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Convex separable optimization

Stefan M. Stefanov
Department of Informatics
Neofit Rilski South-West University
Blagoevgrad 2700
Bulgaria

Abstract

The special case of *convex* separable optimization is considered in this paper. Relations between an admissible solution and an optimal solution of the initial separable optimization problem (*SP*), approximate separable problem (*ASP*), and linear approximate separable problem (*LASP*) are considered. The problem of mesh (grid) point generation, which is important about accuracy of the approximation process for the separable problem, is also considered. Estimates of the accuracy of the particular piecewise linear approximation, which is used, are presented.

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1. Introduction

Consider problems (SP), (ASP), and $(LASP)$, formulated in [25] as follows, respectively.

(SP)

$$\min\{f(x) \stackrel{def}{=} \sum_{j=1}^{n} f_j(x_j)\} \tag{S1}$$

s.t.

$$g_i(x) \stackrel{\text{def}}{=} \sum_{j=1}^{n} g_{ij}(x_j) \le \gamma_i, \forall i = 1, \dots, m \tag{S2}$$

$$x_j \ge 0, j = 1, \dots, n, \tag{S3}$$

(ASP)

$$\min\{\hat{f}(x) \stackrel{\text{def}}{=} \sum_{j\in\Lambda} f_j(x_j) + \sum_{j\notin\Lambda} \hat{f}_j(x_j)\} \tag{A1}$$

s.t.

E-mail: **stefm@swu.bg**

$$\hat{g}_i(x) \stackrel{\text{def}}{=} \sum_{j\in\Lambda} g_{ij}(x_j) + \sum_{j\notin\Lambda} \hat{g}_{ij}(x_j) \le \gamma_i, \forall i = 1,\dots,m \qquad (A2)$$

$$x_j \ge 0, j = 1,\dots,n, \qquad (A3)$$

$(LASP)$

$$\min\{\hat{f}(x) \stackrel{\text{def}}{=} \sum_{j\in\Lambda} f_j(x_j) + \sum_{j\notin\Lambda} \sum_{l=0}^{k_j} \lambda_{lj} f_j(x_{lj})\} \qquad (L1)$$

s.t.

$$\hat{g}_i(x) \stackrel{\text{def}}{=} \sum_{j\in\Lambda} g_{ij}(x_j) + \sum_{j\notin\Lambda} \sum_{l=0}^{k_j} \lambda_{lj} g_{ij}(x_{lj}) \le \gamma_i, \forall i = 1,\dots,m \qquad (L2)$$

$$\sum_{l=0}^{k_j} \lambda_{lj} = 1, j \notin \Lambda \qquad (L3)$$

$$\lambda_{lj} \ge 0, l = 0,\dots,k_j;\ j \notin \Lambda \qquad (L4)$$

$$x_j \ge 0, j \in \Lambda, \qquad (L5)$$

where

$$\Lambda = \{j\text{: functions } f_j, g_{ij} \text{ are linear}, i = 1,\dots,m\}.$$

Various aspects of separable optimization, including convex/concave separable optimization, are considered in [1]-[26], etc.

In Section 2, relations between an admissible solution and an optimal solution for the separable problem (SP), of the approximate separable problem (ASP), and of the linear approximate separable problem $(LASP)$ are stated and proved. In Section 3, the problem of mesh (grid) point generation, which is important about accuracy of the approximation process for the separable problem, is considered.

2. Relations between an admissible solution and an optimal solution for problems (SP), (ASP) and $(LASP)$

In this section, some results regarding the relations between an admissible solution and an optimal solution of the separable problem (SP), of the approximate separable problem (ASP), and of the linear approximate separable problem $(LASP)$ are formulated and proved.

Theorem 1: *Let* (SP) *be the initial separable problem,* $(LASP)$ *be its approximating problem. Suppose that functions* f_j *are strictly convex, functions* g_{ij} *are convex for indices* $i = 1,\dots,m$ *and* $j \notin \Lambda$. *If* $\hat{x}_j$ *for* $j \in \Lambda$ *and* $\hat{\lambda}_{lj}$ *for* $l = 0,\dots,k_j$ *and* $j \notin \Lambda$, *constitute a solution of problem* $(LASP)$, *then for every index* $j \notin \Lambda$, *no more than two values of* $\hat{\lambda}_{lj}$ *are positive, and they must be adjacent.*

Proof: Assume that $\exists\ \hat{\lambda}_{sj} > 0$ and $\hat{\lambda}_{tj} > 0$, with x_{sj}, x_{tj} not adjacent. Therefore $\exists$ mesh point $x_{rj} \in (x_{sj}, x_{tj})$ such that

$$x_{rj} = \lambda x_{sj} + (1-\lambda) x_{tj}, \lambda \in (0,1),$$

i.e., x_{rj} is a convex combination of x_{sj} and x_{tj}.

Denote by $u_i \geq 0$ for $i = 1, \dots, m$ the dual variables, which are associated with (L2) of problem $(LASP)$, and by v_j the dual variables (the Lagrange multipliers), which are associated with constraints (L3) of problem $(LASP)$. The Karush-Kuhn-Tucker (KKT in short) optimality conditions for program $(LASP)$ are as follows

$$f_j(x_{sj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{sj}) + v_j = 0 \quad (1)$$

$$f_j(x_{tj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{tj}) + v_j = 0 \quad (2)$$

$$f_j(x_{lj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{lj}) + v_j \geq 0, \text{for } l = 0, \dots, k_j. \quad (3)$$

From strict convexity of functions f_j and convexity of functions g_{ij} by assumption we obtain

$$\begin{aligned} & f_j(x_{rj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{rj}) + v_j \\ & < \delta f_j(x_{sj}) + (1-\delta) f_j(x_{tj}) + \sum_{i=1}^{m} u_i [\delta g_{ij}(x_{sj}) + (1-\delta) g_{ij}(x_{tj})] + v_j \\ & \equiv \delta\left[f_j(x_{sj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{sj}) + v_j\right] \\ & \quad + (1-\delta)\left[f_j(x_{tj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{tj}) + v_j\right] = 0 \end{aligned}$$

This strict inequality contradicts (3) with $l = r$. Hence, the assumption made is wrong, therefore x_{sj} and x_{tj} have to be adjacent.

Theorem 2: *If assumptions of Theorem 1 are fulfilled and* $\hat{x}_j = \sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj}$, $j \notin \Lambda$, *then* $\hat{x} = (\hat{x}_j)_n$ *is an admissible solution of the initial separable program* (SP).

Proof: Since functions g_{ij} are convex for indices $j \notin \Lambda$ and indices $i = 1, \dots, m$, and since $\hat{x}_j$, $j \in \Lambda$, $\hat{\lambda}_{lj}$, $l = 0, \dots, k_j$, $j \notin \Lambda$, form a solution of problem $(LASP)$ (and, hence, these variables are admissible for problem $(LASP)$), we obtain

$$\begin{aligned} g_i(\hat{x}) & \stackrel{\text{def}}{=} \sum_{j \in \Lambda} g_{ij}(\hat{x}_j) + \sum_{j \notin \Lambda} g_{ij}(\hat{x}_j) \\ & = \sum_{j \in \Lambda} g_{ij}(\hat{x}_j) + \sum_{j \notin \Lambda} g_{ij}\left(\sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj}\right) \\ & \leq \sum_{j \in \Lambda} g_{ij}(\hat{x}_j) + \sum_{j \notin \Lambda} \sum_{l=0}^{k_j} \hat{\lambda}_{lj} g_{ij}(x_{lj}) \leq \gamma_i, \forall i = 1, \dots, m. \end{aligned}$$

Furthermore, since $\hat{x}_j \geq 0$ for indices $j \in \Lambda$, which follows from the supposition that $\hat{x}_j$, $j \in \Lambda$, and $\hat{\lambda}_{lj}$, $l = 0, \dots, k_j$, $j \notin \Lambda$, constitute a solution of problem $(LASP)$, and because $\hat{\lambda}_{lj} \geq 0$, $x_{lj} \geq 0$, $l = 0, \dots, k_j$, $j \notin \Lambda$, it follows that $\hat{x}_j = \sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj} \geq 0$ for all indices $j \notin \Lambda$. Hence, $\hat{x} = (\hat{x}_j)_n$ is an admissible solution for the original program (SP).

Theorem 3: *Suppose that functions* f_j, g_{ij} *are convex and differentiable for all indices* $i = 1, \dots, m$; $j \notin \Lambda$ *in problems* (SP), $(LASP)$, *and* $\hat{f}_j$ *and* $\hat{g}_{ij}$ *are the piecewise linear approximations of* f_j *and* g_{ij} *in* $[a_j, b_j]$, *respectively. For* *indices* $j \not\in \Lambda$ *and for* $x_j \in [a_j, b_j]$, *let* $|f_j'(x_j)| \le M_j$. *For indices* $j \not\in \Lambda$, $i = 1, \dots, m$, *and for* $x_j \in [a_j, b_j]$, *let* $|g_{ij}{}'(x_j)| \le K_{ij}$. *For indices* $j \not\in \Lambda$, *let* ε_j *be the maximal mesh length, which is used for the variable* x_j.

Denote

$$\overline{\xi}_0 = 2\sum_{j \not\in \Lambda} M_j \varepsilon_j, \overline{\xi}_i = 2\sum_{j \not\in \Lambda} K_{ij}\varepsilon_j \forall i = 1, \dots, m, and\ \xi \overset{\text{def}}{=} \max_{0 \le i \le m} \{\overline{\xi}_i\}.$$

Then

$$\hat{f}(x) \ge f(x) \ge \hat{f}(x) - \xi \text{ and } \hat{g}_i(x) \ge g_i(x) \ge \hat{g}_i(x) - \xi, \forall i = 1, \dots, m.$$

Proof: We have to establish that

$$\hat{f}_j(x_j) \ge f_j(x_j) \ge \hat{f}_j(x_j) - 2M_j\varepsilon_j, j \not\in \Lambda.$$

Let $j \not\in \Lambda$, and let $x_j \in [a_j, b_j]$. Then $\exists$ mesh points μ_k and μ_{k+1} with $x_j \in [\mu_k, \mu_{k+1}]$. Hence $x_j = \delta\mu_k + (1-\delta)\mu_{k+1}$ for a $\delta \in [0,1]$. Therefore

$$\hat{f}_j(x_j) = \delta f_j(\mu_k) + (1-\delta) f_j(\mu_{k+1}) \ge f_j(\delta\mu_k + (1-\delta)\mu_{k+1}) = f_j(x_j),$$

and this is the first inequality to be proved. The definition of $\hat{f}_j$, the assumption of convexity of f_j, $\delta \in [0,1]$, and the expression of x_j are used in order to draw the above conclusions.

Because $\hat{f}_j(x_j)$ is a piecewise linear approximation of $f_j(x_j)$, and μ_k, μ_{k+1} are mesh points by assumption, we have that

$$\hat{f}_j(x_j) = f_j(\mu_k) + (x_j - \mu_k)\frac{f_j(\mu_{k+1}) - f_j(\mu_k)}{\mu_{k+1} - \mu_k}. \tag{4}$$

If we use the following property of the convex differentiable functions

$$f(x_1) - f(x_2) \ge \langle f'(x_2), x_1 - x_2 \rangle,$$

we obtain

$$f_j(x_j) \ge f_j(\mu_k) + (x_j - \mu_k) f_j'(\mu_k). \tag{5}$$

Then (4) and (5) imply

$$\hat{f}_j(x_j) - f_j(x_j) \le (x_j - \mu_k)\left[\frac{f_j(\mu_{k+1}) - f_j(\mu_k)}{\mu_{k+1} - \mu_k} - f_j'(\mu_k)\right]. \tag{6}$$

A calculus theorem states that $\exists$ a value $z \in [\mu_k, \mu_{k+1}]$ with

$$\frac{f_j(\mu_{k+1}) - f_j(\mu_k)}{\mu_{k+1} - \mu_k} = f_j'(z).$$

By the assumption made,

$$\frac{f_j(\mu_{k+1}) - f_j(\mu_k)}{\mu_{k+1} - \mu_k} - f_j'(\mu_k) \le 2M_j,$$

and $x_j - \mu_k \le \varepsilon_j$. From (6) we have

$$\hat{f}_j(x_j) - f_j(x_j) \le 2M_j\varepsilon_j.$$

Hence

$$\hat{f}_j(x_j) \ge f_j(x_j) \ge \hat{f}_j(x_j) - 2M_j\varepsilon_j, \quad j \not\in \Lambda \text{ and } x_j \in [a_j, b_j]. \tag{7}$$

Then (7) implies

$$\sum_{j\notin\Lambda} \hat{f}_j(x_j) + \sum_{j\in\Lambda} f_j(x_j) \geq \sum_{j\notin\Lambda} f_j(x_j) + \sum_{j\in\Lambda} f_j(x_j) \geq \sum_{j\notin\Lambda} \hat{f}_j(x_j) + \sum_{j\in\Lambda} f_j(x_j) - 2\sum_{j\notin\Lambda} M_j\varepsilon_j,$$

i.e.,

$$\hat{f}(x) \geq f(x) \geq \hat{f}(x) - \overline{\xi}_0.$$

Analogously, we obtain

$$\hat{g}_i(x) \geq g_i(x) \geq \hat{g}_i(x) - \overline{\xi}_i \text{ for } i = 1, \dots, m.$$

Using the expression of ξ, the proof of Theorem 3 is complete.

Proposition 1: Let problems (SP), (ASP), $(LASP)$ and the set Λ be defined as in the statement of Theorem 1. For indices $j \notin \Lambda$, $i = 1, \dots, m$, let $\hat{g}_{ij}$ be the linear piecewise approximations of the convex functions g_{ij}. Assume that $\hat{x}_j$, $j \in \Lambda$, and $\hat{\lambda}_{lj}$, $l = 0, \dots, k_j$, $j \notin \Lambda$, constitute an optimal solution of program $(LASP)$ with $\hat{x}$, whose elements, defined as follows: $\hat{x}_j$, $j \in \Lambda$, and $\hat{x}_j = \sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj}$, $j \notin \Lambda$, is the optimal solution of (ASP).

Then $\hat{x}$ is an admissible solution for the initial separable problem (SP).

Proof: Since $\hat{x}$ is optimal solution of problem (ASP) by assumption, then $\hat{x}$ is an admissible solution of problem (ASP), i.e., $\hat{g}_i(\hat{x}) \leq \gamma_i$, $i = 1, \dots, m$, and $\hat{x} \geq 0$. From Theorem 3 we have the following implication

$$\hat{g}_i(\hat{x}) \leq \gamma_i \Rightarrow g_i(\hat{x}) \leq \hat{g}_i(\hat{x}) \leq \gamma_i, i = 1, \dots, m,$$

and because $\hat{x} \geq 0$, then $\hat{x}$ is an admissible solution of the initial separable problem (SP).

3. Mesh Point Generation

In this section, the problem of mesh point generation is considered.

Assumptions I.

Let x_{lj}, $l = 0, \dots, k_j$, $j \notin \Lambda$, be the mesh points.

Consider problem $(LASP)$, which approximates the initial separable problem (SP), using the mesh points x_{lj}, $l = 0, \dots, k_j$ for indices $j \notin \Lambda$.

The function of Lagrange of program $(LASP)$ is

$$L(x, \lambda, u, v) = \sum_{j\in\Lambda} f_j(x_j) + \sum_{j\notin\Lambda} \sum_{l=0}^{k_j} \lambda_{lj} f_j(x_{lj}) + \sum_{i=1}^{m} u_i \left[\sum_{j\in\Lambda} g_{ij}(x_j) + \sum_{j\notin\Lambda} \sum_{l=0}^{k_j} \lambda_{lj} g_{ij}(x_{lj}) - \gamma_i\right] + \sum_{j\notin\Lambda} v_j \left[\sum_{l=0}^{k_j} \lambda_{lj} - 1\right], \tag{8}$$

with $x_j \geq 0$ for $j \in \Lambda$, $\lambda_{lj} \geq 0$ for $l = 0, \dots, k_j$ with $j \notin \Lambda$, and $u_i \geq 0$ for all $i = 1, \dots, m$.

The optimal solution $\hat{x}_j, \hat{\lambda}_{lj}$ of problem $(LASP)$ as well as the corresponding Lagrange multipliers $\hat{u}_i, \hat{v}_j$ fulfil the KKT optimality conditions for problem $(LASP)$.

Using that $f_j(x_j) = c_j x_j$ and $g_{ij}(x_j) = a_{ij} x_j$ for indices $j \in \Lambda$, the KKT optimality conditions for problem $(LASP)$ can be written as follows:

$$c_j + \sum_{i=1}^{m} u_i a_{ij} \geq 0, \forall j \in \Lambda \tag{9}$$

$$x_j \left[c_j + \sum_{i=1}^{m} u_i a_{ij}\right] = 0, \forall j \in \Lambda \tag{10}$$

$$f_j(x_{lj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{lj}) + v_j \geq 0, \forall l = 0, \dots, k_j, j \not\in \Lambda \tag{11}$$

$$\lambda_{lj}\left[f_j(x_{lj}) + \sum_{i=1}^{m} u_i g_{ij}(x_{lj}) + v_j\right] = 0, \forall l = 0, \dots, k_j, j \not\in \Lambda \tag{12}$$

$$x_j \geq 0, j \in \Lambda, \lambda_{lj} \geq 0, \forall l = 0, \dots, k_j, j \not\in \Lambda \tag{13}$$

$$\sum_{j \in \Lambda} a_{ij} x_j + \sum_{j \not\in \Lambda} \sum_{l=0}^{k_j} \lambda_{lj} g_{ij}(x_{lj}) - \gamma_i \leq 0, \forall i = 1, \dots, m \tag{14}$$

$$u_i \left[\sum_{j \in \Lambda} a_{ij} x_j + \sum_{j \not\in \Lambda} \sum_{l=0}^{k_j} \lambda_{lj} g_{ij}(x_{lj}) - \gamma_i\right] = 0, \forall i = 1, \dots, m \tag{15}$$

$$\sum_{l=0}^{k_j} \lambda_{lj} = 1, \forall j \not\in \Lambda \tag{16}$$

$$u_i \geq 0, \forall i = 1, \dots, m. \tag{17}$$

Since $f_j(x_j) \stackrel{\text{def}}{=} c_j x_j$ and $g_{ij}(x_j) \stackrel{\text{def}}{=} a_{ij} x_j$ for indices $j \in \Lambda$, if we multiply both sides of (9) by $x_j \geq 0$, $j \in \Lambda$, we obtain

$$f_j(x_j) + \sum_{i=1}^{m} u_i g_{ij}(x_j) \geq 0 \; \forall x_j \geq 0, j \in \Lambda. \tag{9'}$$

Let $x_{l_0 j}$ be additional mesh point. If

$$f_j(x_{l_0 j}) + \sum_{i=1}^{m} \hat{u}_i g_{ij}(x_{l_0 j}) + \hat{v}_j \geq 0, \tag{18}$$

then, setting $\hat{\lambda}_{l_0 j} = 0$, the KKT optimality conditions for program $(LASP)$ will be fulfilled.

Because it is not known where the new mesh point is to be located, we must solve next subproblem for every index $j \not\in \Lambda$ in order to establish if all variables x_j, which satisfy the box constraints $a_j \leq x_j \leq b_j$ for $j \not\in \Lambda$, would also satisfy condition (18).

Subproblem of problem (SP).

$$\min\left\{d_j(x_j) = f_j(x_j) + \sum_{i=1}^{m} \hat{u}_i g_{ij}(x_j) + \hat{v}_j\right\}$$

s.t.

$$a_j \leq x_j \leq b_j,$$

where $[a_j, b_j]$ with $b_j > a_j \geq 0$ denotes the admissible interval for the variable x_j for $j \not\in \Lambda$.

If the minimal objective function value of $d_j(x_j)$ of the Subproblem is nonnegative for all indices $j \not\in \Lambda$, it turns out that a new mesh point cannot be found, which violates condition (18). The next Theorem 4 states that if this is fulfilled, the current admissible solution will be optimal for the initial separable program (SP), and if the minimal objective function value for this problem turns out to be negative for some index $j \notin \Lambda$, a better approximation for program (SP) can be obtained.

Theorem 4: *Suppose that Assumptions* ***I.*** *are satisfied.*

For every index $j \notin \Lambda$, let consider Subproblem of problem (SP) stated above. Denote by d_j^* the optimal objective function value for this problem.

Then we have

$$\sum_{j \notin \Lambda} (d_j^* - \hat{v}_j) - \sum_{i=1}^m \hat{u}_i \gamma_i \le \sum_{j=1}^n f_j(x_j^*) \le \sum_{j=1}^n f_j(\hat{x}_j) \le \hat{f},$$

where $\hat{x}_j = \sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj}$, $j \notin \Lambda$, and $x^* = (x_j^*)_n$ is an optimal solution of the initial program (SP).

Proof: Because $\hat{u}_i$, $\hat{v}_j$ are the dual variables, which are associated with problem $(LASP)$, the following KKT optimality conditions (9) hold true:

$$c_j + \sum_{i=1}^m \hat{u}_i a_{ij} \ge 0, \forall j \in \Lambda.$$

Multiply the above inequalities by $x_j \ge 0$ and take into consideration that $f_j(x_j) \overset{\text{def}}{=} c_j x_j$, $g_{ij}(x_j) \overset{\text{def}}{=} a_{ij} x_j$ for indices $j \in \Lambda$, we obtain

$$f_j(x_j) + \sum_{i=1}^m \hat{u}_i g_{ij}(x_j) \ge 0 \ \forall x_j \ge 0, j \in \Lambda. \tag{19}$$

Using the definition of d_j^*, we get

$$f_j(x_j) + \sum_{i=1}^m \hat{u}_i g_{ij}(x_j) + \hat{v}_j \ge d_j^*, \forall j \notin \Lambda, a_j \le x_j \le b_j. \tag{20}$$

Summing (19) over all indices $j \in \Lambda$ and (20) over all indices $j \notin \Lambda$, and after that subtracting $\sum_{i=1}^m \hat{u}_i \gamma_i$ from the obtained sum, we get

$$\sum_{j=1}^n f_j(x_j) + \sum_{i=1}^m \hat{u}_i \left[\sum_{j=1}^n g_{ij}(x_j) - \gamma_i\right] \ge \sum_{j \notin \Lambda} (d_j^* - \hat{v}_j) - \sum_{i=1}^m \hat{u}_i \gamma_i \tag{21}$$

for the variables $x_j \in [a_j, b_j]$. If we use that

$$a_j \le x_j^* \le b_j, \sum_{j=1}^n g_{ij}(x_j^*) \le \gamma_i, \hat{u}_i \ge 0,$$

from (21) we get

$$\sum_{j=1}^n f_j(x_j^*) \ge \sum_{j \notin \Lambda} (d_j^* - \hat{v}_j) - \sum_{i=1}^m \hat{u}_i \gamma_i,$$

which is the first inequality we had to prove. In accordance with Proposition 1, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ is an admissible solution for the initial program (SP), therefore $\sum_{j=1}^n f_j(x_j^*) \le \sum_{j=1}^n f_j(\hat{x}_j)$, which is the second inequality to be proved.

From the convexity of functions f_j for indices $j \notin \Lambda$ as well as from definition of $\hat{f}$ we obtain

$$\sum_{j=1}^n f_j(\hat{x}_j) = \sum_{j \in \Lambda} f_j(\hat{x}_j) + \sum_{j \notin \Lambda} f_j(\hat{x}_j)$$

$$= \sum_{j \in \Lambda} f_j(\hat{x}_j) + \sum_{j \notin \Lambda} f_j\left(\sum_{l=0}^{k_j} \hat{\lambda}_{lj} x_{lj}\right) \le \sum_{j \in \Lambda} f_j(\hat{x}_j)$$

$$+ \sum_{j \notin \Lambda} \sum_{l=0}^{k_j} \hat{\lambda}_{lj} f_j(x_{lj}) = \hat{f},$$

which is the third inequality to be proved.

Theorem 5 *(Generation of mesh points for problem* $(LASP)$*). Suppose that Assumptions **I.** are fulfilled and for every* $j \notin \Lambda$, d_j^* *is the optimal objective function value of Subproblem for problem* (SP)*. Then:*

1. *If* $d_j^* \geq 0$ *for all* $j \notin \Lambda$, *then* $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ *is optimal solution of program* (SP), *and* $\sum_{j=1}^{n} f_j(\hat{x}_j) = \hat{f}$.
2. *Else, if* $d_j^* < 0$ *for an index* $j \notin \Lambda$ *and* $x_{l_0 j}$ *is the optimal solution of the corresponding Subproblem for problem* (SP), *then, if insert the mesh point* $x_{l_0 j}$ *in program* $(LASP)$, *we obtain a new approximating problem like problem* $(LASP)$, *whose minimum objective function value is not greater than the value* $\hat{f}$.

The proof of Theorem 5 is omitted.

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