Journal of Information & Optimization Sciences
ISSN 0252-2667 (Print), ISSN 2169-0103 (Online)
DOI : 10.47974/JIOS-2037

Mathematical analysis of novel soliton solutions to a nonlinear wave equation

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Abstract

This paper seeks to develop comprehensive soliton solutions by employing the generalized modified equal width (MEW) equation as the governing framework. To achieve that, we have chosen the (δ'/δ)-expansion approach, which is an effective, reliable, and compatible method. The outcomes achieved in this instance are defined in terms of a mix of rational, trigonometric, and hyperbolic functions. The corresponding physical features

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of these resultant soliton solutions and influence of parameters are presented by carving two-dimensional and three-dimensional and contour graphs by means of MATLAB. For estimated values of these parameters, bell-shaped soliton, singular soliton, parabolic soliton, periodic soliton and solitons of other types are formulated. By examining the solutions across carefully chosen parameter regimes, the work offers a clearer perspective on the system's complex behavior. The results introduce new soliton phenomena and uncover previously unreported aspects of the governing equation.

Subject Classification: *35C07, 35C08, 35C09, 35B35.*

Keywords: *Nonlinear evolution equations (NLEEs), Generalized modified equal width (MEW) equation, (δ'/δ)-expansion approach, Soliton solutions.*

Introduction

The research and analysis of nonlinearity has established itself as a particularly intriguing field for scholars due to the fact that a great number of circumstances could potentially be described mathematically using nonlinear evolution equations (NLEEs). Within the NLEEs, linear terms induce dispersion, whereas nonlinear terms facilitate data intensification. Identifying the travelling wave solutions of these equations is quite a difficult assignment. A nonlinear evolution equation can be mathematically represented in its generic form as

$$p_t = K(p, p_x, p_{xx}, \dots, p_{xxx\dots x})$$

Here p represents a function of one or more parameters (typically expressing spatial coordinates) and time, and K represents a nonlinear function that defines the development of the system and relies on the physical or mathematical context from which the equation originated. The NLEEs strike across numerous branches of science, biology, and engineering that includes fluid dynamics, quantum mechanics, relativity, astrophysics, electromagnetism, plasma physics, ecology, gas dynamics, biophysics, elasticity, biomechanics, atmospheric problems, chemical kinematics, and many others. With regard to NLEEs, traveling wave equations have exceptional characteristics, since they can perfectly explain the wave propagation across a medium without having any distortion. In any case, we can't disregard the influence of nonlinearities by any means, as they cause substantial shifts in the wave propagation characteristics, which gives rise to fascinating wave phenomena such as solitons, chaos, pattern formation, etc.

The gMEW equation was first developed by Wang, Li, and Zhang, [1] in 2008. The equation is mainly a PDE that is built on the equal width wave (EW) equation [2] and features solitary waves that have both positive and negative amplitudes, with identical width. The gMEW equation is associated with the mKdV equation [3], and with the mRLW equation [4] since, each one of them represents a nonlinear wave equation that possesses cubic nonlinearities and contains solitary wave solutions, referred to as wave packets or pulses. In this article, we examine and solve the aforesaid equation to assist academics in formulating more accurate wave models and forecasting software. Many professionals, including coastal engineers, ocean engineers, coastal planners, tsunami wave pattern forecasters, and environmental scientists, depend on this information since this data can be used to build structures, supervise coastal areas, and make informed decisions about the management of coastal areas. Recently, Lu, Seadawy, and Ali studied the aforementioned equation using two different methods [5].

In order to solve an NLEE, we utilize the concept of homogeneous balancing [6], which involves establishing an equilibrium between the top-order derivative and the highest-power nonlinear component. Hence, the outcomes derived from this approach manifest as solitons, which are a distinct category of traveling waves capable of covering significant distances without deviating. Researchers have historically achieved a range of soliton forms, among which are lump-stripe, lump-periodic, periodic, V shaped, singular, kink, anti-bell, bell, etc. Currently, there are multiple methods that can be utilized for determining the solution of an NLEE and some of these well-known techniques are the exp-function method [7], [8], [9], the sine cosine method [10], [11], the homogeneous balance method [12], [13], generalized Kudryashov method [14], [15], the Kudryashov method [16], the improved f-expansion method [17], [18], the extended Kudryashov method [19], hyperbolic function method [20], the extended hyperbolic function method [21], the tanh function method [22], [23], [24], the first integral method [25], the Jacobi elliptic method [26], [27], [28], the improved Bernoulli sub-equation method [29], Hirota bilinear method [30], the Bucklund transform method [31], (G'/G) −expansion method [32], [33], [34], [35], [36], [37], [38], [39], the two variables $(G'/G, 1/G)$- expansion method [40], [41], [42], [43], [44] and many more.

The material here employs the (δ'/δ) −expansion approach to solve the generalized MEW equation for determining precise traveling wave solutions. This approach employs a linear ordinary differential equation (LODE) of second-order as an auxiliary equation, coupled with constant coefficients. The (δ'/δ) −expansion approach is a very effective and straightforward mathematical technique for generating traveling soliton solutions and applicable in real-world circumstances, mathematical physics, and engineering segments. To the best of our knowledge, it is currently unknown, if the (δ'/δ) −expansion approach has been employed to provide soliton solutions for the generalized MEW problem. One case has been examined where the modified equal width equation is studied utilizing the one variable $(\frac{G'}{G})$-expansion method [45], but this equation differs from ours. The resulting soliton solutions will be categorized into three different families, comprising periodic solutions and solitons, and are exhibited in three-dimensional, two-dimensional, and contour graphs to deliver anatomical explanations of the phenomenon using commercially available tools such as MATLAB and Maple.

The remaining article is arranged in the following manner: The methodology is presented in Segment 2. In Segment 3, the application of the utilization of the generalized MEW equation is discussed. The graphical depiction is offered in Segment 4. In Segment 5, we compare our work with a known paper, and subsequently, we retrieve conclusions in Segment 6.

Methodology

We begin by considering the nonlinear PDE

$$K(p, p_t, p_{xt}, p_{tt}, p_{xx}, \dots) = 0, \tag{2.1}$$

where $p = p(x,t)$ is an unknown function representing the wave envelope or the quantum mechanical wave function, and K denotes a polynomial in $p(x,t)$.

The subsequent sections outline the key steps of the (δ'/δ)-expansion method used to construct exact solutions.

Phase I: Consider the travelling wave transformation

$$p(x,t) = p(\psi), \psi = x - \omega t. \tag{2.2}$$

Substituting Equation (2.2) into Equation (2.1), together with the wave speed parameter ω, converts the governing equation into an ODE of the form

$$L(p, p', p'', p''', \dots) = 0. \tag{2.3}$$

Phase II: Equation (2.3) is then integrated term by term, either once or multiple times as required, introducing an integration constant during the process. For simplicity and without loss of generality, this constant is taken to be zero.

Phase III: The answer to Eq. (2.3) may be seen as a travelling wave, which presents itself in a unique way

$$p(\psi) = \sum_{\ddot{\imath}=0}^{V} z_{\ddot{\imath}} \, (\delta'/\delta)^{\ddot{\imath}}, \tag{2.4}$$

with $\delta = \delta(\psi)$, which fulfils the linear ODE of second-order

$$\delta'' + \lambda\delta' + \mu\delta = 0, \tag{2.5}$$

where the constants are $z_{\ddot{\imath}}$ ($\ddot{\imath} = 0,1,2,\dots,n$), λ and μ and

$$(\delta'/\delta) = \begin{cases} \dfrac{-\lambda}{2} + \dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{\beta_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \beta_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{\beta_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \beta_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right), & \lambda^2 - 4\mu > 0 \\ \dfrac{-\lambda}{2} + \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-\beta_1 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \beta_2 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{\beta_1 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \beta_2 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right), & \lambda^2 - 4\mu < 0 \\ \left(\dfrac{-\lambda}{2} + \dfrac{\beta_2}{\beta_1 + \beta_2\xi}\right), & \lambda^2 - 4\mu = 0 \end{cases}$$

Phase IV: Equations (2.4) and (2.5) are substituted into Equation (2.3) to determine the integer value of n by solving the resulting relation. Subsequently, a homogeneous balance is performed between the highest-order derivative term and the leading nonlinear term in Equation (2.3).

Phase V: Using the value of $n$obtained in Phase IV, Equations (2.4) and (2.5) are substituted back into Equation (2.3). By equating the coefficients of $\left(\frac{\delta'}{\delta}\right)^i$, where i ranges from zero to infinity, an algebraic system of equations is constructed with the unknown parameters d_j $(j = 0,1,2, \dots, n)$, ω, λ, and μ.

Phase VI: The algebraic system derived in Phase V is solved using the symbolic computation software *Maple*, yielding the explicit values of d_j $(j = 0,1,2, \dots, n)$, ω, λ, and μ. Substituting these values, along with the obtained n, into Equations (2.4) and (2.5) leads to the travelling-wave solution of Equation (2.1).

Utilization the generalized MEW Equation

This study focuses on solving the generalized MEW Eq. by employing the (δ'/δ) −expansion approach. The following is the equation:

$$p_t + 3p^2 p_x - \alpha p_{xxt} = 0, -\infty < x < \infty, t > 0. \tag{3.1}$$

At this point, applying the travelling wave transformation:

$$\psi = x - \omega t. \tag{3.2}$$

where ω stands for traveling wave speed, along with $p(x,t) = p(\psi)$, the PDE in Eq. (3.1) translates to an ODE in single variable ψ as:

$$-\omega \frac{dp}{d\psi} + 3p^2 \frac{dp}{d\psi} + \alpha\omega \frac{d^3p}{d\psi^3} = 0. \tag{3.3}$$

By integrating both sides of Eq. (3.3) w.r.to ψ, we get the following result:

$$-\omega p + p^3 + \alpha\omega p'' = 0. \tag{3.4}$$

In the case the integral constant is zero, it is because we're specifically looking for solitary wave solutions. Therefore, to solve Eq. (3.4), we employ the approach called homogeneous balancing and it gives $V + 2 = 3V$ that is $V = 1$.

Putting $V = 1$ in Eq. (2.4) we obtain the solution of ODE (3.4) which gives

$$\mathrm{p}(\psi) = \mathrm{z}_0 + \mathrm{z}_1(\delta'/\delta), \tag{3.5}$$

where z_0 *and* z_1 acts as the arbitrary constants which will be calculated.

When equations (3.5) and (3.4) are combined, the left side of Eq. (3.2) becomes a polynomial in powers of $(\delta'/\delta)^{i}$. Now, according to phase (5) mentioned in Eq. (2.2), we collect phrases having equivalent powers of (δ'/δ). Afterwards, we equal this polynomial's coefficients to zero, producing algebraic equations set.

The previously described algebraic system is solved with the help of mathematical tool Maple to arrive at the following solutions:

$$z_0 = \sqrt{\omega - 2\mu\alpha\omega}, z_1 = \sqrt{-2\alpha\omega}, \lambda = -\frac{\sqrt{\omega - 2\mu\alpha\omega}\sqrt{-2\alpha\omega}}{\alpha\omega}.$$

Putting the aforesaid answers (3.5), in the equation (3.4), we demonstrated the solutions for three distinct families, which are provided below

Family 1: In the case of Hyperbolic function solution (when $\lambda^2 - 4\mu > 0$), we acquire

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{\beta_1 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \beta_2 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}{\beta_1 cosh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi + \beta_2 sinh\frac{\sqrt{\lambda^2-4\mu}}{2}\xi}\right)\right). \tag{3.6}$$

Here $\psi = x - \omega t$. In particular if $\beta_1 \neq 0$ and $\beta_2 = 0$, then the aforesaid solution transforms into:

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\right), \tag{3.7}$$

and if $\beta_2 \neq 0$ and $\beta_1 = 0$, then

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(coth\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\right). \tag{3.8}$$

Family 2: In the case of Trigonometric function solution (when $\lambda^2 - 4\mu < 0$), we acquire:

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-\beta_1 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \beta_2 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}{\beta_1 cos\frac{\sqrt{4\mu-\lambda^2}}{2}\xi + \beta_2 sin\frac{\sqrt{4\mu-\lambda^2}}{2}\xi}\right)\right), \tag{3.9}$$

where, $\psi = x - \omega t$. In particular if $\beta_1 \neq 0\ and\ \beta_2 = 0$, then the aforesaid solution transforms into:

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(-tan\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\right), \tag{3.10}$$

and if $\beta_2 \neq 0\ and\ \beta_1 = 0$, then

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\left(cot\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\right). \tag{3.11}$$

Family 3: In the case of Rational function solution (when $\lambda^2 - 4\mu = 0$), we acquire:

$$p(\psi) = \sqrt{\omega - 2\mu\alpha\omega} + \sqrt{-2\alpha\omega}\left(\frac{-\lambda}{2} + \frac{\beta_2}{\beta_1 + \beta_2\xi}\right). \tag{3.12}$$

Here $\psi = x - \omega t$.

Graphical Representation

This segment aims to illustrate the travelling wave solutions that resulted from solving the generalized MEW problem. The mathematical tool MATLAB is used to show the two- and three-dimensional and contour graphs as well as to illustrate the estimated values of the arbitrary constants related to the solutions. It is to be notated that, there may be changes in wave profile if we alter the auxiliary and physical parameters. The physical characteristics of the solitons obtained during the investigation are described as follows.

Firstly, Eq. (3.7) exhibits a V shaped soliton solution which is shown in figure 1 for the parameter's values $\alpha = -6, \omega = 1, \mu = 1$. But, for the parameter's values $\alpha = 15, \omega = -3, \mu = 3$ the Eq. (3.7) changes its shape to a N-soliton solution as shown in figure 2. The N-solitons have limited energy and momentum, depending on the wave's amplitude and speed. When we take the parameter's values as $\alpha = 5, \omega = -13, \mu = 3$ for Eq. (3.8), figure 3 exhibits a singular soliton solution. Moreover, Eq. (3.8) exhibits a cuspon for the values $\alpha = -5, \omega = 1, \mu = 2$ as shown in figure 4.

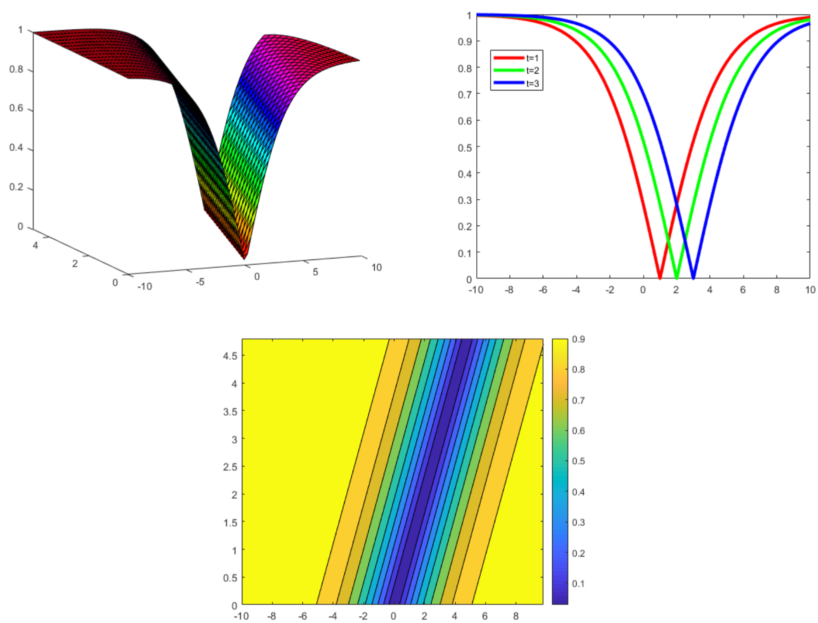
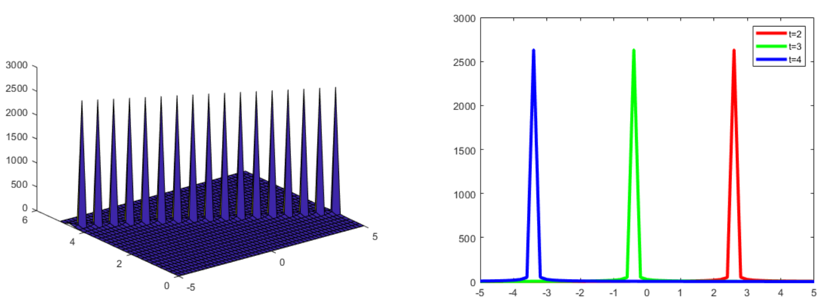


Fig 1
V-shape of the Eq. (3.7) for $\boldsymbol{\alpha = -6, \omega = 1, \mu = 1}$.



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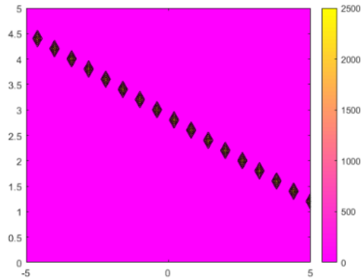


Fig 2
N-soliton shape of the Eq. (3.7) for $\alpha = 15, \omega = -3, \mu = 3$.

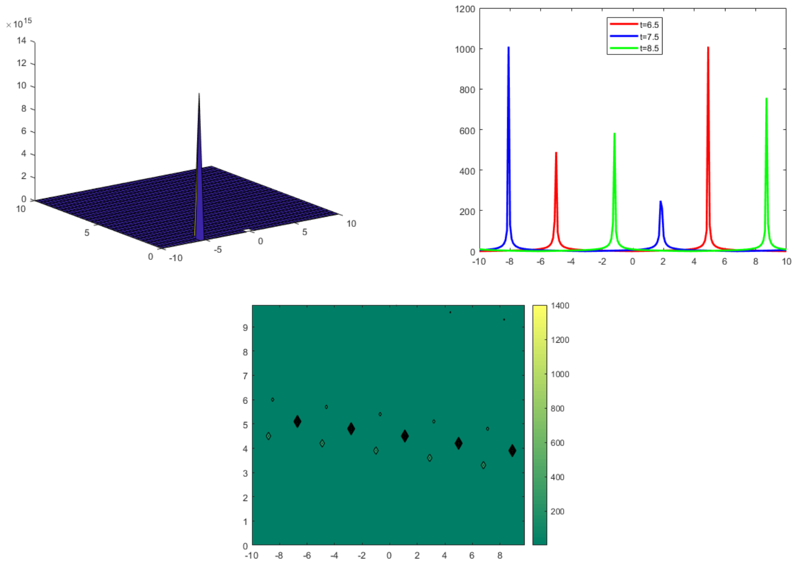
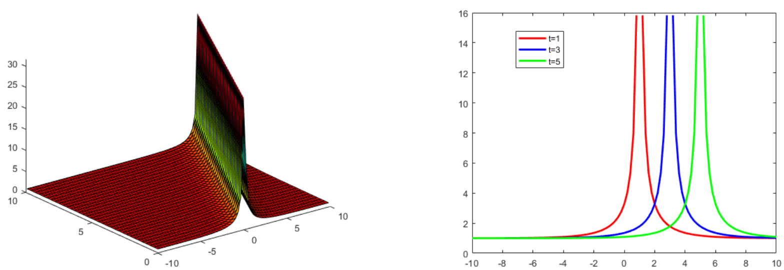


Fig 3
Singular soliton shape of the Eq. (3.8) for $\alpha = 5, \omega = -13, \mu = 3$.



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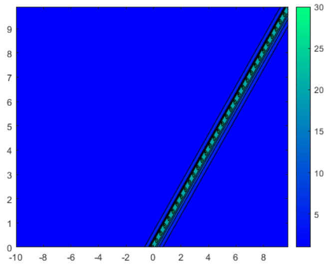


Fig 4
Cuspon shape of the Eq. (3.8) for $\alpha = -5, \omega = 1, \mu = 2$.

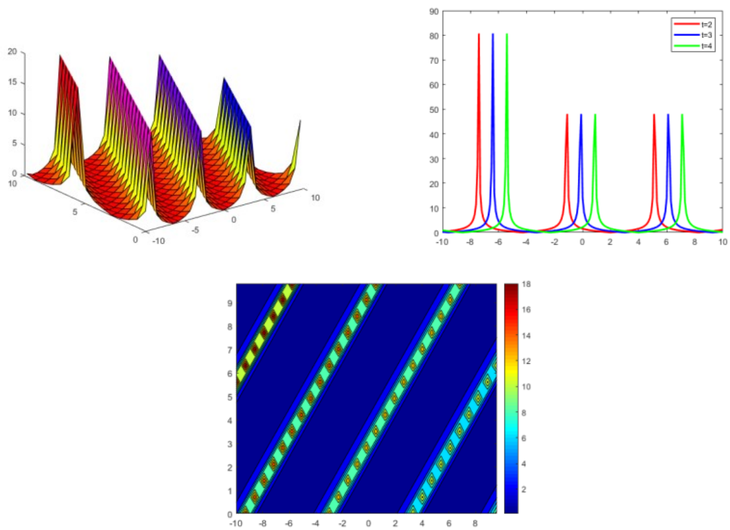


Fig 5
Periodic shape of the Eq. (3.10) for $\alpha = 2, \omega = 1, \mu = 1$.

In the case of figure 5 a periodic soliton solution is noticed for Eq. (3.10), where the parameter's values are $\alpha = -2, \omega = 1, \mu = 1$. The periodic-shape soliton solution is utilized for transferring information for an extended distance without much distortion, since it is a highly consistent and durable wave solution. We observe a V shaped soliton solution for the aforementioned equation when the parameters change to $\alpha = -4, \omega = -1, \mu = 2$ shown in figure 6. Moreover, for the values $\alpha = 1, \omega = -10, \mu = 3$ the figure 7 exhibits a N-soliton solution for the same Eq. (3.10). For Eq. (3.11), the periodic soliton solution is shown in figure 8 where the values are $\alpha = 1, \omega = 1, \mu = 1$. Also, it represents a peakon for the values $\alpha = -5, \omega = 1, \mu = 2$ as shown in figure 9. Furthermore, the solution Eq. (3.11) exhibits a singular soliton solution, shown in figure 10 for the parameter's values $\alpha = 5, \omega = -13, \mu = 3$.

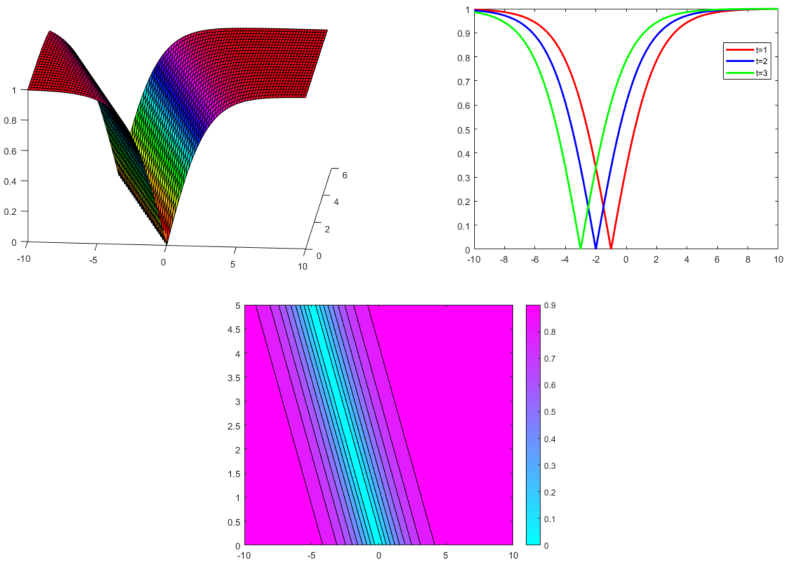


Fig 6
V-shape of the Eq. (3.10) for $\alpha = -4, \omega = -1, \mu = 2$.

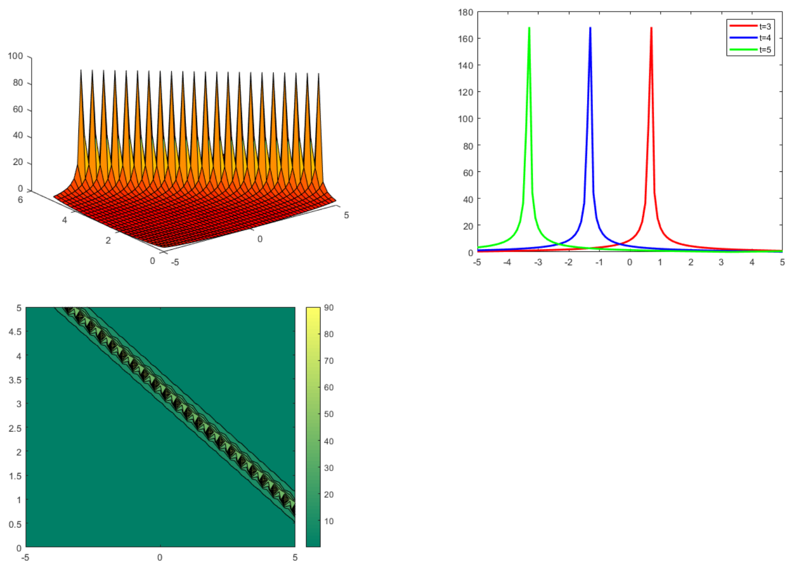


Fig 7
N-soliton shape of the Eq. (3.10) for $\alpha = 1, \omega = -10, \mu = 3$.

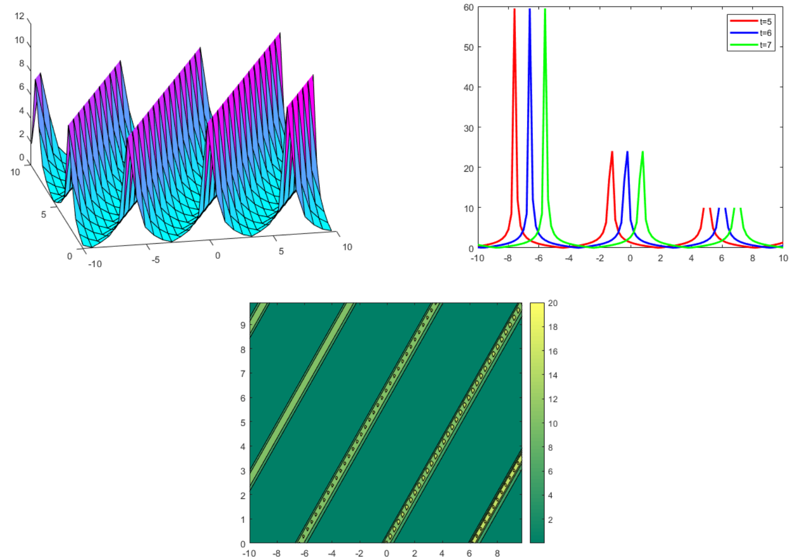


Fig 8
Periodic shape of the Eq. (3.11) for $\alpha = 1, \omega = 1, \mu = 1$.

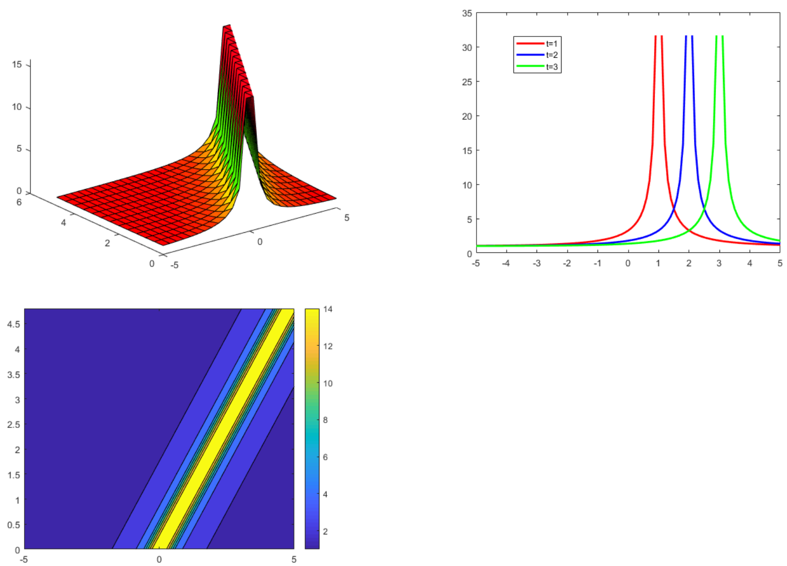


Fig 9
Peakon shape of the Eq. (3.11) for $\alpha = -5, \omega = 1, \mu = 2$.

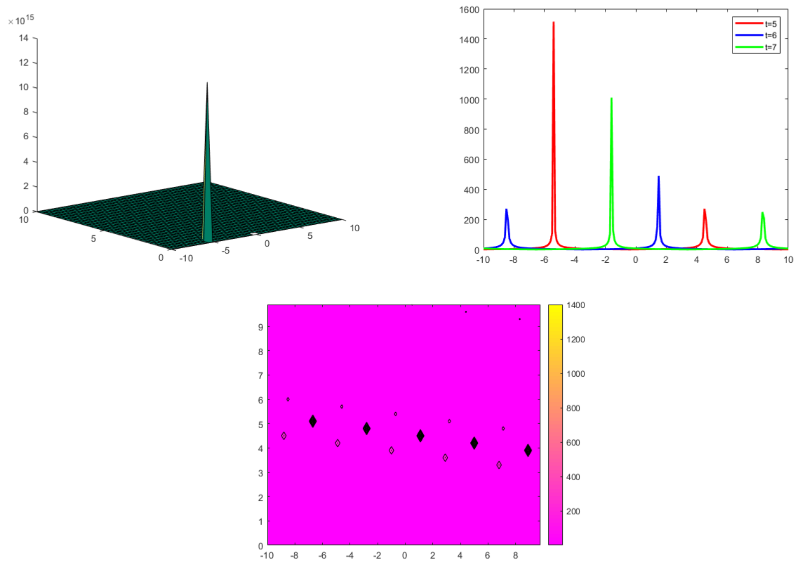
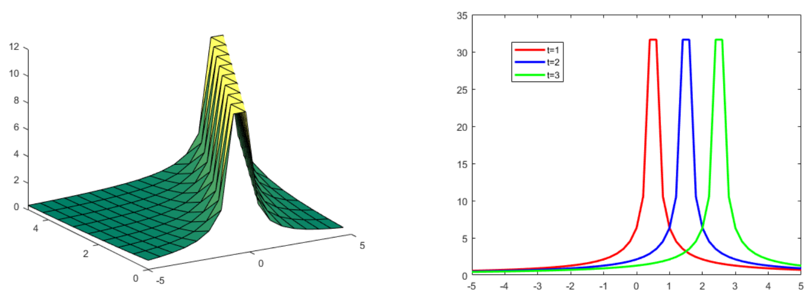


Fig 10
Singular soliton shape of the Eq. (3.11) for $\boldsymbol{\alpha = 5, \omega = -13, \mu = 3}$.

Finally, for the parameter's values $\beta_1 = 2, \beta_1 = 4, \alpha = 5, \omega = 1, \mu = 1$ the Eq. (3.12) exhibits a bell-shaped soliton solution as shown in figure 11. The bell-shaped soliton solution is a strong wave with a high energy content that may travel great distances. The bell-shaped soliton is a crucial subject to study for forecasting the behavior of coastal engineering and tsunami weaving. Also, the Eq. (3.12) represents a singular soliton solution which is exhibited in figure 12 for the values $\beta_1 = 15, \beta_1 = -1, \alpha = 5, \omega = -17, \mu = 2$. Moreover, the equation shows a parabolic soliton solution displayed in figure 13 for the parameter's values $\beta_1 = -20, \beta_1 = -1, \alpha = -5, \omega = -7, \mu = 3$.



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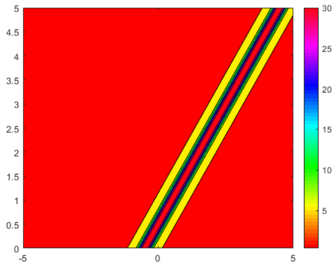


Fig 11
Bell shape of the Eq. (3.12) for $\beta_1 = 2, \beta_1 = 4, \alpha = 5, \omega = 1, \mu = 1$.

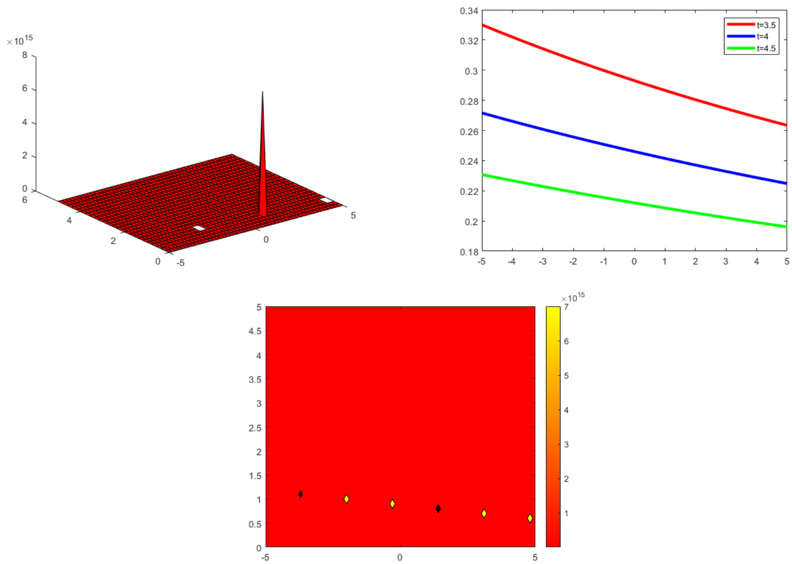
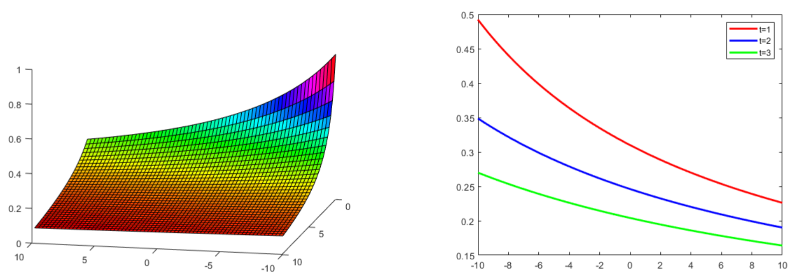


Fig 12
Singular soliton of the Eq. (3.12) for $\beta_1 = 15, \beta_1 = -1, \alpha = 5, \omega = -17, \mu = 2$.



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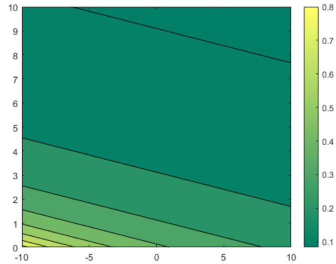


Fig 13
Parabolic shape of the Eq. (3.12) for $\beta_1 = -20, \beta_1 = -1, \alpha = -5, \omega = -7, \mu = 3$.

Comparison

In this segment, we shall analyze and contrast the solutions produced by using (δ'/δ) −expansion approach applied to Eq. (3.4) with the solutions found by Dianchen LU et al. [5], where two different methods have been implemented. Employing the extended simple equation method, Dianchen LU et al. derived six distinct solutions. However, only four of these solutions (specifically denoted as (26), (28), (31), and (32)) were examined. In addition, the authors have investigated a total of three solutions using the $exp(-\varphi(\xi))$ expansion approach. In contrast, we have produced and analyzed five novel soliton solutions to the generalized MEW problem. We have also shown 3D, 2D, and contour graphs to visually represent the solutions, which adds diversity and richness to the solutions.

Solutions found using extended simple equation method [5]	**Solutions obtained using $\boldsymbol{exp(-\varphi(\xi))}$ expansion approach [5]**	**The solutions we derived**

a) If $b_3 = 0$, then
$U_5(x,t) = \frac{\beta^{1/4}\sqrt{\omega}}{\sqrt{2\alpha}}\,(\pm\sqrt{b_1} + (\frac{3b_1^2}{(1-\sqrt{2}\,tan(b_1^2)(\xi+\varepsilon_0))})),$
$4b_0b_2 > b_1^2$
b) If $b_0 = b_3 = 0$, then
$U_6(x,t) = a_0 + (\frac{2a_0b_2e^{b_1(\xi+\varepsilon_0)}}{1-b_2e^{b_1(\xi+\varepsilon_0)}}), b_1 > 0$
c) If $b_1 = b_3 = 0$, then
$U_8(x,t) = -\frac{a_1b_0}{\sqrt{b_0b_2}\,tan(\sqrt{b_0b_2}(\xi+\varepsilon_0))},$
$b_0b_2 > 0, \beta > 0$
and
$U_9(x,t) = -\frac{a_1b_0}{\sqrt{b_0b_2}\,tan(\sqrt{b_0b_2}(\xi+\varepsilon_0))},$
$b_0b_2 < 0, \beta < 0$
	If $\lambda^2 - 4\mu > 0, \mu \neq 0$ then
	$u_{11}(\xi) = a_0 + \frac{4a_0\mu}{\lambda(-\sqrt{\lambda^2-4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+\xi_0))-\lambda)}$
	If $\lambda^2 - 4\mu > 0, \mu = 0$ then
	$u_{22}(\xi) = a_0 + (\frac{2a_0}{\exp(\lambda(\xi+\xi_0))-1})$
	If $\lambda^2 - 4\mu < 0$, then
	$u_{33}(\xi) = a_0 + \frac{4a_0\mu}{\lambda(\sqrt{4\mu-\lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi+\xi_0))-\lambda)}$
		If $\beta_1 \neq 0\ and\ \beta_2 = 0\ and\ \mu = 0$, then solution (3.6) transforms into
		$u_1(\xi) = Y_1 + Z_1\left[\Delta\left(tanh\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\right]$
		in this case λ< 0 and Δ acts as the arbitrary parameter.
		If $\beta_1 = 0\ and\ \beta_2 \neq 0\ and\ \mu = 0$, then solution (3.6) transforms into
		$u_2(\xi) = Y_2 + Z_2\left[\Delta\left(coth\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\right]$
		in this case $\lambda < 0$ and Δ acts as the arbitrary parameter.
		If $\beta_1 \neq 0\ and\ \beta_2 = 0\ and\ \mu = 0$, then solution (3.9) transforms into
		$u_3(\xi) = Y_3 + Z_3\left[\Delta\left(-tan\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\right]$
		in this case $\lambda > 0$and Δ acts as the arbitrary parameter.
		If $\beta_1 = 0\ and\ \beta_2 \neq 0\ and\ \mu = 0$,

Contd...

		then solution (3.9) transforms into $u_4(\xi) = Y_4 + Z_4\left[\Delta\left(cot\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\right]$ in this case $\lambda > 0$ and Δ acts as the arbitrary parameter. If $\lambda = 0$, then solution (3.12) transforms into $u_5(\xi) = Y_5 + \left[Z_5\left(\frac{-\lambda}{2} + \frac{\beta_2}{\beta_1 + \beta_2\xi}\right)\right]$

Stability Analysis

Performing a thorough stability analysis of the governing models is crucial for fully capturing and interpreting the fundamental dynamical behavior of the system. In this section, the stability properties of the generalized modified equal-width (gMEW) equation have been systematically investigated through the application of the linear stability framework. Within this analytical approach, a perturbed solution is introduced to the governing equation (3.1), expressed in the following functional form:

$$p(x,t) = \epsilon_0\, u(x,t) + u_0. \tag{6.1}$$

Here, ϵ_0 denotes an arbitrary constant, while u_0 represents the steady-state solution of the governing equation.

By substituting the perturbed solution given in Equation (6.1) into the governing Equation (3.1) and subsequently linearizing the terms, the resulting linearized equation is obtained as:

$$\epsilon_0 u_t + 3\epsilon_0 u_0^2 u_x - \alpha\epsilon_0 u_{xxt} = 0. \tag{6.2}$$

As the resulting equation is linear, the trial solution for Equation (6.2) is assumed to be of the form:

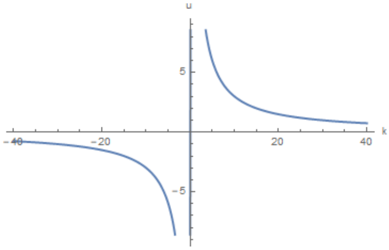
$$u(x,t) = e^{i(k_0 x - c_0 t)}. \tag{6.3}$$

Here, k_0 and c_0 denote the wave number and phase velocity of the perturbed wave function. Substituting Equation (6.3) into the linearized form of Equation (6.2) leads to the following analytical expression for the phase velocity:

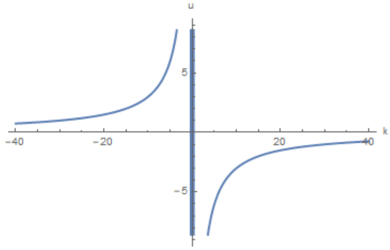
$$c_0 = \frac{3k_0{u_0}^2}{1+\alpha{k_0}^2} \tag{6.4}$$

The phase velocity analysis reveals that the perturbed wave function is stable for all valid parameter values except at $\alpha = 0$ and $k_0 = 0$, confirming the strong stability of the gMEW model. Furthermore, the stability characteristics are highly sensitive to variations in α, as shown in Figure 14.

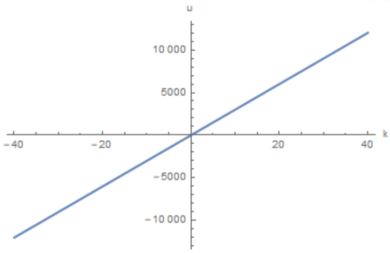
Figure 14(i) corresponds to $\alpha = 10$, $q_0 = -10$, and Figure 14(ii) to $\alpha = -10$, $q_0 = -10$; in both cases, the system remains stable except at the critical point $k_0 = 0$. Furthermore, when $\alpha = 0$, a linear relationship emerges between c_0 and k_0, with the velocity c_0 exhibiting a linear increase as k_0 increases. Under these conditions, the model transitions into an unstable regime, indicating its susceptibility to instability for $\alpha = 0$.



(i) α = 10



(ii) α = -10



(iii) α = 0

Figure 14
Stability representation

Conclusion

This article demonstrates the use of the (δ'/δ) −expansion approach to the Generalized MEW equation to showcase soliton solutions, which have potential applications in signal processing, ocean physics, optical fibers, sound waves, and other related topics. Multiple forms of soliton solutions, such as bell-shaped soliton, singular soliton, parabolic soliton, V shaped, cuspon, peakon and periodic soliton have been derived in our study. The soliton solutions generated from our research are visually represented using MATLAB, showcasing remarkable contour, three-dimensional, and two-dimensional graphs. These visual depictions demonstrate that the solutions we have achieved are solitary wave solutions, that may be utilized for transmitting data over great distances with minimal energy consumption. Furthermore, we have conducted a comparative analysis of our achieved solutions with other solutions available in the existing body of the literature. It is crucial to observe that, this novel form of verified solution has not been replicated in any prior literature. Our research showcases that the (δ'/δ) −expansion technique is a practical and valuable mathematical technique to investigating different NLESs in the fields of engineering and science.

Credit Authorship Contribution Statement

Nasir Uddin, and **Md. Antajul Islam** Conceptualization, Resources, Methodology, Investigation, Writing-Original Draft. **Md. Musa Miah,** and **Md. Mahmud Alam**: Data Curation, Project administration, co-supervision, Writing-Review Editing. **Nasrin Nahar Rimu**: Software, Data Curation, Visualization, Writing-Review Editing. **Pinakee Dey**: supervision, Data Curation, Formal Analysis, Validation, Writing-Review Editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

Data Availability

No datasets were generated or analyzed during the current study.

Funding

This research received no external funding.

Author Contributions

All authors contributed to the conception and design of the study. All authors reviewed, revised, and approved the final manuscript.

Acknowledgements

The authors gratefully acknowledge the anonymous reviewers for their constructive comments and valuable suggestions, which helped improve the quality of this manuscript. The authors also appreciate the support provided by Mawlana Bhashani Science and Technology University.

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Received January, 2025