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Parametric linear programming : The case of parameterized objective function

Stefan M. Stefanov
Department of Informatics
Neofit Rilski South-West University
Blagoevgrad 2700
Bulgaria

Abstract

In this paper, the parametric linear programming problems are considered, and in particular, the case of parameterized objective function. Numerical solution of such problems is discussed, an illustrative numerical example is included, and postoptimal and sensitivity analysis of the obtained solution is made.

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1. Introduction

In many mathematical programming problems, the input data depends on parameter(s). Such problems are called *parametric programming problems*.

Consider the dependency of the data of the canonical linear programming problem on parameter(s).

Linear programming problem, whose coefficients of the objective function depend linearly on the parameter t, consists in solving problem (1)-(3) for each value of the parameter $t \in [\alpha, \beta]$:

$$\max\{L(x) = \sum_{j=1}^{n} (c_j' + c_j'' t)x_j\} \tag{1}$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \forall i = 1, \dots, m, \tag{2}$$

$$x_j \geq 0, \forall j = 1, \dots, n, \tag{3}$$

where c_j', c_j'', a_{ij}, and b_i are given coefficients.

If the constraints' right-hand sides of a linear programming problem depend linearly on the parameter t, then this problem consists in solving problem (4)-(6) for each value of the parameter $t \in [\alpha, \beta]$:

E-mail: stefm@swu.bg

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$$\max\{L(x) = \sum_{j=1}^{n} c_j x_j\} \tag{4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i' + b_i'' t, \forall i = 1, \dots, m, \tag{5}$$

$$x_j \geq 0, \forall j = 1, \dots, n, \tag{6}$$

where c_j, a_{ij}, b_i', b_i'' are given coefficients.

In the case, when both coefficients of the objective function and the right-hand sides of the constraints of a linear optimization problem depend linearly on the parameter t, then this problem consists in solving problem (7)-(9) for each value of the parameter $t \in [\alpha, \beta]$:

$$\max\{L(x) = \sum_{j=1}^{n} (\mathrm{c}_j' + \mathrm{c}_j'' t) x_j\} \tag{7}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i' + b_i'' t, \forall i = 1, \dots, m, \tag{8}$$

$$x_j \geq 0, \forall j = 1, \dots, n. \tag{9}$$

Problem (7)-(9) is called the *general parametric linear optimization problem*.

Generalization of the three problems (1)-(3), (4)-(6), and (7)-(9), considered above, is the so-called *general problem of the parametric linear programming*, in which coefficients of the objective function, coefficients of variables x_j in the constraints' left-hand sides, and the constraints' right-hand sides depend linearly on the parameter t. Statement of this problem is as follows: for each feasible value of the parameter $t \in [\alpha, \beta]$ solve the optimization problem

$$\max\{L(x) = \sum_{j=1}^{n} (\mathrm{c}_j' + \mathrm{c}_j'' t) x_j\} \tag{10}$$

subject to

$$\sum_{j=1}^{n} (a_{ij}' + a_{ij}'' t) x_j = b_i' + b_i'' t, i = 1, \dots, m \tag{11}$$

$$x_j \geq 0, \quad j = 1, \dots, n. \tag{12}$$

All parametric linear programming problems, considered above, are solvable by using the linear optimization methods.

Before that, consider the geometric interpretation of the parametric linear programming problems, focusing mainly on problem (1)-(3).

Suppose that the set of nonnegative solutions (satisfying constraint (3)) of the system of equations (2), which is the feasible polyhedral set (2)-(3) of problem (1)-(3), is nonempty and that it contains more than one feasible point (more than one feasible solution). Then problem (1)-(3) consists in finding - for each value of the parameter $t \in [\alpha, \beta]$ - a feasible point (feasible solution), for which the objective function $L(\mathrm{x})$, defined by (1), assumes a maximum value. In order to find such a feasible point, assume that $t = t_0$, and using the geometric interpretation of the linear optimization problems, solve the linear optimization problem (1)-(3) with $t = t_0$, that is, either find a vertex of the feasible polyhedral

set of the considered linear problem, such that the linear objective function assumes a maximum value at this vertex, or establish that the considered linear problem has no optimal solution for the value $t = t_0$.

Once a point is found, for which the objective function $L(\mathrm{x})$, defined by (1), assumes a maximum value with $t = t_0$, we are looking for a set of values of the parameter t, for which the coordinates of the found point constitute optimal solution of the linear problem (1)-(3) with $t = t_0$. The values of the parameter t, which we have found, are excluded from further consideration, and we take another value t_1 of t in the interval $[\alpha, \beta]$. In the same manner, find an optimal solution of the considered linear problem for $t = t_1$ or establish that this problem has no optimal solution for $t = t_1$. After that we find a subinterval of $[\alpha, \beta]$ for t, for which the found point constitutes an optimal solution of the considered linear problem or for which this problem has no optimal solution.

After a finite number of steps, for each value of the parameter $t \in [\alpha, \beta]$, we either find an optimal solution of problem (1)-(3) or establish that this problem has no optimal solution.

The following Example 1 shows that some real-world problems are described by parametric linear programming problems.

Example 1: A company produces two types of products, P_1 and P_2, using three types of materials, I, II, and III. Amounts of materials of each of the three types, necessary for producing a unit quantity of each of the two products, and the supply (available quantities) of each of the three types of materials, which can be used for producing a unit quantity of each of the two products, are presented in Figure 1.

Figure 1

Material \ Product	P_1	P_2	Supply of the material
I	4	1	16
II	2	2	22
III	6	3	36

It is known that the price of a unit product can vary from 2 to 12 for product of type P_1 and from 3 to 13 for product of type P_2, and these variations are determined by the relations $c_1 = 2 + t$ and $c_2 = 13 - t$, respectively, where $0 \leq t \leq 10$.

For each of the possible values of the price of a unit product find a production plan, such that the total cost of the production be maximal.

Mathematical model for Example 1.

Denote by x_1 and x_2 the quantities of products of types P_1 and P_2 to be produced, respectively.

Then the *mathematical model* for the problem of Example 1 is as follows

$$\max\{L(x) = (2 + t)x_1 + (13 - t)x_2\} \tag{13}$$

subject to

$$\begin{array}{rcl} 4x_1 + x_2 & \leq & 16 \\ 2x_1 + 2x_2 & \leq & 22 \\ 6x_1 + 3x_2 & \leq & 36 \end{array} \tag{14}$$

$$x_1 \geq 0, x_2 \geq 0, \tag{15}$$

where $0 \leq t \leq 10$.

Problem (13)-(15) is a parametric linear programming problem of the form (1)-(3).

Parametric linear programming problems, sensitivity analysis, and related topics are studied in [1-15], etc.

The case of the parameterized objective function is considered and the solution of an illustrative example is presented in Section 2 of the paper. Contents of the paper is outlined and an open problem for research is formulated in Section 3.

2. Solving the linear problem with parameterized objective function

Consider again problem (1)-(3). Assign the parameter t some value $t_0 \in [\alpha, \beta]$, and using the simplex method (or the Big M method, if necessary), we solve the canonical linear programming problem P_{t_0}, obtained from problem (1)-(3) with $t = t_0$. As a result, we either find optimal solution of problem P_{t_0} or establish that this problem is unsolvable. In the first case, using elements of the last, $m+1$-th, row of the last (optimal) simplex table, in which the values of $\Delta_j(t_0) \stackrel{\text{def}}{=} \Delta_j' + t_0\Delta_j''$ are included, we find

$$\underline{t} = \begin{cases} \max\left(-\frac{\Delta_j'}{\Delta_j''}\right), & \text{if there is } \Delta_j'' < 0, \\ -\infty, & \text{if all } \Delta_j'' \geq 0, \end{cases} \tag{16}$$

$$\overline{t} = \begin{cases} \min\left(-\frac{\Delta_j'}{\Delta_j''}\right), & \text{if there is } \Delta_j'' > 0, \\ \infty, & \text{if all } \Delta_j'' \leq 0. \end{cases} \tag{17}$$

For all $\underline{t} \leq t \leq \overline{t}$ problem (1)-(3) has the same optimal solution, as well as for $t = t_0$.

In this case, if problem (1)-(3) is unsolvable for $t = t_0$, in the $m+1$-th row of the last simplex table there is a value $\Delta_k = \Delta_k' + t_0\Delta_k'' > 0$, where $\alpha_{ik} \leq 0$, $i = 1, \ldots, m$. Then:

1) If $\Delta_k'' = 0$, then problem (1)-(3) is unsolvable for each value of t;
2) If $\Delta_k'' < 0$, then problem (1)-(3) is unsolvable for all $t < t_1 \stackrel{\text{def}}{=} -\frac{\Delta_k'}{\Delta_k''}$;
3) If $\Delta_k'' > 0$, then problem (1)-(3) is unsolvable for all $t > t_1$.

Determining all values of the parameter $t \in [\alpha, \beta]$, for which problem (1)-(3) has the same optimal solution or for which this problem is unsolvable, we

obtain an interval of varying the parameter t, which interval is excluded from further consideration. Set the parameter t be equal to some value of the interval $[\alpha, \beta]$ and solve the problem, obtained from problem (1)-(3) with this value of t.

After a finite number of iterations we either find an interval, such that for each value of t in this interval, the considered problem has the same optimal solution, or find an interval, such that for all feasible values of t in this interval, the considered problem is unsolvable.

Algorithm for solving problem (1)-(3).

1. Assign the parameter t a value, equal to some number $t_0 \in [\alpha, \beta]$, and find an optimal solution x^* of problem (1)-(3) with $t = t_0$ or establish that this problem is unsolvable. Go to 2.
2. Determine all values of the parameter $t \in [\alpha, \beta]$, for which the optimal solution of step 1. remains optimal or for which the considered problem is unsolvable. These values of the parameter are excluded from further consideration. Go to 3.
3. Assign the parameter t a value, equal to some number, belonging to the remaining part of the interval $[\alpha, \beta]$, and using the simplex method, we find an optimal solution of the linear optimization problem, obtained from problem (1)-(3) with this value of t. Go to 4.
4. Determine all values of the parameter t, for which the new optimal solution remains optimal or for which the considered problem is unsolvable. The calculations continue for all values of the parameter $t \in [\alpha, \beta]$. Go to 5.
5. End.

Example 2: For all values of parameter $t \in (-\infty, \infty)$ solve the problem

$$\max\{L(x) = 2x_1 + (3 + 4t)x_2\} \tag{18}$$

subject to

$$\begin{array}{rrrrrcl} x_1 & +x_2 & +x_3 & & & = & 12 \\ x_1 & -x_2 & & +x_4 & & = & 10 \\ -x_1 & +x_2 & & & +x_5 & = & 6 \end{array} \tag{19}$$

$$x_j \geq 0, \quad j = 1, \ldots, 5. \tag{20}$$

Solution: Set, for example, $t = 0$ in the objective function $L(x)$, defined by (18), and solve the obtained canonical linear optimization problem by the simplex method; note that the system of equality constraints of this problem is in basic form with respect to variables x_3, x_4, and x_5:

Table 1

Simplex table for Example 2.

Basis	$\mathbf{c}_B$	β	x_1	x_2	x_3	x_4	x_5
			2	$3+4t$	0	0	0
x_3	0	12	1	1	1	0	0
x_4	0	10	1	-1	0	1	0
x_5	0	6	-1	**1**	0	0	1
Δ		0	2	$3+4t$	0	0	0
x_3	0	6	**2**	0	1	0	-1
x_4	0	16	0	0	0	1	1
x_2	$3+4t$	6	-1	1	0	0	1
Δ		$-18-24t$	$5+4t$	0	0	0	$-3-4t$
x_1	2	3	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
x_4	0	16	0	0	0	1	**1**
x_2	$3+4t$	9	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$
Δ		$-33-36t$	0	0	$-\frac{5}{2}-2t$	0	$-\frac{1}{2}-2t$

The unique optimal solution of the considered problem is $x_0^* \equiv (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (3, 9, 0, 16, 0)$.

Determine values of the parameter t, for which the optimal solution x_0^* remains optimal. This is satisfied when $\Delta_j \leq 0$ for all $j = 1, \ldots, 5$ in the last (3rd) simplex table of Table 1, that is, when $\Delta_3 \equiv -\frac{5}{2} - 2t \leq 0$ and $\Delta_5 \equiv -\frac{1}{2} - 2t \leq 0$, therefore $t \geq -0{,}25$. Hence, for $t \in [-0.25, \infty)$ the given problem has an optimal solution $x_0^* = (3, 9, 0, 16, 0)$, and $L_{\max} = L(x_0^*) = 33 + 36t$.

Consider now a value of the parameter t, less than $-0{,}25$. In this case, we would obtain $\Delta_5 > 0$ in the last (3rd) simplex table of Table 1. Therefore x_0^* would not be an optimal solution for the value $t = -0,25$. In the new pivot transformation, the basic variable x_4 leaves the basis, and the nonbasic variable x_5 becomes a basic variable in the next basis:

Table 2

Simplex table for Example 2: 1st variation of parameter values.

Basis	$\mathbf{c}_B$	β	x_1	x_2	x_3	x_4	x_5
			2	$3+4t$	0	0	0
x_1	2	11	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0
x_5	0	16	0	0	0	1	1
x_2	$3+4t$	1	0	1	$\mathbf{\frac{1}{2}}$	$-\frac{1}{2}$	0
Δ		$-25-4t$	0	0	$-\frac{5}{2}-2t$	$\frac{1}{2}+2t$	0

The obtained new feasible corner point (vertex of the feasible polyhedral set of the considered problem) $x_1^* = (11,1,0,0,16)$ is an optimal solution for

$\Delta_3 \equiv -\frac{5}{2} - 2t \leq 0$ and $\Delta_4 \equiv \frac{1}{2} + 2t \leq 0$, that is, for $t \in [-1.25, -0.25]$. Therefore, if $t \in [-1.25, -0.25]$, then $x_1^* = (11,1,0,0,16)$ is an optimal solution of the given problem (18)-(20) and $L_{\max} = L(x_1^*) = 25 + 4t$.

Find an optimal solution for $t < -1{,}25$. In this case, we would obtain $\Delta_3 > 0$ in Table 2. Therefore x_1^* would not be an optimal solution for these values of t. That is why we find a new feasible corner point by a pivot transformation, such that the basic variable x_2 leaves the basis, and the nonbasic variable x_3 becomes a basic variable in the next basis:

Table 3

Simplex table for Example 2: 2nd variation of parameter values.

Basis	$\mathbf{c}_B$	β	x_1	x_2	x_3	x_4	x_5
			2	$3+4t$	0	0	0
x_1	2	10	1	-1	0	1	0
x_5	0	16	0	0	0	1	1
x_3	0	2	0	2	1	-1	0
Δ		-20	0	$5+4t$	0	-2	0

The obtained new feasible corner point $x_2^* = (10,0,2,0,16)$ is an optimal solution for all values of t, such that $\Delta_2 \equiv 5 + 4t \leq 0$, that is, for all $t \leq -1{,}25$. Therefore, for $t \in (-\infty, -1.25]$, the given problem (18)-(20) has an optimal solution $x_2^* = (10,0,2,0,16)$ and $L_{\max} = L(x_2^*) = 20$.

Summary of the obtained optimal solutions of problem (18)-(20):

for $t \in (-\infty, -1.25]$: $x_2^* = (10,0,2,0,16)$, $L_{\max} = 20$,

for $t \in [-1.25, -0.25]$: $x_1^* = (11,1,0,0,16)$, $L_{\max} = 25 + 4t$,

for $t \in [-0.25, \infty)$: $x_0^* = (3,9,0,16,0)$, $L_{\max} = 33 + 36t$.

3. Conclusion

In this paper, the parametric linear programming problems are considered, and in particular, the case of parameterized objective function. Numerical solution of such problems is discussed, an illustrative numerical example is included, and postoptimal and sensitivity analysis of the obtained solution is made.

Open problem for research is to study other parametric linear programming problems, along with sensitivity analysis of these problems.

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