ |
| $x_2$ | $-3t+9$ | $-2t+6$ | 0 | 1 | 1 | 1 |
| $x_1$ | $-5t+8$ | $-14t+30$ | 1 | 0 | 2 | 1 |
| $\Delta$ | | $-76t^2+298t-294$ | 0 | 0 | $18t-28$ | $4t-19$ |

From Table 6 it follows that if $t \in \left(-\infty, \frac{14}{9}\right]$, then $\mathbf{x}_2^* = (-14t + 30, -2t + 6,0,0)$ is an optimal solution of the problem, and $L_{\max} = L(\mathbf{x}_2^*) = 76t^2 - 298t + 294$.

Therefore:

for $t \in \left(-\infty, \frac{14}{9}\right]$: $\mathbf{x}_2^* = (-14t + 30, -2t + 6,0,0), L_{\max} = 76t^2 - 298t + 294$,

for $t \in \left[\frac{14}{9}, \frac{9}{5}\right]$: $\mathbf{x}_1^* = (-10t + 18,0, -2t + 6,0), L_{\max} = 40t^2 - 134t + 126$,

for $t \in \left[\frac{9}{5}, \frac{15}{7}\right]$: $\mathbf{x}_0^* = (0,5t - 9, -7t + 15,0), L_{\max} = -50t^2 + 168t - 126$,

for $t \in \left(\frac{15}{7}, \infty\right)$: problem (16)-(18) is unsolvable.

## 4. Conclusion

In this paper, the parametric linear programming problems are considered, in particular, the case of parameterized right-hand sides of constraints and the general case, with parameterized both the objective function and right-hand sides of constraints. Methods for solving such problems are presented, illustrative numerical examples are included, and postoptimal and sensitivity analysis is made. This paper is a continuation of paper [Stefanov 14].

An open problem for future research is to study other parametric linear programming problems, along with sensitivity analysis of these problems.

## References

[1] I. Adler and R. Monteiro, "A geometric view of parametric linear programming", *Algorithmica*, vol. 8, pp. 161-176 (1992).

[2] J. F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems, Springer Series in Operations Research, Springer-Verlag, New York (2000).

[3] T. Gal and H. J. Greenberg, Advances in Sensitivity Analysis and Parametric Programming, Kluwer Academic Publishers, Norwell, Massachusetts (1997).

[4] A. G. Hadigheh and T. Terlaky, "Generalized support set invariancy sensitivity analysis in linear optimization", *Journal of Industrial and Management Optimization*, vol. 2, pp. 1-18 (2006).

[5] A. Ghaffari-Hadigheh, Ghaffari-Hadigheh, and T. Terlaky, "Bi-parametric optimal partition invariancy sensitivity analysis in linear optimization", *Central European Journal of Operations Research*, vol. 16, no. 2, pp. 215-238 (2008).

[6] A. G. Hadigheh, K. Mirnia, and T. Terlaky, "Active constraint set invariancy sensitivity analysis in linear optimization", *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 303-315 (2007).

[7] A. G. Hadigheh and T. Terlaky, "Sensitivity analysis in linear optimization: Invariant support set intervals", *European Journal of Operational Research*, vol. 169, no. 3, pp. 1158-1175 (2006).

[8] B. Jansen, J. J. de Jong, C. Roos, and T. Terlaky, "Sensitivity analysis in linear programming: Just be careful!", *European Journal of Operational Research*, vol. 101, no. 1, pp. 15-28 (1997).

[9] S. M. Stefanov, Convex Separable Programming: Theory and Methods, Kluwer Academic Publishers, Dordrecht-Boston-London (2000).

[10] S. M. Stefanov, "Well-posedness and primal-dual analysis of some convex separable optimization problems", *Advances in Operations Research*, vol. 2013, 10 pages (2013) Article ID 279030.

[11] S. M. Stefanov, Separable Programming: Theory and Methods, 4th rev. enld. ed., Springer Science+Business Media, B.V., Dordrecht (2016).

[12] S. M. Stefanov, Separable Optimization: Theory and Methods, Springer Optimization and Its Applications, vol. 177, Springer, Cham (2021).

[13] S. M. Stefanov, "Continuous linear knapsack problems revisited", *Journal of Information and Optimization Sciences*, vol. 44, no. 5, pp. 909-922 (2023).

[14] S. M. Stefanov, "Parametric linear programming: the case of the parameterized objective function", *Journal of Information and Optimization Sciences*, published online.

[15] S. M. Stefanov, "On the dual simplex method", *Journal of Information and Optimization Sciences*, to appear.

[16] O. Stein, Grundzüge der Parametrischen Optimierung, Springer, Berlin (2021).

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