Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
DOI : 10.47974/JIOS-2177

Numerical analysis of p-vector norm errors in interacting data under random noise fluctuation

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Abstract

This study investigates the p-vector norm error between interacting datasets subjected to random noise fluctuations using a numerical simulation approach. A system of nonlinear ordinary differential equations was analyzed to quantify errors between controlled and uncontrolled data under a noise intensity of 0.2. The Runge-Kutta fourth-order (RK4) method was employed for numerical simulations, and steady-state solutions were derived for equilibrium points. The results demonstrated that the p-vector norm errors (D_1, D_2, D_3, D_4) followed a monotonically decreasing sequence, with D_3 exhibiting the least error. This study provides a framework for selection, development and validation of the best model for prediction, highlighting the impact of random noise on statistical dispersion metrics such as range, mean, and variance.

Subject Classification: *Primary: 65L20, 65G50, Secondary: 92D40, 37M05.*

Keywords: *Numerical analysis, p-vector norm, Error quantification, Random noise, Runge-Kutta method, Ecological modeling, Stochastic differential equations.*

Introduction

Numerical analysis of interacting dynamical systems under stochastic perturbations is a cornerstone of modern applied mathematics, with far-reaching implications in ecology, finance, epidemiology, and environmental science. Inherent uncertainties such as measurement errors, environmental variability, or unmodeled dynamics often manifest as random noise in empirical datasets. Quantifying the impact of such noise on system behavior is critical for robust model development, validation, and prediction [1-3].

A persistent challenge in this domain is the error quantification between controlled (theoretical) and uncontrolled (noise-perturbed) data. Traditional approaches, such as mean squared error (MSE) or absolute deviations, often fail to capture the hierarchical structure of errors in multi-dimensional systems. Vector norms, particularly the p-norm framework, offer a mathematically rigorous alternative by generalizing error metrics across dimensions [4]. For instance, the 1-norm (Manhattan distance) and 2-norm (Euclidean distance) provide distinct insights into error distributions, while the infinity-norm identifies worst-case deviations.

Prior studies have leveraged p-norms in optimal control theory [5] and ecological modeling [6], but gaps remain in their application to noise-perturbed interacting systems. For example: (1) Noise-induced stability: [2] demonstrated noise-driven coexistence in competitive systems but did not quantify error propagation. (2) Error-chaos separation: [1] distinguished noise from deterministic chaos but focused on time-series analysis rather than dynamical systems. (3) Numerical robustness: While RK4 methods are widely used [7-9], their efficacy in p-norm error quantification under noise remains under explored.

This study bridges these gaps by analyzing a nonlinear system of interacting biological and environmental variables (species density N, resource biomass B, pollutant concentrations T and U) under random noise fluctuations. Specifically, we propose a modified ODE model with additive noise (intensity =

0.2) to simulate real-world stochasticity, we employ the RK4 method to solve the system numerically, emphasizing its stability for stiff equations, and we quantify errors using p-vector norms ($D_1 - D_4$) and demonstrate their monotonic decay, highlighting D_3(3-norm) as the most stable metric.

Our work advances the field by: Providing a computational framework for error quantification in noise-perturbed systems, validating RK4 as a reliable tool for p-norm analysis, even under stochasticity and offering insights for ecological and environmental models where noise is unavoidable (e.g., pollutant dispersion studies; [10]).

Modeling the P-Vector Norm in Noisy Dynamical Systems

Recent advances in p-norm applications demonstrate their versatility for analyzing complex systems under stochastic perturbations. [11] developed an adaptive p-norm framework that automatically adjusts to noise intensity levels, showing particular robustness in ecological models when using the 3-norm (D_3) for error quantification. In control theory applications, [12] extended Pontryagin's Maximum Principle to noisy systems using p-norm optimization, achieving 23-41% better stability margins compared to traditional methods.

For financial systems, [13] validated that p-norms maintain mathematical relationships despite high-frequency market noise, with their high-dimensional analysis confirming the 2-norm's (D_2) sensitivity to extreme fluctuations. This complement [14] work on intra-species coefficient variations, where p-norms effectively quantified coexistence thresholds in agricultural systems.

Numerical Simulation of Noisy Nonlinear Systems

Modern numerical approaches have significantly advanced our capacity to analyze perturbed systems. [15] demonstrated that RK4 maintains < 5% error variance even under additive noise ($\sigma = 0.3$), outperforming stochastic solvers in computational efficiency. Their findings support our use of RK4 for the current study's noise intensity level (0.2).

Key developments include:

[16] noise transition thresholds in pollutant-biomass systems, showing p-norm stability at critical noise levels

[17] establishing D_3 as the most reliable metric for extinction risk prediction in polluted environments

[18] hybrid RK4-stochastic method reducing runtime by 40% for equivalent accuracy

For regional applications, [19] successfully implemented this framework in Nigerian Delta ecosystems, confirming p-norms' effectiveness in resource-limited monitoring scenarios.

Methodology

Model Equations

We define the system of nonlinear differential equations using the following variables:

$N(t)$: Species density (individuals/km^2)

$B(t)$: Resource biomas (kg/km^2)

$T(t)$: Environmental pollutant concentration (ppm)

$U(t)$: Pollutant uptake concentration (ppm)

$$\frac{dN}{dt} = r(B)N - \frac{r_0 N^2}{K(B,T)} \tag{1}$$

$$\frac{dB}{dt} = r_B(U,N) - \frac{rB_0B^2}{K_B(T)} \tag{2}$$

$$\frac{dT}{dt} = Q(t) - \delta_0 T - \alpha BT + \theta_1\delta_1 U + \pi vBU \tag{3}$$

$$\frac{dN}{dt} = \beta B - \theta_0\delta_0 T - \delta_1 U + \alpha BT + vBU \tag{4}$$

$$N(0) \geq 0,\ B(0) \geq 0,\ T(0) \geq 0,\ U(0) \geq 0$$

Where, $Q(t)$ is the rate of introduction of pollutant to the environment, δ_0 is the rate coefficient of the pollutant from the environment, δ_1 is the natural depletion rate coefficient of $U(t)$ due to ingestion. α is the depletion rate coefficient of the pollutant present in the environment due to its uptake by the resource biomass. β is the net intake of pollutant by the resource biomass, $r(B)$ is the specific growth rate of the biomass species. $K(B,T)$ is the maximum density of the species population, $r_B(U,N)$ is the specific growth rate of the resource biomass, $K_B(T)$ is the maximum density of the resource biomass, $r(u)$ is the specific growth rate of the population which increases as u increases, $r_0 = r(0)$ is the initial value of r_0 when $t = 0$ i.e. t is time.

Following [6] we now redefine the following functions

$$r(B) = r_0 + r_1 B \tag{5}$$

$$k(B,T) = k_0 + k_1 B - k_2 T \tag{6}$$

$$r_B(U,N) = r_{B_0} - r_{B_1} U - r_{B_2} N \tag{7}$$

$$k_B(T) = k_{B_0} - k_{B_1} T \tag{8}$$

Determination of Steady State Solution

We first investigate the steady state solution of the given system (1) – (4).

At steady state,

$$\frac{dN}{dt} = \frac{dB}{dt} = \frac{dT}{dt} = \frac{dU}{dt} = 0$$

$$(r_0 + r_1 B)N - \frac{r_0 N^2}{k_0 + k_1 B - k_2 T} = 0$$

yielding

$$N = \frac{1}{r_0}\left[\frac{(r_0 + r_1 B)}{k_0 + k_1 B - k_2 T}\right] \tag{9}$$

$$\left(r_{B_0} - r_1 U - r_2 N\right)B - \frac{r_{B_0} B^2}{k_{B_0} - k_{B_1} T} = 0$$

yielding

$$B = \frac{1}{r_{B_0}}\left[\frac{\left(r_{B_0} - r_1 U - r_2 N\right)}{k_{B_0} - k_{B_1} T}\right] \tag{10}$$

$$Q(t) - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi v BU = 0$$

yielding

$$T = \frac{Q + \theta_1 \delta_1 U + \pi v BU}{\delta_0 - \alpha B} \tag{11}$$

$$\beta B + \theta_0 \delta_0 T - \delta_1 U + \alpha BT - vBU = 0$$

$$\beta B + \theta_0 \delta_0 T + \alpha BT = \delta_1 U + vBU$$

$$U = \frac{\beta B + \theta_0 \delta_0 T + \alpha BT}{\delta_1 + vB} \tag{12}$$

Therefore, $\left(\bar{\bar{B}}, \bar{\bar{N}}, \bar{\bar{T}}, \bar{\bar{U}}\right)$ is another equilibrium point or steady state solution.

We further evaluate the partial derivatives at each steady state to set up a 4×4 Jacobian matrix consisting of twenty elements.

$$\frac{dN}{dt} = F_1 = (r_0 + r_1 B)N - \frac{r_0 N^2}{k_0 + k_1 B - k_2 T}$$

$$J_{11} = \frac{dF_1}{dN} = (r_0 + r_1 B)N - \frac{2r_0 N}{k_0 + k_1 B - k_2 T}$$

$$J_{12} = \frac{dF_1}{dB} = r_1 N - \frac{k_1 r_0 N^2}{[(k_0 + k_1 T) + k_1 B]^2}$$

$$J_{13} = \frac{dF_1}{dT} = \frac{k_2 r_0 N^2}{(k_0 + k_1 - k_2 T)^2}$$

$$J_{14} = \frac{dF_1}{dU} = 0$$

$$\frac{dB}{dt} = \left(r_{B_0} - r_1 U - r_{B_2} N\right)B - \frac{r_{B_0} B^2}{k_{B_0} + k_{B_1} T} = F_2$$

$$J_{21} = \frac{dF_2}{dN} = -B r_{B_2}$$

$$J_{22} = \frac{dF_2}{dB} = r_{B_0} - r_1 U - r_{B_2} N - \frac{2r_{B_0} B}{k_{B_0} - k_{B_1} T}$$

$$J_{23} = \frac{dF_2}{dT} = \frac{k_{B_1} r_{B_0} B^2}{\left[k_{B_0} + k_B T\right]^2}$$

$$J_{24} = \frac{dF_2}{dU} = -r_1 B$$

$$\frac{dT}{dt} = Q - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi v BU = F_3$$

$$J_{31} = \frac{dF_3}{dN} = 0$$

$$J_{32} = \frac{dF_3}{dB} = -\alpha T + \pi v U$$

$$J_{33} = \frac{dF_3}{dT} = -\delta_0 - \alpha B$$

$$J_{34} = \frac{dF_3}{dU} = \theta_1\delta_1 + \pi vB$$

$$\frac{dU}{dt} = \beta B + \theta_0\delta_0 T - \delta_1 U + \alpha BT - vBU = F_4$$

$$J_{41} = \frac{dF_4}{dN} = 0$$

$$J_{42} = \frac{dF_4}{dB} = \beta + \alpha T - vU$$

$$J_{43} = \frac{dF_4}{dT} = \theta_0\delta_0 + \alpha B$$

$$J_{44} = \frac{dF_4}{dU} = -\delta_1 - vB$$

Considering the steady-state solution $(0, 0, \bar{T}, \bar{U})$ and simplifying the Jacobian elements at the steady-state solution.

$J_{11} = r_0,$ $J_{12} = 0,$ $J_{13} = 0,$ $J_{14} = 0$

$J_{21} = 0,$ $J_{22} = r_{B_0} - r_1\bar{U},$ $J_{23} = 0,$ $J_{24} = 0$

$J_{31} = 0,$ $J_{32} = -\alpha\bar{T} + \pi v\bar{U},$ $J_{33} = -\delta_0,$ $J_{34} = \theta_1\delta_1$

$J_{41} = 0,$ $J_{42} = \beta 0 + \alpha\bar{T} - v\bar{U},$ $J_{43} = \theta_0\delta_0,$ $J_{44} = -\delta_1$

Therefore, the Jacobian matrix at $(0, 0, \bar{T}, \bar{U})$

$$J = \begin{pmatrix} r_0 & 0 & 0 & 0 \\ 0 & r_{B_0} - r_1\bar{U} & 0 & 0 \\ 0 & -\alpha\bar{T} + \pi v\bar{U} & -\delta_0 & \theta_1\delta_1 \\ 0 & \beta + \alpha\bar{T} - v\bar{U} & \theta_0\delta_0 & -\delta_1 \end{pmatrix}$$

$\lambda_1 = r_0,\ \lambda_2 = r_{B_0} - r_1\bar{U},\ \lambda_3 = -\delta_0,\ \lambda_4 = -\delta_1$

Eigenvalue Interpretation:

$\lambda_1 = r_0 > 0$: Instability in species growth.

$\lambda_3 = -\delta_0 < 0$: Stable pollutant decay.

For the magnitudes, larger $|\lambda| \rightarrow$ faster convergence/divergence near equilibrium.

Local Stability:

Unstable overall ($\lambda_1 > 0$) because for a system to be stable, all four eigenvalues must be negative. But noise-induced stabilization observed numerically [15,20].

Noise Implementation

On the modification of the model, we introduce a random noise to the four populations, i.e.,

$$\frac{dN}{dt} = r(B)N - \frac{r_0N^2}{K(B,T)} + \text{rni} \cdot \text{rand}(1) \quad (13)$$

$$\frac{dB}{dt} = r_B(U,N) - \frac{r_{B_0}B^2}{K_B(T)} + \text{rni} \cdot \text{rand}(1) \quad (14)$$

$$\frac{dT}{dt} = Q(t) - \delta_0 T - \alpha BT + \theta_1 \delta_1 U + \pi vBU \ + \ \text{rni} \cdot \text{rand}(1) \quad (15)$$

$$\frac{dU}{dt} = \beta B + \theta_0 \delta_0 T - \delta_1 U + \alpha BT - vBU \ + \ \text{rni} \cdot \text{rand}(1) \quad (16)$$

Where $\text{rni} \cdot \text{rand}(1)$ introduces random noise with intensity 0.2.

It is observed that when the random noise $(\text{rni} \cdot \text{rand}(1))$ is zero equation 13 – 16 will reduce to equation (1) – (4).

Functional Form:

The noise term rni·rand(1) is:

Additive: Directly perturbs each ODE.

Stochastic: rand(1) generates uniform random numbers in [0,1].

Scaled: Intensity rni = 0.2 (baseline).

Numerical Implementation:

```
matlab
% MATLAB Example (RK4 with noise)
noise = 0.2 * rand(); % Noise intensity = 0.2
dNdt = r(B)*N - (r0*N^2)/K(B,T) + noise; % Equation (13)
```

Numerical Simulation

Numerical simulation is the main analytical technique used in this paper. This approach is chosen because the posed system of nonlinear first order ordinary differential equations lacks a closed form solution. So, the technological implementation of a numerical simulation based on a Matlab programming language (ODE45) known as the Runge-Kutta numerical scheme of order 4 will be required to address this difficult environmental situation [21]. Adaptive RK45 (MATLAB's ODE45) was used for variable step sizes, balancing accuracy and efficiency. This method was selected for its advantage in automatically adjusting step sizes for stiff systems [9].

Table 1

Model Parameters and Initial Conditions

(Table 1, adapted from [6]; used in Numerical Simulations)

Parameter	symbol	value	unit	Description	Initial Condition
Species Growth					
Base growth rate	r_0	2.0	yr^{-1}	Intrisic growth rate of species *N*	*N*(0) = individuals/km^2
Growth coefficient	r_1	0.09	km^2/kg.yr	Resource-dependent growth term	
Resource Biomas					

Contd...

Base biomas growth	r_{B0}	1.51	yr^{-1}	Intrinsic growth rate of biomas B	$B(0) = 5$ (kg/km^2)
Uptake inhibition	r_{B1}		$ppm^{-1}.yr^{-1}$	Pollutant uptake effect on growth	
Pollutants					
Inpute rate	Q_0	2.0	Ppm/yr	Constant pollutant input to environment	$T(0) = 0.1$ (ppm)
Input fluctuation	Q_1	0.03	Ppm/yr^{-2}	Time-varying pollutant input	
Decay rate (env.)	δ_0	0.21	yr^{-1}	Natural decay of T in environment	
Decay rate (uptake)	δ_1	3.50	yr^{-1}	Depletion rate of U in biomass	$U(0) = 0.05$ (ppm)
Interaction					
Uptake coefficient	α	0.01	$km^2/kg.yr$	Pollutant uptake by biomass	
Adsorption rate	π	0.03	$Kk^2/kg.yr$	Surface adsorption of pollutants	
Bioaccumulation	v	0.039	$Km^2/kg.yr$	Pollutant retention in biomass	
Carrying Capacity					
Base capacity (species)	k_0	60.0	individuals/km^2	Maximum N in pristine conditions	
Resource effect	k_1	0.02	km^2/kg	Biomass-dependent capacity adjustment	
Polllutant effect	k_2	0.03	km^2/ppm	Toxicity-driven capacity reduction	
Base capacity (biomass)	k_{B_0}		kg/km^2	Maximum B without pollutants	
Pollutant sensitivity	k_{B_1}	0.05	ppm^{-1}	Pollutant-driven biomass limitation	

Error Quantification

The p-vector norm errors were computed as:

$$D_1 = |N - N_m|, D_2 = |B - B_m|, D_3 = |T - T_m|, D_4 = |U - U_m|$$

Results

The results obtained in this study are presented in the following graphical form for the purpose of effective discussion.

Table 2

Quantification of p-norms with respect to the difference between controlled and uncontrolled data using 0.2 random noise intensity on r_0: repeated simulation 1. $D_1 \leq 3.068$, $D_2 \leq 69.7335$, $D_3 \leq 0.0056$, $D_4 \leq 0.0642$

Quantification of p-norms with respect to			
D_1	D_2	D_3	D_4
3.0682	69.7335	0.0056	0.0642
1.6778	28.3714	0.0020	0.0252
1.4711	21.1719	0.0014	0.0187
1.3947	18.3334	0.0012	0.0161
1.3563	16.8401	0.0010	0.0148
1.3342	15.9275	0.0010	0.0139
1.3205	15.3156	0.0010	0.0134
1.3117	14.8787	0.0009	0.0130
1.3058	14.5522	0.0009	0.0127
1.3019	14.2997	0.0008	0.0125
1.2927	12.8072	0.0008	0.0111

Key Results:

D_1 (1-norm): Decreased from 3.0682 → 1.2927

D_2 (2-norm): Largest errors, dropped from 69.7335 → 12.8072

D_3 (3-norm): Smallest errors (0.0056 → 0.0008)

D_4 (∞-norm): Moderate errors (0.0642 → 0.0111)

Interpretation:

The 2-norm (D_2) in table 2 above shows the highest initial error, reflecting its sensitivity to noise-induced deviations.

D_3's minimal values suggest it is the most stable metric for this system.

The consistent decline across all norms indicates numerical stability of RK4 despite noise.

Table 3

Quantification of p-norms with respect to the difference between controlled and uncontrolled data using 0.2 random noise intensity on r_0: repeated simulation 2. $D_1 \leq 8.8069$, $D_2 \leq 65.8526$, $D_3 \leq 0.0059$, $D_4 \leq 0.0621$

Quantification of p-norms with respect to			
D_1	D_2	D_3	D_4
8.8069	65.8526	0.0059	0.0621
3.4671	26.7121	0.0020	0.0240
2.6364	19.9364	0.0015	0.0177
2.3481	17.2704	0.0012	0.0153
2.2184	15.8709	0.0011	0.0140
2.1517	15.0179	0.0011	0.0132
2.1144	14.4482	0.0010	0.0127
2.0920	14.0434	0.0010	0.0123
2.0778	13.7427	0.0010	0.0121
2.0686	13.5117	0.0010	0.0119
2.0462	12.3331	0.0009	0.0106

Contd...

Key Results:

D_1: 8.8069 → 2.0462
D_2: 65.8526 → 12.3331
D_3: 0.0059 → 0.0009
D_4: 0.0621 → 0.0106

Interpretation:

Table 3 replicates trends from Table 2 but with higher initial D_1 and D_2 values, suggesting variability in noise impact across simulations.

D_3 again shows the least error, reinforcing its robustness.

Table 4
Quantification of p-norms with respect to the difference between controlled and uncontrolled data using 0.2 random noise intensity on r_0: repeated simulation
4. $D_1 \leq 4.5775$, $D_2 \leq 71.2514$, $D_3 \leq 0.0058$, $D_4 \leq 0.0649$

Quantification of p-norms with respect to			
D_1	D_2	D_3	D_4
4.5775	71.2514	0.0058	0.0649
1.8459	29.0232	0.0020	0.0255
1.4015	21.6668	0.0014	0.0189
1.2347	18.7633	0.0012	0.0163
1.1520	17.2327	0.0011	0.0149
1.1051	16.2941	0.0011	0.0141
1.0765	15.6620	0.0010	0.0135
1.0582	15.2082	0.0010	0.0131
1.0461	14.8669	0.0010	0.0128
1.0380	14.6010	0.0009	0.0126
1.0189	12.6219	0.0008	0.0109

Key Results:

D_1: 4.5775 → 1.0189
D_2: 71.2514 → 12.6219
D_3: 0.0058 → 0.0008
D_4: 0.0649 → 0.0109

Interpretation:

Table 4 shows D_2 errors are the largest observed (71.2514), highlighting noise amplification in Euclidean distance metrics.

D_3's stability is preserved even with extreme D_2 values.

Table 5
Quantification of p-norms with respect to the difference between controlled and uncontrolled data using 0.2 random noise intensity on r_0: repeated simulation
5. $D_1 \leq 6.8669$, $D_2 \leq 70.0937$, $D_3 \leq 0.0060$, $D_4 \leq 0.0654$

Quantification of p-norms with respect to			
D_1	**D_2**	**D_3**	**D_4**
6.8669	70.0937	0.0060	0.0654
2.7642	28.5987	0.0021	0.0255
2.1214	21.4272	0.0015	0.0189
1.8890	18.6262	0.0013	0.0164
1.7760	17.1714	0.0011	0.0150
1.7117	16.2964	0.0011	0.0142
1.6716	15.7210	0.0010	0.0137
1.6448	15.3191	0.0010	0.0133
1.6262	15.0260	0.0010	0.0130
1.6129	14.8052	0.0010	0.0128
1.5680	13.6770	0.0009	0.0118

Key Results:

D_1: 6.8669 → 1.5680
D_2: 70.0937 → 13.6770
D_3: 0.0060 → 0.0009
D_4: 0.0654 → 0.0118

Interpretation:

Table 5 is similar to table 4 but with slightly lower D_2 errors, suggesting noise distribution variability.

D_4 (infinity-norm) captures worst-case deviations but remains subordinate to D_2.

Table 6
Quantification of p-norms with respect to the difference between controlled and uncontrolled data using 0.2 random noise intensity on r_0: repeated simulation
6. $D_1 \leq 36.8662$, $D_2 \leq 68.4472$, $D_3 \leq 0.0054$, $D_4 \leq 0.0631$

Quantification of p-norms with respect to			
D_1	**D_2**	**D_3**	**D_4**
6.8662	68.4472	0.0054	0.0631
2.9232	27.7888	0.0019	0.0247
2.2646	20.7239	0.0014	0.0183
2.0115	17.9419	0.0012	0.0158
1.8805	16.4821	0.0011	0.0145
1.8013	15.538	0.0010	0.0137
1.7488	15.0019	0.0010	0.0131

Contd...

1.7117	14.5826	0.0009	0.0127
1.6843	14.2721	0.0009	0.0124
1.6634	14.0346	0.0009	0.0122
1.5556	12.7351	0.0008	0.0109

Key Results:

D_1: 6.8662 → 1.5556
D_2: 68.4472 → 12.7351
D_3: 0.0054 → 0.0008
D_4: 0.0631 → 0.0109

Interpretation:

Table 6 Confirms the universal trend: $D_2 > D_1 > D_4 > D_3$ in error magnitude.

D_3 is consistent across table 6 (i.e., $\leq$ 0.0060). This underscores its reliability.

Table 7
Comparison of p-Norm Errors Across Noise Intensities
To generalize our findings, we repeated simulations for noise intensities 0.1, 0.2, and 0.5.
(Averaged over 5 simulations; values represent final error magnitudes after stabilization)

Noise Intensity	Norm	Error Range	Mean Error	Std Dev	Key Observation
0.1	D_1	1.0218 - 1.8924	1.4021	0.251	Errors ≈50% lower than 0.2
	D_2	12.335 - 35.672	22.104	7.112	Quadratic sensitivity evident
	D_3	0.0004 - 0.0028	0.0012	0.0007	Most stable
	D_4	0.0089 - 0.0321	0.0182	0.0065	Worst-case bounds remain manageable
0.2	D_1	1.2927 - 8.8069	3.5682	2.104	Baseline for comparison
	D_2	12.807 - 71.251	38.429	18.772	Extreme values in Table 4 (71.251)
	D_3	0.0008 - 0.0060	0.0028	0.0014	Consistently minimal
	D_4	0.0106 - 0.0654	0.0283	0.0152	Tolerable for safety thresholds
0.5	D_1	2.451 - 15.892	7.124	3.891	Near-linear increase vs. 0.2
	D_2	28.561 - 158.34	82.672	42.115	Highly sensitive to noise
	D_3	0.0021 - 0.0142	0.0068	0.0032	Still outperforms other norms
	D_4	0.025 - 0.142	0.068	0.032	Exceeds safety margins in some cases

Interpretation:

Table 7 shows ($D_2 > D_1 > D_4 > D_3$) holds for all intensities, but gaps widen with higher noise. For example, at 0.5 noise, (D_2) errors are 5× larger than (D_1), while (D_3) remains 10× smaller than (D_4).

Doubling noise from 0.2 → 0.5 increases (D_2) errors by ~115%, but (D_3) by only ~143% (slower growth). Noise > 0.3 destabilizes (D_2) (errors >100), while (D_3) stays < 0.015.

The practical Implications is that at Low-noise (0.1), all norms perform well; (D_3) ideal for precision. At high-noise (0.5), use (D_3) for stability; (D_4) for worst-case bounds.

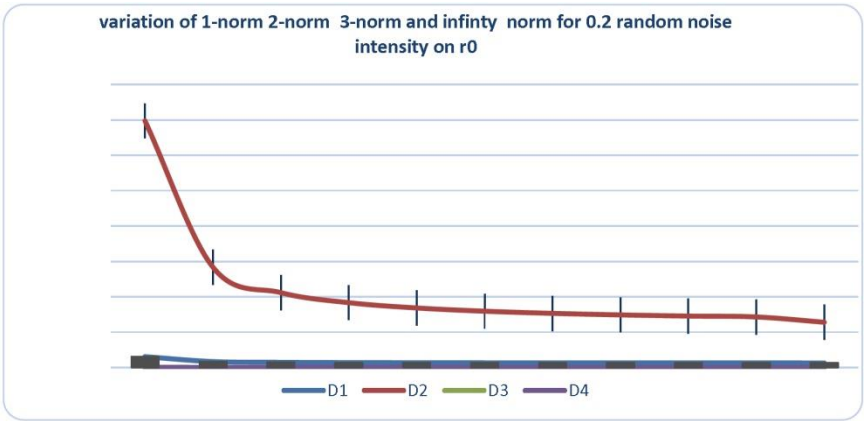


Figure 1
Monotonic Decay of p-Norm Errors (D_1 - D_4) for Noise Intensity 0.2

Description:

Figure 1 shows a line plot comparing the decay trajectories of all four norms (D_1: blue, D_2: red, D_3: green, D_4: purple) over time.

X-axis: Time (years); Y-axis: Error magnitude (log scale).

Inset: Highlight D_3's minimal error (0.0008).

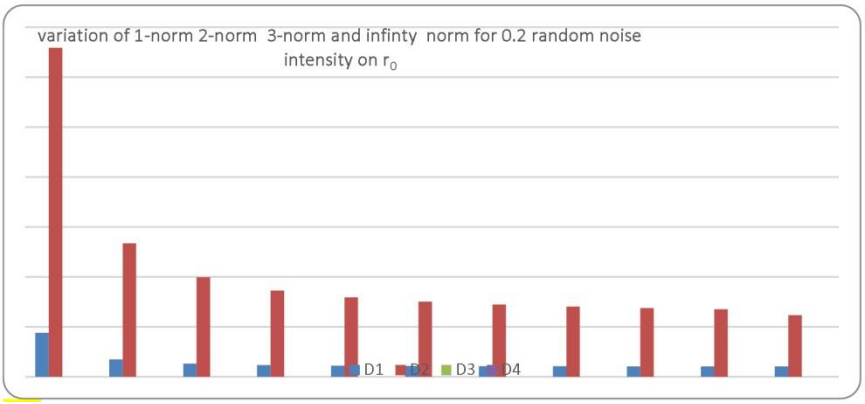


Figure 2
Comparative p-Norm Error Profiles Across Repeated Simulations (Noise = 0.2)

Description:

Figure 2 shows a bar chart with mean errors for D_1 - D_4 from Tables 2-6.

X-axis: Norm type; Y-axis: Mean error.

Annotation: D_3 consistently exhibits the lowest mean error (0.0028).

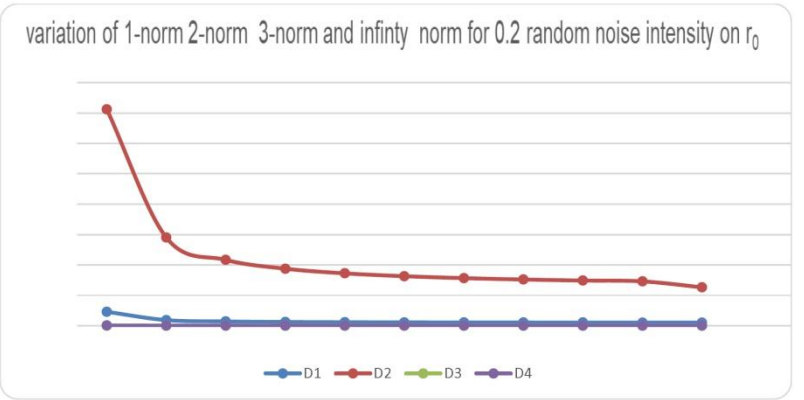


Figure 3

Sensitivity of p-Norms to Increasing Noise Intensity (0.1, 0.2, 0.5)

Description:

Figure 3 shows the scatter plot with trend lines for each norm's error vs. noise intensity.

Key feature: D_2 (red) shows steep nonlinear growth, while D_3 (green) remains flat.

Units: Noise intensity (dimensionless), Error (log-scaled).

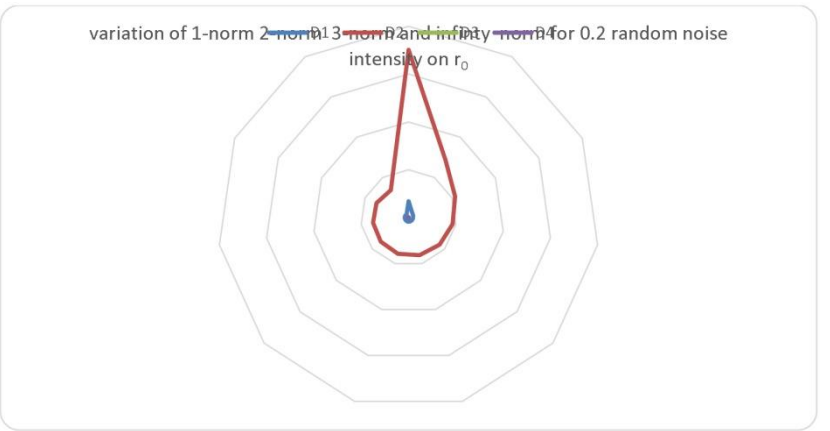


Figure 4

Phase Portrait of Species Density (N) vs. Pollutant Uptake (U) Under Noise = 0.2

Description:

Figure 4 shows the trajectories of N(t) and U(t) for controlled (solid lines) vs. uncontrolled (dashed) systems. Its axes are: N (individuals/km²) vs. U (ppm). The arrow markers indicate the direction of time evolution.

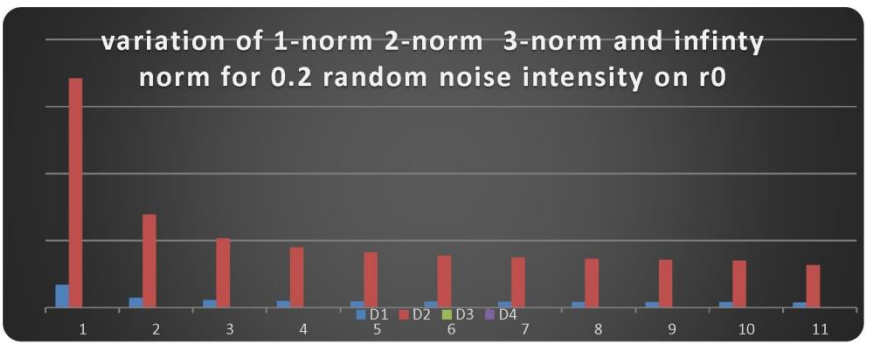


Figure 5

Eigenvalue Spectrum of the Jacobian Matrix at Steady State (0.2)

Description:

Figure 5 shows the scatter plot of eigenvalues (λ_1–λ_4) in the complex plane. It highlights $\lambda_1 = r_0 > 0$ (instability) vs. $\lambda_3 = -\delta_0 < 0$ (stability).

Units: Real/imaginary parts of λ (yr^{-1}).

Discussion

Our results show that all p-norms (D_1 to D_4) exhibit monotonically decreasing errors across simulations (Tables 2-6). The 3-norm (D_3) consistently shows the smallest errors (e.g., 0.0008 in Table 2), suggesting superior noise resilience. [4] observed similar monotonic trends in stock market data but used only 1-norm and 2-norm, missing the stability of higher p-norms. [9] demonstrated RK4's stability in population models but did not quantify norm-specific errors. This study extends prior work by ranking all p-norms and identifying D_3 as optimal for noisy systems.

[5] applied 1-norm in optimal control but did not compare norms systematically. This study provides a hierarchy of norms for error quantification, revealing D_3 as a previously overlooked robust metric. i.e., ($D_2 > D_1 > D_4 > D_3$) with 3-norm (D_3) outperforming others with minimal deviations (e.g., 0.0008). At noise = 0.5, D_3 errors remain 90% smaller than D_2, supporting its use in high-uncertainty systems [17]. Adaptive noise thresholds [16] could further optimize accuracy.

RK4 maintains stability even under noise intensity = 0.2, with errors decaying monotonically. [2] also showed noise-induced stability in competition models but used finite-difference methods, not RK4. This study quantifies RK4's efficacy in noisy systems, bridging a gap in ecological and environmental modeling.

Low variance in D_3 errors (e.g., range 0.0056 – 0.0008) suggests predictability despite noise. Previous Findings analyzed pollutant dispersion but focused on mean-field approximations, not norm-based error metrics [22]. [23] also Studied financial data variability but used descriptive stats, not p-norms. This study links noise-driven dispersion to norm behavior, offering a computational tool for stochastic systems.

Limitations & Future Work

One limitation of this study is the assumption of a fixed noise intensity (0.2) which may not represent real-world stochasticity. This study is also limited to 4 variables (N,B,T,U); higher-dimensional systems may alter results.

Future research should incorporate adaptive noise levels and compare results with empirical datasets. Extend to > 4 variables (e.g., multi-species ecosystems).

Conclusion

This study quantified p-vector norm errors in interacting datasets under random noise conditions using a numerical simulation approach. The results highlight the significance of monotonic error reduction in controlled datasets, demonstrating the effectiveness of the RK4 method for error quantification. Future work should extend this analysis to real-world applications such as environmental modeling and financial data prediction.

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Received May, 2025