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# On the dual simplex method

Stefan M. Stefanov
*Department of Informatics*
*Neofit Rilski South-West University*
*Blagoevgrad 2700*
*Bulgaria*

**Abstract**

Some theoretical issues about the dual simplex method are discussed, an algorithm for solving linear optimization problems by this method is considered, and illustrative numerical example is presented in this article.

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## 1. Introduction. Duality in linear optimization

The pair of dual problems, the Canonical problem and its Dual canonical problem (DCP), can be written in vector-matrix form as follows

(CP)

$$\max\{L(\mathrm{x}) = \langle c, \mathrm{x}\rangle\} \tag{1}$$

subject to

$$A\mathrm{x} = \mathrm{b} \tag{2}$$

$$\mathrm{x} \geq 0 \tag{3}$$

and

(DCP)

$$\min\{D(\mathrm{y}) = \langle \mathrm{b}, \mathrm{y}\rangle\} \tag{4}$$

subject to

$$\mathrm{y}A \geq \mathrm{c}, \tag{5}$$

where $\mathrm{c} = (c_j)_n$, $A = (a_{ij})_{m\times n}$, $\mathrm{b} = (b_i)_m^T$, $\mathrm{x} = (x_j)_n^T$, $\mathrm{y} = (y_j)_n$, $\langle \mathrm{c}, \mathrm{x}\rangle$ is notation of the scalar (inner, dot) product of vectors $c$ and $x$.

Assume that system (2) is compatible and rank $A = m$.

*E-mail:* **stefm@swu.bg**

The considered problem (1)-(3) is said to be the *primal problem*, and problem (4)-(5) is said to be its *dual problem*.

Recall that column vectors

$$\mathrm{a}^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}, j = 1, \dots, n,$$

of coefficients of system (2), that is, column vectors of matrix $A$ of system (2), are called *vectors of constraints*.

*Basic vectors* of the corner point $\mathrm{y}$ of the dual problem (4)-(5) is a set of $m$ linearly independent vectors $\mathrm{a}_{*s_1}, \dots, \mathrm{a}_{*s_m}$ of coefficients of the constraints of the primal problem (CP) (1)-(3), satisfying the conditions

$$\langle \mathrm{y}, \mathrm{a}_{*s_i} \rangle = c_{s_i}, i = 1, \dots, m. \tag{6}$$

Here, the index " $*$ " in the notation $\mathrm{a}_{*s_i}$ means that this is a vector of constraints of the dual problem (DCP) (4)-(5).

The corner point $y$ of the dual problem (4)-(5) is said to be *non-degenerate* if for each nonbasic vector $\mathrm{a}_{*j}$, the following relations are satisfied

$$\langle \mathrm{y}, \mathrm{a}_{*j} \rangle > c_{s_i}, j \neq s_i, i = 1, \dots, m. \tag{7}$$

Otherwise the corner point is said to be *degenerate*.

Assume that the dual problem (DCP), defined by (4)-(5), is non-degenerate.

Here it is not required nonnegativity of the right-hand sides of system (2) for the concept of ``basic form'' of system (2), unlike this requirement for the classical form of the simplex method.

Recall the statement of the following

**Duality theorem:** Dual problems (1)-(3) and (4)-(5) are both solvable or both unsolvable. If they are solvable and $\mathrm{x}^*$ is an optimal solution of the primal problem (CP) defined by (1)-(3) and $\mathrm{y}^*$ is an optimal solution of the dual problem (DCP) defined by (4)-(5), then $L(\mathrm{x}^*) = D(\mathrm{y}^*)$.

Problems, studied in this paper, and related to them are considered in reference titles [1]-[20], etc.

## 2. The dual simplex method

Let the basic form of system (2) with respect to variables $x_{s_1}, x_{s_2}, \dots, x_{s_m}$ be

$$\textstyle\sum_{j=1}^{n} \alpha_{ij} x_j = \beta_i, i = 1, \dots, m, \tag{8}$$

where

$$\alpha_{is_i} = 1;\ \alpha_{is_j} = 0, i \neq j, \tag{9}$$

that is, the variable $x_{s_i}$ is involved in $i$-th equation and $s_1 < s_2 < \cdots < s_m$.

Vector $\mathrm{x} = (x_1, \dots, x_n)$ with components

$$x_{s_i} = \beta_i, i = 1, \dots, m;\ x_j = 0, j \neq s_i, i = 1, \dots, m, \tag{10}$$

is said to be a *pseudo-feasible solution* of the primal problem (1)-(3), generated by the corner point $\mathrm{y}$ of the dual problem (4)-(5).

The basis of x is $B_{\mathrm{x}} = \{x_{s_1}, \dots, x_{s_m}\}$, and the variables $x_{s_1}, \dots, x_{s_m}$ are called *basic variables*.

In the general case, x is not a feasible solution of the primal problem (1)-(3) because its components do not satisfy the conditions $\beta_i \geq 0,\ i = 1, \dots, m$.

Denote

$$B = (\mathrm{a}_{*s_1}, \dots, \mathrm{a}_{*s_m}) = \begin{pmatrix} a_{1s_1} & \cdots & a_{1s_m} \\ \cdots & \cdots & \cdots \\ a_{ms_1} & \cdots & a_{ms_m} \end{pmatrix},$$

$$\tilde{\mathrm{c}} = (c_{s_1}, \dots, c_{s_m}), \widetilde{\Delta} = (\Delta_1, \dots, \Delta_n),$$

where, as it is known, the basic matrix $B$ is invertible, that is, the inverse matrix $B^{-1}$ exists.

Then (6) becomes $\mathrm{y}B = \tilde{\mathrm{c}}$, that is,

$$\mathrm{y} = \tilde{c}B^{-1}. \tag{11}$$

It is known that after expressing basic variables of the primal problem (1)-(3) from the basic form (8) and replacing these basic variables in the objective function (1), we obtain the following form of objective function (1):

$$L(\mathrm{x}) = -\Delta_0 + \sum_{j=1}^{n} \Delta_j x_j,$$

where

$$\Delta_0 \overset{\text{def}}{=} \sum_{i=1}^{m} c_{s_i}\beta_i,$$

$$\Delta_j \overset{\text{def}}{=} c_j - \sum_{i=1}^{m} c_{s_i}\alpha_{ij}, j \neq s_i, i = 1, \dots, m,$$

$$\Delta_{s_i} = 0, i = 1, \dots, m.$$

The basic form (8) can be presented in the form

$$B^{-1}A\mathrm{x} = B^{-1}\mathrm{b}, \tag{8'}$$

where

$$B^{-1}A = (\alpha_{*1}, \dots, \alpha_{*n}) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix},$$

$$B^{-1}\mathrm{b} = (\beta_1, \dots, \beta_m)^T.$$

From (10) it follows that

$$(x_{s_1}, \dots, x_{s_m})^T = (\beta_1, \dots, \beta_m)^T = B^{-1}\mathrm{b}$$

and

$$\alpha_{*j} = B^{-1}\mathrm{a}_{*j}, j = 1, \dots, n.$$

From the above notation we get

$$\widetilde{\Delta}_j = c_j - \tilde{\mathrm{c}}B^{-1}\mathrm{a}_{*j}, j = 1, \dots, n,$$

i.e.,

$$\widetilde{\Delta} = \mathrm{c} - \tilde{\mathrm{c}}B^{-1}A. \tag{12}$$

Therefore

$$\widetilde{\Delta} \overset{(12)}{=} \mathrm{c} - \tilde{\mathrm{c}}B^{-1}A \overset{(11)}{=} \mathrm{c} - \mathrm{y}A \overset{(5)}{\leq} 0,$$

that is, $\Delta_j \leq 0,\ j = 1, \dots, n$.

Conversely, let (8) be an arbitrary basic form of system (2), with an associated vector

$$\mathrm{x} = (0, \dots, 0, x_{s_1}, 0, \dots, 0, x_{s_2}, 0, \dots, 0, x_{s_m}, 0, \dots, 0)^T = \\ = (0, \dots, 0, \beta_1, 0, \dots, 0, \beta_2, 0, \dots, 0, \beta_m, 0, \dots, 0)^T,$$

such that $\widetilde{\Delta} = \mathrm{c} - \tilde{\mathrm{c}}B^{-1}A \leq 0$. Set $\tilde{\mathrm{c}}B^{-1} = \mathrm{y}$. Then $\mathrm{c} - \mathrm{y}A \leq 0$, that is, vector $y$ satisfies (5). Hence, y turns out to be a feasible solution to the dual problem (4)-(5).

Since

$$\Delta_{s_i} = c_{s_i} - \langle \mathrm{y}, \mathrm{a}_{*s_i} \rangle = 0, i = 1, \dots, m,$$

then the feasible solution y turns out to be a corner point for the dual problem (4)-(5) by definition (see (6)). Then x is a pseudo-feasible solution for the primal problem (1)-(3).

We have shown that

$$\Delta_j \leq 0, j = 1, \dots, n, \tag{13}$$

is a necessary and sufficient condition for the vector

$$\mathrm{x} = (0, \dots, 0, \beta_1, 0, \dots, 0, \beta_2, 0, \dots, 0, \beta_m, 0, \dots, 0)^T$$

to be a pseudo-feasible solution of the primal problem (1)-(3).

For a given basic form (8) of system (2), conditions (13) are necessary and sufficient condition for a vector $\mathrm{y} = \tilde{\mathrm{c}}B^{-1}$ to be a feasible solution of the dual problem (4)-(5).

The following optimality criterion holds true.

**Optimality criterion:** If the basic elements of the pseudo-feasible solution x are nonnegative, then x turns out to be an optimal solution to the primal problem (1)-(3) and y turns out to be an optimal solution to the dual problem (4)-(5).

Indeed, if the basic components $x_{s_i}$, $i = 1, \dots, m$, of vector x are nonnegative, then x is a corner feasible point for the primal problem (1)-(3), and since $\Delta_{s_i} = 0,\ i = 1, \dots, m$, then

$$\langle \widetilde{\Delta}, \mathrm{x} \rangle \overset{(12)}{=} \langle \mathrm{c} - \mathrm{y}A, \mathrm{x} \rangle = 0,$$

hence $\langle \mathrm{c}, \mathrm{x} \rangle = \langle \mathrm{y}A, \mathrm{x} \rangle$ and

$$L(\mathrm{x}) \overset{\text{def}}{=} \langle \mathrm{c}, \mathrm{x} \rangle = \langle \mathrm{y}, A\mathrm{x} \rangle \overset{(2)}{=} \langle \mathrm{y}, \mathrm{b} \rangle \overset{\text{def}}{=} D(\mathrm{y}).$$

From the Duality theorem it follows that x is an optimal solution to the primal problem (1)-(3) and y is an optimal solution to the dual problem (4)-(5).

The Optimality criterion of the dual simplex method is also implied by the Optimality criterion of the classical simplex method: if a pseudo-feasible solution turns out to be a feasible solution to the primal problem (1)-(3), then it is also an optimal solution to this problem, because it satisfies conditions (13).

Analyze the behavior of the dual objective function $D(\mathrm{y})$, defined by (4), in the move from a feasible corner point to another feasible corner point of the dual problem (4)-(5).

Consider the vector $\mathrm{y}' \overset{\text{def}}{=} \mathrm{y} + T\mathrm{e}^{(p)}B^{-1}$, where

$$T \in \mathbb{R}, \quad \mathrm{e}^{(p)} = (\underbrace{0, \dots, 0, 1}_{p}, 0, \dots, 0)_m.$$

We have

$$\mathrm{y}'A = (\mathrm{y} + T\mathrm{e}^{(p)}B^{-1})A = \mathrm{y}A + T\mathrm{e}^{(p)}B^{-1}A.$$

We have established that the basic form (8) can also be presented in the form (8′): $B^{-1}A\mathrm{x} = B^{-1}\mathrm{b}$, that is, $(\alpha_{ij})_{m\times n} = B^{-1}A$. Therefore

$$\mathrm{e}^{(p)}B^{-1}A = \mathrm{e}^{(p)}(\alpha_{ij})_{m\times n} = (\alpha_{p_1}, \alpha_{p_2}, \dots, \alpha_{p_n}) \overset{\text{def}}{=} \alpha_{p*},$$

hence

$$\mathrm{y}'A = \mathrm{y}A + T\alpha_{p*}. \tag{14}$$

Furthermore

$$\widetilde{\Delta}' = \mathrm{c} - \mathrm{y}'A = c - (\mathrm{y}A + T\alpha_{p*}) = (c - \mathrm{y}A) - T\alpha_{p*} = \widetilde{\Delta} - T\alpha_{p*}.$$

Taking into consideration (9) and the fact that $\Delta_{s_i} = 0,\ i = 1, \dots, m$, we get

$$\Delta_j' = \begin{cases} 0, & j = s_1, s_2, \dots, s_m, j \neq s_p \\ -T, & j = s_p \\ \Delta_j - T\alpha_{pj}, & j \neq s_1, \dots, s_m. \end{cases} \tag{15}$$

Since y is a feasible solution to the dual problem (4)-(5), then $\Delta_j \leq 0$, $j = 1, \dots, n$, according to (13).

In order that $\mathrm{y}'$ be a feasible solution of the dual problem (4)-(5), we must have

$$\Delta_j' \leq 0, j = 1, \dots, n. \tag{16}$$

From (15) with $j = s_p$ it follows that $T \geq 0$.

The following two cases are possible.

a) $\alpha_{pj} \geq 0,\ j = 1, \dots, n$. Then condition (16) will be satisfied regardless the value of $T$, and according to the above conclusion, $\mathrm{y}'$ will be a feasible solution of the dual problem (4)-(5) for every $T \geq 0$.

b) There exists $\alpha_{pj} < 0$. Let

$$\frac{\Delta_q}{\alpha_{pq}} = \min_{\alpha_{pj}<0} \frac{\Delta_j}{\alpha_{pj}}. \tag{17}$$

Then $y'$ is a feasible solution to the dual problem (4)-(5) with

$$0 \leq T \leq \frac{\Delta_q}{\alpha_{pq}}. \tag{18}$$

Then the value of the dual objective function $D(\mathrm{y})$ for $\mathrm{y}'$ is

$$D(\mathrm{y}') \overset{(4)}{=} \langle \mathrm{y}', \mathrm{b} \rangle = \langle \mathrm{y} + Te^{(p)}B^{-1}, \mathrm{b} \rangle = \langle \mathrm{y}, \mathrm{b} \rangle + T\langle \mathrm{e}^{(p)}, B^{-1}\mathrm{b} \rangle = D(\mathrm{y}) + T\beta_p. \tag{19}$$

Depending on the signs of $\beta_i$, $i = 1, \dots, m$, in (8), there are two possible cases for the pseudo-feasible solution $\mathrm{x}$:

1. Basic components $x_{s_i}$, $i = 1, \dots, m$, of $\mathrm{x}$ are nonnegative. According to the Optimality criterion, $\mathrm{x}$ turns out to be an optimal solution to the primal problem (CP) defined by (1)-(3).
2. The pseudo-feasible solution $\mathrm{x}$ has negative basic component(s). This case itself has two subcases:

2.1 *There exists* $x_{s_p}$ *such that* $x_{s_p} = \beta_p < 0$, $1 \leq p \leq m$, *and* $\alpha_{pj} \geq 0$, $j = 1, \dots, n$.

We have already established that in this case, the vector $\mathrm{y}' \overset{\text{def}}{=} \mathrm{y} + T\mathrm{e}^{(p)}B^{-1}$ is a feasible solution to the dual problem (4)-(5) for each $T$. Then from (19) with $T \to +\infty$ and $\beta_p < 0$ it follows that the dual objective function $D(\mathrm{y})$ is unbounded below in the feasible set defined by (5), hence the dual problem (4)-(5) has no optimal solution. Then Duality theorem implies that the primal problem (1)-(3) also has no optimal solution in this case.

2.2 *For each negative basic component* $x_{s_i} = \beta_i < 0$ *of* $\mathrm{x}$ *there exists at least one* $\alpha_{ij} < 0$.

Take, for example, the negative basic component $x_{s_p} = \beta_p < 0$. Vector $\mathrm{y} + T\mathrm{e}^{(p)}B^{-1}$ turns out to be a feasible solution to the dual problem (4)-(5) for every $T$, which satisfies (18).

Let

$$T = \frac{\Delta_q}{\alpha_{pq}} = \min_{\alpha_{pj}<0} \frac{\Delta_j}{\alpha_{pj}}.$$

From (15) we obtain

$$\Delta'_q = \Delta_q - T\alpha_{pq} = \Delta_q - \frac{\Delta_q}{\alpha_{pq}}.\alpha_{pq} = 0,$$

that is, the feasible solution $\mathrm{y}'$ satisfies the $q$-th equation of system (5) as equality. According to (15), for this choice of $T$, $\mathrm{y}'$ will satisfy as equalities the equations with numbers $s_1, \dots, s_{p-1}, q, s_{p+1}, \dots, s_m$ of system (5).

The corresponding vectors of constraints $\mathrm{a}_{*s_1}, \ldots, \mathrm{a}_{*s_{p-1}}, \mathrm{a}_{*q}, \mathrm{a}_{*s_{p+1}}$, $\ldots, a_{*s_m}$ are linearly independent. Therefore the feasible solution $y'$ is a feasible corner point for the dual problem (4)-(5) by definition.

Basic vectors of the feasible corner point $\mathrm{y}'$ are obtained by the basic vectors of the feasible corner point $y$ by replacing $\mathrm{a}_{*s_p}$ by $\mathrm{a}_{*s_q}$.

Thus, the move from the feasible corner point $\mathrm{y}$ to the feasible corner point $\mathrm{y}'$ leads to decrease of the value of the objective function $D(\mathrm{y})$, defined by (4):

$$D(\mathrm{y}') = D(\mathrm{y}) + T\beta_p < D(\mathrm{y}),$$

that is, the feasible point $\mathrm{y}'$ is *better* than the feasible corner point $\mathrm{y}$ because the dual problem (4)-(5) is a *minimization* problem. We have taken into consideration that the dual problem (4)-(5) is non-degenerate by assumption, therefore $\beta_i > 0$, $i = 1, \ldots, m$. The choice of $T$ is unique for a non-degenerate problem (4)-(5) because $\Delta_j > 0,\ j \neq s_i,\ i = 1, \ldots, m$.

The move from the feasible corner point $\mathrm{y}$ to the feasible corner point $\mathrm{y}'$ is a move from the pseudo-feasible solution $\mathrm{x}$ to the pseudo-feasible solution $\mathrm{x}'$ with a basis

$$B_{\mathrm{x}'} = \{x_{s_1}, \ldots, x_{s_{p-1}}, x_q, x_{s_{p+1}}, \ldots, x_{s_m}\},$$

such that the value of the dual objective function increases. This move is done through a pivot transformation (pivoting) of system (8) with pivot element $\alpha_{pq}$, using the known formulas of the classical simplex method.

Let a feasible corner point $y$ for the dual problem (4)-(5) be known, therefore a pseudo-feasible solution $x$ to the primal problem (1)-(3) also be known.

**Algorithm of the Dual simplex method for solving the primal problem (1)-(3)**

**1.** Check the signs of the right-hand sides $\beta_i,\ i = 1, \ldots, m$, of system (8).

a) If all $\beta_i \geq 0,\ i = 1, \ldots, m$, then the pseudo-feasible solution $\mathrm{x}$ turns out to be a feasible solution to the primal problem (1)-(3). From the Optimality criterion it follows that $\mathrm{x}$ is an optimal solution to the primal problem (1)-(3). Go to step 5.

b) If there exist $\beta_p < 0$ and $\alpha_{pj} \geq 0,\ j = 1, \ldots, n$, then the dual problem (4)-(5) has no optimal solution because its objective function $D(\mathrm{y})$ is unbounded below in the considered feasible set, defined by (5). From a property of the pair of dual problems it follows that the feasible set of the primal problem (1)-(3) is empty, that is, problem (1)-(3) has no feasible solution, therefore problem (1)-(3) has no optimal solution. Go to step 5.

c) If for each $i$, such that $\beta_i < 0$, there is at least one $\alpha_{ij} < 0$, then the pseudo-feasible solution $\mathrm{x}$ is not a feasible solution of the primal problem (1)-(3). Go to step 2.

**2.** Choose a row $p$ (pivot row) such that $\beta_p < 0$. Go to step 3.

**3.** Choose a column $q$ (pivot column) such that

$$\frac{\Delta_q}{\alpha_{pq}} = \min_{\alpha_{pj}<0} \frac{\Delta_j}{\alpha_{pj}}.$$

Go to step 4.

**4.** Pivot transformation (pivoting) with pivot element $\alpha_{pq}$:

$$\alpha_{pj}{}' = \frac{\alpha_{pj}}{\alpha_{pq}}, j = 1, \dots, n,$$

$$\alpha_{ij}{}' = \alpha_{ij} - \frac{\alpha_{pj}}{\alpha_{pq}} \alpha_{iq}, i = 1, \dots, m, i \neq p; j = 1, \dots, n,$$

$$\beta_p' = \frac{\beta_p}{\alpha_{pq}},$$

$$\beta_i' = \beta_i - \frac{\beta_p}{\alpha_{pq}} \alpha_{iq}, i \neq p, i = 1, \dots, m,$$

$$(-\Delta_0)' = (-\Delta_0) - \frac{\beta_p}{\alpha_{pq}} \Delta_q,$$

$$\Delta_j' = \Delta_j - \frac{\alpha_{pj}}{\alpha_{pq}} \Delta_q, j = 1, \dots, n.$$

Go to step 1.

**5.** End.

When applying the dual simplex method, the simplex table of the classical simplex method is used; the only difference is that the pivot row and the pivot column are selected according to other criteria.

At step 4 of the Algorithm, by superscript “prime”, the new elements of the next simplex table are denoted. For example, $\alpha_{ij}{}'$ is notation of the new element in cell $(i, j)$ of the new simplex table, which replaces the element $\alpha_{ij}$ of the previous simplex tableau. Similarly for other elements of the new simplex table.

**Example 1:** Solve the linear program

(LP)

$$\max\{L(\mathrm{x}) \stackrel{\text{def}}{=} -x_1 - 4x_2\}$$

subject to

$$\begin{array}{rrcr} x_1 & -3x_2 & \leq & -6 \\ -2x_1 & +x_2 & \leq & 6 \\ 2x_1 & -x_2 & \leq & -1 \\ 2x_1 & +x_2 & \leq & 20 \end{array}$$

$$x_1 \geq 0, x_2 \geq 0.$$

**Solution:** Transform the given problem into canonical form:

(CP)

$$\max\{\overline{L}(\overline{\mathrm{x}}) \overset{\text{def}}{=} -x_1 - 4x_2\}$$

subject to

$$\begin{array}{cccccccc} x_1 & -3x_2 & +x_3 & & & & = & -6 \\ -2x_1 & +x_2 & & +x_4 & & & = & 6 \\ 2x_1 & -x_2 & & & +x_5 & & = & -1 \\ 2x_1 & +x_2 & & & & +x_6 & = & 20 \end{array}$$

$$x_j \geq 0, j = 1, \dots, 6.$$

The dual problem of this canonical problem is
(DCP)

$$\min\{D(y) \overset{\text{def}}{=} -6y_1 + 6y_2 - y_3 + 20y_4\}$$

subject to

$$\begin{array}{cccccc} y_1 & -2y_2 & +2y_3 & +2y_4 & \geq & -1 \\ -3y_1 & +y_2 & -y_3 & +y_4 & \geq & -4 \end{array}$$

$$y_i \geq 0, i = 1,2,3,4.$$

The (DCP) has a feasible corner point $\mathrm{y}^{(0)} = (0,0,0,0)$, which satisfies the nonnegativity requirements $y_i \geq 0,\ i = 1,2,3,4$ as equalities and whose basis is $B_{\mathrm{y}^{(0)}} = \{\mathrm{a}_{*3}, \mathrm{a}_{*4}, \mathrm{a}_{*5}, \mathrm{a}_{*6}\}$. The initial pseudo-feasible solution of (CP) is $\overline{\mathrm{x}}^{(0)} = (0,0,-6,6,-1,20)$, with basic variables $x_3, x_4, x_5, x_6$. The objective function value for $\overline{\mathrm{x}}^{(0)}$ is $\overline{L}(\overline{\mathrm{x}}^{(0)}) = 0$.

Construct the initial simplex table and the next one (see Table 1).

**Table 1**
**Simplex table for Example 1.**

| Basis | $\mathbf{c}_B$ | $\beta$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|---|
| | | | $-1$ | $-4$ | $0$ | $0$ | $0$ | $0$ |
| $x_3$ | $0$ | $-6$ | $1$ | $\mathbf{-3}$ | $1$ | $0$ | $0$ | $0$ |
| $x_4$ | $0$ | $6$ | $-2$ | $1$ | $0$ | $1$ | $0$ | $0$ |
| $x_5$ | $0$ | $-1$ | $2$ | $-1$ | $0$ | $0$ | $1$ | $0$ |
| $x_6$ | $0$ | $20$ | $2$ | $1$ | $0$ | $0$ | $0$ | $1$ |
| $\Delta$ | | $0$ | $-1$ | $-4$ | $0$ | $0$ | $0$ | $0$ |
| $x_2$ | $-4$ | $2$ | $-\frac{1}{3}$ | $1$ | $-\frac{1}{3}$ | $0$ | $0$ | $0$ |
| $x_4$ | $0$ | $4$ | $-1$ | $0$ | $\frac{1}{3}$ | $1$ | $0$ | $0$ |
| $x_5$ | $0$ | $1$ | $1$ | $0$ | $-\frac{1}{3}$ | $0$ | $1$ | $0$ |
| $x_6$ | $0$ | $18$ | $3$ | $0$ | $\frac{1}{3}$ | $0$ | $0$ | $1$ |
| $\Delta$ | | $8$ | $-\frac{7}{3}$ | $0$ | $-\frac{4}{3}$ | $0$ | $0$ | $0$ |

**Explanations:** The initial pseudo-feasible solution $\mathrm{x}^{(0)}$ of (CP) is not an optimal solution of (CP) because $x_3^{(0)} = -6 < 0,\ x_5^{(0)} = -1 < 0$.

In the first simplex table, either the first or the third row can be selected as a pivot row because $\beta_3 = -6 < 0$, $\beta_5 = -1 < 0$. We choose, for example, the first row, that is, the row of $x_3$. Since only one element of this row is negative: $-3$, then the pivot column and, therefore, the pivot element $\alpha_{12} = -3$ is uniquely determined.

After pivot transformation we obtain the second simplex table, with associated pseudo-feasible solution $\mathrm{x}^{(1)} = (0,2,0,4,1,18)$. Since all its components are nonnegative, then $\mathrm{x}^{(1)}$ turns out to be an optimal solution to the canonical problem (CP) according to the Optimality criterion.

Therefore, after eliminating the so-called slack variables $x_3, x_4, x_5, x_6$ in $\mathrm{x}^{(1)}$, the optimal solution to the given linear program (LP) is $\mathrm{x}^* = (0,2)$, and the optimal value of the objective function of (LP) is $L_{\max} = L(\mathrm{x}^*) = -8$.

**Conclusion**

Some theoretical issues about the dual simplex method are discussed in this paper, an algorithm for solving linear optimization problems by this method is considered, and illustrative numerical example is presented.

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