

Independent 2-domination polynomials in complete bipartite graphs

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Abstract

This paper investigates the independent 2-domination polynomial of complete bipartite graphs $K_{m,n}$. We establish closed formulas for this polynomial and describe all independent 2-dominating sets in such graphs. The findings illustrate how the bipartite structure determines the enumeration of these sets and provide further insight into domination-type polynomials in special graph families.

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1. Introduction

Graph theory, a fundamental area within discrete mathematics, offers versatile methods for representing and studying complex structures. Among its central themes are domination concepts, which play an important role in diverse fields such as computer science, network design, biology, and security systems. [1], [2]

In a simple graph $G = (V, E)$, a subset $S \subseteq V(G)$ is called a dominating set if every vertex not in S has at least one neighbor in S . This notion can be

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extended to the concept of a k -dominating set, where each vertex outside S must be adjacent to at least k members of S . A particularly significant case is when $k = 2$, known as a 2-dominating set. Furthermore, if such a 2-dominating set is independent, it is referred to as an independent 2-dominating set. The formal definition of this concept is given below. [3], [4]

Definition 1.1: Consider a simple undirected graph $G = (V, E)$. A subset $S \subseteq V(G)$ is said to be an independent 2-dominating set in G if it satisfies the following two properties simultaneously:

1. **2-Domination:** Each vertex $v \in V(G) \setminus S$ must be adjacent to at least two different vertices belonging to S . Equivalently, the number of neighbors of v inside S is not less than two:

$$\forall v \in V(G) \setminus S, |N(v) \cap S| \geq 2,$$

where $N(v)$ denotes the neighborhood of the vertex v .

2. **Independence:** The vertices in S are pairwise non-adjacent, which means S forms an independent set:

$$\forall u, v \in S, \{uv\} \notin E(G).$$

Independent 2-dominating sets appear in various applications, including fault-tolerant network design, biological system analysis, and the study of networks that require particular coverage and distribution properties.

The study of independent 2-dominating sets offers important perspectives on vertex distribution within a graph and their local interactions. This notion is significant in several practical domains, such as fault-tolerant network construction, efficient resource allocation, and the representation of biological systems. The independent 2-domination number, denoted by $\gamma_i^2(G)$, is defined as the smallest cardinality of a set with this property. Identifying this parameter and investigating its behavior in different classes of graphs is regarded as a central problem in graph theory. [5], [6]

We now present a formal definition of the independent 2-domination number.

Definition 1.2: The independent 2-domination number of a graph G , denoted by $\gamma_i^2(G)$, is defined as the minimum cardinality of an independent 2-dominating set in G .

Equivalently, $\gamma_i^2(G)$ corresponds to the least number of vertices that can be chosen so that the resulting set is both independent and 2-dominating.

For a deeper investigation of independent 2-dominating sets, researchers have also introduced related polynomial formulations.

Definition 1.3: The independent 2-domination polynomial of a graph G is given by

$$D_i^2(G, x) = \sum_{k=\gamma_i^2(G)}^{|V(G)|} d_i^2(G, k) x^k,$$

where:

- $\gamma_i^2(G)$ denotes the independent 2-domination number of G , i.e., the minimum size of such a set.
- $d_i^2(G, k)$ is the number of independent 2-dominating sets of cardinality k in G .
- x is an indeterminate, and the exponent reflects the size of the corresponding sets.

This polynomial encodes how independent 2-dominating sets are distributed according to their cardinalities in the graph G , and it serves as a useful tool for exploring both structural and combinatorial aspects of graphs. [7], [8], [9], [10], [11], [12]

In recent years, several classes of domination-related polynomials have been studied to better understand structural characteristics of graphs. For example, Jafari and Alikhani examined the roots of total domination polynomials, providing deeper insights into their behavior [13]. Similarly, Mojdeh and Emadi introduced the hop domination polynomial and analyzed its combinatorial properties [14].

In addition to domination polynomials, further developments related to bipartite structures have also appeared in the literature. Reddy and Hemavathi proposed a generalized form of bipartite graphs and discussed their fundamental structural aspects [15]. More recently, Swadi and Najim investigated the generalized k -connectivity in equally complete bipartite graphs and their line graphs, highlighting new directions within this class of graphs [16]. These works demonstrate that bipartite graphs and domination-type invariants continue to provide a rich ground for theoretical advancement.

Recent research indicates that the study of such polynomials provides valuable information about the structural characteristics of graphs. In certain families of graphs, investigations of independent 2-domination polynomials have yielded explicit formulas as well as deeper theoretical understanding. Nevertheless, a wide range of graphs and their associated polynomial behaviors remain unexamined, presenting opportunities for further exploration. Independent 2-domination polynomials provide a useful framework for addressing problems in network optimization, biological modeling, and combinatorial analysis. In the context of fault-tolerant or resilient networks, they assist in determining critical vertex subsets that guarantee both connectivity and adequate coverage. Moreover, within biological applications, these polynomials contribute to understanding the interactions and functional behavior of various system components. [4], [17], [18], [19], [20], [21], [22], [23], [24], [25]

Although significant progress has been achieved, determining the independent 2-domination number and studying the corresponding polynomials in more complex graph families still presents considerable difficulties. Extending these notions to broader contexts such as weighted, dynamic, or random graphs may create promising avenues for future investigations in this area [4], [25], [26], [27], [28].

In this paper, we focus on the independent 2-domination polynomials of complete bipartite graphs. Before presenting our main results, we recall the formal definition of a complete bipartite graph.

Definition 1.4: A complete bipartite graph $K_{m,n}$ is a bipartite graph whose vertex set can be partitioned into two disjoint subsets U and V . Every vertex in U is adjacent to all vertices in V , while no edges exist within the same part. The graph has $m \times n$ edges, vertices in U have degree n , and vertices in V have degree m . By construction, $K_{m,n}$ is both bipartite and connected. [29]

Examples: $K_{3,3}$: A bipartite graph with six vertices and nine edges, which is a well-known non-planar graph.

- $K_{1,n}$: A star graph consisting of one central vertex connected to n peripheral vertices.

Complete bipartite graphs play an important role in practical applications such as the design of computer architectures (e.g., server-client models) and the modeling of biological or social networks.

2. Independent 2-domination sets in the complete bipartite graph

In this section, we determine the independent 2-dominating sets of $K_{3,4}$ and extend the discussion to the general case $K_{m,n}$.

Theorem 2.1: *For the complete bipartite graph $K_{3,4}$ with partition sets $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2, v_3, v_4\}$, the independent 2-dominating sets are:*

$$S = \{u_1, u_2, u_3\} \text{ and } S = \{v_1, v_2, v_3, v_4\}.$$

Proof: To qualify as an independent 2-dominating set, a subset S must satisfy two requirements: (i) every vertex in $V(G) \setminus S$ has at least two neighbors in S , and (ii) S is independent, which implies it is entirely contained in either U or V .

We analyze the following possibilities:

Case 1: Choosing all three vertices of U . The set $S = \{u_1, u_2, u_3\}$ is independent since vertices in U are mutually non-adjacent. Each v_j is adjacent to all three u_i , so condition (i) holds. Thus, S is an independent 2-dominating set.

Case 2: Choosing three vertices from V . Take $S = \{v_1, v_2, v_3\}$. Although independent, this set fails condition (i) because vertex v_4 has only one neighbor in S . Therefore, it is not an independent 2-dominating set.

Case 3: Choosing all four vertices of V . For $S = \{v_1, v_2, v_3, v_4\}$, independence is clear, and each u_i is adjacent to all four vertices of V , giving at least two neighbors in S . Hence, this set satisfies both conditions.

Consequently, the only independent 2-dominating sets in $K_{3,4}$ are $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3, v_4\}$. +

Theorem 2.2: *Let $K_{m,n}$ be the complete bipartite graph with partition sets $U = \{u_1, u_2, \dots, u_m\}$ and $V = \{v_1, v_2, \dots, v_n\}$. The independent 2-dominating sets can be described as follows:*

- *When $m \leq n$, the set U forms an independent 2-dominating set.*
- *When $n \leq m$, the set V forms an independent 2-dominating set.*
- *More generally, any subset lying entirely in one partition that ensures every vertex in the opposite part has at least two neighbors within S is also acceptable.*

Proof: We verify each scenario separately.

Case 1: All vertices of U chosen, with $m \leq n$. The set $S = U$ is independent, since vertices within U share no edges. Every vertex v_j is adjacent to all u_i , and hence has at least two neighbors in S . Thus, U itself is an independent 2-dominating set.

Case 2: All vertices of V chosen, with $n \leq m$. Similarly, $S = V$ is independent. Each vertex u_i connects to all vertices of V , so every u_i has at least two neighbors in S . Therefore, V is also an independent 2-dominating set.

Case 3: Proper subsets of the smaller partition. If S is taken as a subset of the smaller part, then in order for vertices in the opposite part to have two neighbors in S , the subset must contain all vertices of that part. Hence, no smaller choice of S satisfies the 2-domination requirement.

Consequently, the only possibilities for independent 2-dominating sets in $K_{m,n}$ are (i) $S = U$ when $m \leq n$, (ii) $S = V$ when $n \leq m$, and (iii) any full partition set that guarantees every vertex outside has at least two neighbors within S .

3. The independent 2-domination number in the complete bipartite graph

In this section, we evaluate the independent 2-domination number for the complete bipartite graphs $K_{2,3}$, $K_{3,4}$, and the general case $K_{m,n}$, identifying the smallest independent 2-dominating set in each case.

Theorem 3.1: *For the complete bipartite graph $K_{2,3}$, the independent 2-domination number is equal to 2. The unique minimal independent 2-dominating set is $S = \{u_1, u_2\}$, where the partition $U = \{u_1, u_2\}$ contains the two vertices of smallest size.*

Proof: Consider the bipartition

$$U = \{u_1, u_2\}, V = \{v_1, v_2, v_3\},$$

with each vertex in U adjacent to all vertices in V .

Step 1: Independence. Since vertices within the same part are non-adjacent, any independent 2-dominating set must lie entirely within either U or V .

Step 2: Testing subsets.

- Suppose $S = \{v_1, v_2\}$. Then each vertex in U is adjacent to only one member of S , so the 2-domination requirement fails.
- If $S = \{u_1, u_2\}$, the set is independent, and every v_i is connected to both u_1 and u_2 . Thus, each v_i has two neighbors in S , fulfilling the domination condition.

Step 3: Minimality. No independent set of size smaller than two can satisfy both conditions, hence the independent 2-domination number is 2, and the unique minimal set is

$$S = \{u_1, u_2\}.$$

Therefore,

$$\gamma_i^2(K_{2,3}) = 2.$$

Theorem 3.2: *For the complete bipartite graph $K_{3,4}$, the independent 2-domination number is 3. The minimal independent 2-dominating set is $S = \{u_1, u_2, u_3\}$, where the partition $U = \{u_1, u_2, u_3\}$ contains the three vertices.*

Proof: Consider the bipartition

$$U = \{u_1, u_2, u_3\}, V = \{v_1, v_2, v_3, v_4\},$$

with every vertex of U adjacent to all vertices of V .

Step 1: Independence. Any independent 2-dominating set must be fully contained in either U or V , since vertices within a single part are non-adjacent.

Step 2: Testing possible subsets.

- If $S = \{u_1, u_2\}$, then u_3 has no neighbors in S , so this choice is invalid.
- If $S = \{v_1, v_2\}$, then each vertex in U has only one neighbor in S , which fails the 2-domination condition.
- If $S = \{u_1, u_2, u_3\}$, the set is independent, and each vertex v_i is connected to all three members of S . Hence, every v_i has at least two neighbors in S , satisfying the requirement.

Step 3: Minimality. No set of size smaller than three meets both independence and 2-domination criteria. Thus, the independent 2-domination number equals 3, with

$$S = \{u_1, u_2, u_3\}$$

as the unique optimal set.

Therefore,

$$\gamma_i^2(K_{3,4}) = 3.$$

Theorem 3.3: For the complete bipartite graph $K_{m,n}$, whose vertex set is partitioned into two independent parts U and V of sizes m and n , the independent 2-domination number is

$$\gamma_i^2(K_{m,n}) = \min(m, n).$$

Proof: Let the bipartition of $K_{m,n}$ be defined as

$$U = \{u_1, u_2, \dots, u_m\}, V = \{v_1, v_2, \dots, v_n\},$$

where each vertex in U is adjacent to every vertex in V .

Step 1: Independence. Since vertices within the same part are non-adjacent, any independent 2-dominating set must lie completely inside either U or V .

Step 2: 2-domination. - If the set is taken from U , then every vertex of V must have at least two neighbors in S . - Conversely, if the set is chosen from V , then every vertex of U requires two neighbors in S . Thus, domination is achieved only when the entire smaller part is selected.

Step 3: Minimality. Choosing all vertices from the smaller side of the bipartition ensures independence and guarantees the 2-domination property. No smaller subset can satisfy both conditions simultaneously.

Hence, the independent 2-domination number of $K_{m,n}$ is determined by the size of the smaller partition:

$$\gamma_i^2(K_{m,n}) = \min(m, n).$$

4. Independent 2-Domination Polynomials in Complete Bipartite Graphs

We now compute the independent 2-domination polynomial of the complete bipartite graph $K_{2,3}$. This requires identifying all independent 2-dominating sets of different cardinalities and summing their contributions to the polynomial.

Theorem 4.1: For the complete bipartite graph $K_{2,3}$, the independent 2-domination polynomial is

$$D_i^2(K_{2,3}, x) = x^2 + x^3.$$

Proof: The graph $K_{2,3}$ has bipartition

$$U = \{u_1, u_2\}, V = \{v_1, v_2, v_3\},$$

where each vertex of U is adjacent to every vertex of V . We examine independent 2-dominating sets according to their size.

Sets of size 2. The only independent 2-dominating set of size two is $S = \{u_1, u_2\}$. The vertices u_1 and u_2 are non-adjacent, and each vertex of V is adjacent to both of them, so the domination condition is satisfied. Hence, $d_i^2(K_{2,3}, 2) = 1$.

Sets of size 3. Choosing all three vertices from V , that is $S = \{v_1, v_2, v_3\}$, yields another independent 2-dominating set. Since each vertex of U is adjacent to all three vertices in S , the condition holds, and thus $d_i^2(K_{2,3}, 3) = 1$.

Larger sets. Any set of size 4 or 5 necessarily contains vertices from both U and V , which introduces adjacency and breaks independence. Therefore, no independent 2-dominating sets of such sizes exist.

Combining these cases, the polynomial is

$$D_i^2(K_{2,3}, x) = x^2 + x^3,$$

where the coefficient of x^k counts the independent 2-dominating sets of size k .

Theorem 4.2: *For the complete bipartite graph $K_{m,n}$, the independent 2-domination polynomial is*

$$D_i^2(K_{m,n}, x) = \sum_{k=\gamma_i^2(K_{m,n})}^{\max(m,n)} d_i^2(K_{m,n}, k) x^k = x^m + x^n.$$

Proof: Let the bipartition of $K_{m,n}$ be defined as

$$U = \{u_1, u_2, \dots, u_m\}, V = \{v_1, v_2, \dots, v_n\},$$

with each vertex of U adjacent to all vertices of V . To obtain the polynomial, we count independent 2-dominating sets according to their size.

Case 1: Sets of size m . The entire set U forms an independent set, and each vertex of V is connected to all m vertices of U . Thus, $d_i^2(K_{m,n}, m) = 1$.

Case 2: Sets of size n . Similarly, the set $V = \{v_1, v_2, \dots, v_n\}$ is independent, and every vertex of U has n neighbors in it. Hence, $d_i^2(K_{m,n}, n) = 1$.

Case 3: Other sizes. Any subset that mixes vertices from both partitions would contain adjacent vertices, violating independence. Therefore, no independent 2-dominating sets exist with sizes between m and n , smaller than $\min(m, n)$, or larger than $\max(m, n)$.

Combining these results gives

$$D_i^2(K_{m,n}, x) = x^m + x^n,$$

where the exponent corresponds to the size of the set, and the coefficient indicates the number of independent 2-dominating sets of that size. +

It follows directly that no independent 2-dominating set can be smaller than $\min(m, n)$ or exceed $\max(m, n)$, since either case would contradict the conditions of independence or 2-domination.

Conclusion

In this work we studied the independent 2-domination polynomial for complete bipartite graphs $K_{m,n}$. Recall that $K_{m,n}$ is partitioned into two disjoint

vertex classes U and V , with every vertex of one part adjacent to all vertices of the other. Our objective was to enumerate independent 2-dominating sets by cardinality and to derive the corresponding polynomial invariant for these graphs.

For $K_{m,n}$ we express the polynomial as

$$D_i^2(K_{m,n}, x) = \sum_{k=\gamma_i^2(K_{m,n})}^{\max(m,n)} d_i^2(K_{m,n}, k) x^k,$$

where $\gamma_i^2(K_{m,n})$ denotes the minimum size of an independent 2-dominating set and $d_i^2(K_{m,n}, k)$ counts such sets of cardinality k .

The principal conclusions are summarized below:

- No independent 2-dominating set can have cardinality smaller than $\min(m, n)$ or larger than $\max(m, n)$; such sizes would contradict either independence or the requirement that every outside vertex has at least two neighbors in the set.
- Apart from the extremal choices that take an entire partition, there are no independent 2-dominating sets in the intermediate range of sizes. Consequently, the polynomial has nonzero terms only at the partition sizes, yielding the compact form

$$D_i^2(K_{m,n}, x) = x^m + x^n.$$

These structural observations link the algebraic form of the independent 2-domination polynomial directly to the bipartition sizes of $K_{m,n}$. They also provide a clear starting point for generalizations to other bipartite families and for exploring variants such as weighted, dynamic or random bipartite models.

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