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A tight semidefinite relaxation for portfolio optimization problem with cardinality constraint

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Abstract

We consider the use of mean-variance (MV) portfolio optimization for investors with the risk preference parameter and some real-life trading constraints, such as the cardinality constraint. For the selection of portfolios, semidefinite relaxation (SDR) is used to increase the sparsity as well as to obtain a tight lower bound. Generally, to solve the cardinality constrained portfolio optimization (CCPO), which is NP-hard in nature, one can reformulate it as a mixed-integer quadratic programming problem (MIQP) and solve it with several existing global solvers. During SDR, the substitution leads to a nonconvex rank-one constraint. Many authors dropped it to reduce it to a convex problem, which may sometimes produce a significant discrepancy between the original problem's optimal values and the relaxed problem. In this paper, we propose two SDR models for the MIQP model and a direct SDR model by convexifying the rank-one constraint, which increases the sparsity and provides a tight lower bound. The effectiveness of the proposed model is demonstrated via numerical analysis of historical stock returns.

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1. Introduction

The Markowitz [1, 2] mean-variance (MV) portfolio optimization problems in finance aims to distribute capital among various financial holdings as efficiently as possible. It is mainly centered on finding the best portfolio that trades off between the mean and variance of the asset returns. The main goal of the portfolio optimization model is either maximize return under a specific degree of risk or minimize risk with a particular amount of predicted return. However, Markowitz's model takes two input parameters such as the return vector and the covariance matrix, which are to be estimated from historical finance data. Note that the estimation error and the parameter uncertainty have a great impact on the performance of Markowitz's MV model. Moreover, the classical mean-variance portfolio optimization model ignores many realistic constraints faced by investors from a practical point of view. Thus, several researchers successfully investigated and proposed an alternative model by modifying the MV-model. In [3], Konno and Yamazaki presented the mean absolute deviation model, which uses the absolute deviation of the rate of return on the portfolio as a measure of risk rather than variance. Sharpe [4] defined the Sharp ratio model to measure risk volatility. The omega ratio approach was put forth by Bernadro and Ledoit [5] to examine portfolio performance jointly based on risk and return. Elton et al. [6] studied an optimization model that allows the short sales case that has some weight to be negative. One can also find the traditional optimization models for portfolio selection in [7]. Sasaki, Laamrani, and Yamashiro, [8] proposed method aids investor judgment (i.e., preferences regarding risk and return) by optimizing portfolios to provide a set of solutions that are both Pareto optimal and aligned with the investor's preferences. This system uses an Interactive Genetic Algorithm (iGA), which incorporates the decision maker's subjective preferences.

Portfolio optimization models also have many different practical constraints, including the buy-in threshold, maximal hold constraints, cardinality, tracking error, turnover, sector limit, and round-lot constraints. Markowitz's model gives the advantages of diversification, which is the fundamental concept of modern portfolio selection, but diversification in investment is not completely free. Markowitz's MV model constructs a high diversification by investing in a large number of assets, which raises management fees and transaction expenses. As a result, different practical constraints are added to the portfolio model. A no-short-sales constraint was applied to the MV-model by Jagannathan and Ma [9], who also showed how the "wrong" constraint aids in identifying the solution with superior out-of-sample performance. Additionally, the authors provided some astute justifications, how the incorrect constraint can lead to discovery of a solution with superior out-of-performance. Mencarelli and d'Ambrosio [10] provided a survey on real-life constraints on portfolio selection models.

The present research focuses on the cardinality constrained portfolio optimization (CCPO) problem. Because of the cardinality constraint, which turns the portfolio optimization model into an NP-hard task, these issues are infamously difficult to resolve both theoretically and practically. As a result, the cardinality constraint problem has drawn the attention of numerous scholars. Nevertheless, the mean-variance optimization model with cardinality constraints was reconstructed as a mixed integer quadratic programming problem (MIQP), which is likewise challenging to solve. A lot of work has gone into creating effective algorithms to solve issues with cardinality constraints. Both exact and heuristic methods have been proposed for cardinality constraint portfolio optimization problems. For the exact method, a couple of branch and bound algorithm were developed for the solution of the CCPO to global optimality ([11, 12, 13, 14, 15, 16, 17]). Particularly, Shaw, Liu, and Kopman [18] describe a Lagrangian relaxation method to solve the cardinality constraint portfolio problem by using the branch and bound method. By utilizing Lemkes's algorithm for CCPO, Bertsimas and Shioda [19] created a branch and bound based algorithm that, for randomly generated data, outperforms the quadratic mixed integer solver CPLEX's. Frangioni and Gentile [20, 21] suggested a novel perspective cut reformulation for a mixed integer quadratic program with semi-continuous variables that developed an efficient lower bound for CCPO. Gao and Li [22] investigated an efficient exact solution algorithm for CCPO without relaxing the cardinality constraint directly. They modified the objective function to some separable relaxation, which exempts the cardinality constraint. In a pure cardinality constraint portfolio optimization issue, they enforced the Lagrangian relaxation approach to the primal problem and used their mathematical formulation to study the geometric properties and special structures. Tian et al. [23] proposed a completely positive programming reformulation for the CCPO problem. By employing a sequence of computable cones of non-negative quadratic forms over a combination of second-order cones, SDP approaches can be utilized to obtain an ε-optimal solution to the original problem within a finite number of iterations.The proposed formulation provided a tight lower bound efficiently.

Burdakov, Kanzow, and Schwartz [24] presented a nonlinear mixed integer (NLMI) programming formulation approach for cardinality constrained problems that have still same solution in the terms of global minima, and the reformulation is used as a tool for minimization problems with continuous variables. A mixed integer linear programming (MILP) problem reformulation for cardinality-constrained portfolio optimization (CCPO) was proposed by Hardoroudi et al. [25]. It was demonstrated that the MILP reformulation significantly increases computational effort and robustness when compared to MIQP. Weigele and Zhao [26] showed advantages of the semidefinite relaxation approach over integer and mixed integer nonlinear programming problems considering cardinality constrained portfolio selection problems. Frangioni, Furini, and Gentile [27] proposed an approximated projected perspective reformulation of CCPO that solves the perspective reformulation ones and then uses dual information to apply the new approach, which reduces relaxation computing time and produces quality bounds. Cui et al. [28] proposed a new MIQCQP formulation that is tighter than the standard reformulation via a new Lagrangian decomposition scheme. Zheng et al. [17] present a semidefinite programming (SDP) approach for determining the optimal diagonal decomposition within the perspective reformulation of a relaxed mixed-integer quadratic programming (MIQP) portfolio optimization problem. This approach utilizes a novel Lagrangian decomposition strategy. In [29], Jimbo et al., investigated a comparative study of the CCPO problems. They showed that with a relatively small or large number of assets under a cardinality constraint, they produce similar results with different reliability. Miroforidis [30] demonstrated the concept of upper bounds and lower bounds on objective function values of an efficient portfolio to find a bound for large scale CCPO problems. Furthermore, Tillmann et al. [31] explored generic modeling strategies as well as problem-specific exact and heuristic solutions to CCPO challenges.

Although the majority of the authors researched the CCPO-model and concentrated mostly on creating effective algorithms to address the issue, d'Aspremont et al. [32] developed a method that directly converts the nonconvex cardinality constraint into a convex problem by using semidefinite relaxation (SDR). Kim et al. [33] used the SDR to solve the sparse tangent portfolio optimization problem. Again, Lee et al. [34] developed sparse and robust models using the SDR approach. More specifically, the SDR of the cardinality constraint becomes a semidefinite programming problem that can be easily solved using an existing SDP solver. But in general, the SDR model contains a rank-one constraint which makes the model nonconvex and NP-hard. Most of the authors dropped the rank constraint made it into a convex optimization model. However, dropping a constraint makes the feasible region wider, so the problem should be feasible if the original problem is feasible.

1.1 *Contributions*

The contributions of this paper are mainly on establishing new CCPO models with the aversion parameter attached to the mean term, as listed below.

- MIQP model with bound constraints,
- SDP formulation of the MIQP model described above,
- A direct SDR of CCPO with rank-one relaxations,
- Nonlinear MIQP problem with complementary constraints, maximal hold, and minimal buy-in thresholds constraints and their SDP formulations.

Also, to justify the tightness of the objective value, we compare the objective values of all the models over some real datasets. The structure of the paper is as follows: In Section 2, we introduce the formulation of the portfolio optimization problem with a cardinality constraint. In Section 3, we derive an equivalent MIQP reformulation as well as the SDR approach for the CCPO. Comparative results with real data instances are reported in Section 4. Finally, Section 5 provides some conclusions.

2. Problem formulation

We take n risky assets with an anticipated return vector $\mu \in R^n$ and a covariance matrix of the returns $\mathbf{Q} \in R^{n\times n}$ which is a symmetric positive semidefinite matrix of order n. Therefore, $\mu^T x$ represents the expected return and $x^T\mathbf{Q}x$ represents the variance of the portfolio return, where x indicates the weighted percentage of the individual assets portfolio. Then the classical mean-variance model by Markowitz is to minimize the risk subject to the expected amount of return μ_0 or maximize the profit with a given label of risk σ_0 and vice versa. It can be expressed as :

$$\min_x \quad x^T\mathbf{Q}x \quad \text{s.t} \quad \mu^T x \geq \mu_0, \quad e^T x = 1 \tag{1}$$

or

$$\max_x \quad \mu^T x \quad \text{s.t} \quad x^T\mathbf{Q}x \leq \sigma_0, \quad e^T x = 1. \tag{2}$$

Note that the constraint $e^T x = 1$ denotes budget constraint requiring the investment of all money. Since we want to both maximize the return while minimizing risk, one may express the above MV models (1) and (2) as the return-risk trade-off and express them as a quadratic programming (QP) problem

$$\begin{aligned} \min_x \quad & x^T\mathbf{Q}x - \mu^T x \\ \text{s.t} \quad & e^T x = 1. \end{aligned} \tag{3}$$

However, by introducing a risk preference parameter $\lambda \geq 0$ that controls the trade-off between the risk and return of the asset portfolio, the classical MV model (3) can be represented as follows:

$$\begin{aligned} \min_x \quad & x^T\mathbf{Q}x - \lambda\mu^T x \\ \text{s.t} \quad & e^T x = 1. \end{aligned} \tag{4}$$

where the choice of λ as the risk parameter is crucial to the investment. Fabozzi et al. [35] and Liu and Xu [36] studied a risk aversion parameter MV-model for portfolio optimization of the form

$$\begin{aligned} \max_x \quad & \mu^T x - \lambda x^T\mathbf{Q}x \\ \text{s.t} \quad & e^T x = 1. \end{aligned} \tag{5}$$

When the aversion to risk is low, i.e., λ is small, the portfolio risk is likewise minimal since the penalty from the contribution is minor, which leads to more risky portfolios. In contrast, portfolios with higher risk exposures are more severely penalized when λ is rather large. The authors discussed the impact of the risk aversion parameter on optimization.

Although, from the optimization point of view, assigning a coefficient λ to the mean term is equivalent to doing the same for the variance term, for the portfolio investment point of view, our model (4) is just opposite to the model (5).

Moreover, due to administrative concerns and transaction costs, cardinality constraints (i.e., the maximum number of stocks to be selected) are imported into the portfolio optimization models. Therefore, by introducing the cardinality constraint into the model (4), the number of assets is restricted to being less than or equal to k (positive integer). i.e., introducing the constraint $card(x) \leq k$, the model is

$$\begin{array}{ll} \min\limits_{x} & x^T \mathbf{Q} x - \lambda \mu^T x \\ \text{s.t} & e^T x = 1, \\ & \text{card}(x) \leq k. \end{array} \tag{6}$$

Here, $card(x)$ represents the cardinality of any vector $x \in R^n$. The cardinality constrained optimization model CCPO is an NP-Hard problem [37] and requires relaxation to get an approximate solution. In Section 3, we reformulate CCPO as a mixed-integer quadratic programming problem (MIQP) model, then reformulate again into a semidefinite programming problem. A direct semidefinite relaxation approach to reformulating the CCPO problem into a semidefinite programming (SDP) problem is described in subsection 3.1. In the next subsection, we model CCPO into a relaxed nonlinear mixed integer quadratic programming problem using the idea of Burdakov et al. [24]. Further, a semidefinite relaxation approach is applied to the relaxed nonlinear mixed integer quadratic programming to get a relaxed SDP model.

3. Equivalent formulations and continuous relaxations

Since the problem (6) can be expressed as a mixed-integer problem, we introduce binary variables $y_i \in \{0,1\}$ and reformulating the cardinality constraint as $-My_i \leq x_i \leq My_i$, $\sum_{i=1}^{n} y_i \leq k$, $\forall i, y \in \{0,1\}^n$, with M being an upper bound on $\|x\|_\infty$, the MIQP model of (6) is written as ([19]):

$$\begin{array}{ll} \min\limits_{x} & x^T \mathbf{Q} x - \lambda \mu^T x \\ \text{s.t} & e^T x = 1, \\ & -My_i \leq x_i \leq My_i, \\ & \sum_{i=1}^{n} y_i \leq k, \forall i, \\ & y \in \{0,1\}^n. \end{array} \tag{7}$$

In general, this MIQP reformulation model is an NP-hard problem. When k is small, one can find the optimal solution by using global solvers like GUROBI [38] and CPLEX [39].

Since $\mathbf{Q}$ is positive semidefinite, the objective function of the problem (7) is convex except for the binary constraint $y_i \in \{0,1\}$. The natural continuous convex relaxation of $y_i \in \{0,1\}$ is $y_i \in [0,1]$.

So, the MIQP model (7) can be relaxed as:

$$\begin{array}{ll} \min\limits_{x} & x^T \mathbf{Q} x - \lambda \mu^T x \\ \text{s.t} & e^T x = 1, \\ & -M y_i \leq x_i \leq M y_i, \\ & \sum_{i=1}^{n} y_i \leq k, \forall i, \\ & y \in [0,1]^n. \end{array} \tag{8}$$

If one wishes to view the model (8) as a quadratic constraint quadratic programming (QCQP) problem, the constraint $y_i \in [0,1]$ can be written as $y_i^2 - y_i \leq 0$, and so the QCQP model is

$$\begin{array}{ll} \min\limits_{x} & x^T \mathbf{Q} x - \lambda \mu^T x \\ \text{s.t} & e^T x = 1, \\ & -M y_i \leq x_i \leq M y_i, \\ & \sum_{i=1}^{n} y_i \leq k, \\ & y_i^2 - y_i \leq 0, \forall i = 1,2,3,\cdots,n. \end{array} \tag{9}$$

We can say that the ideal solution of the model (9) provides a lower bound for the model (6).

The SDR of QCQP (9) is obtained by substituting $X = xx^T$ and $Y = yy^T$, which may be relaxed to $X \succcurlyeq xx^T$ and $Y \succcurlyeq yy^T$. Also, we have $x^T \mathbf{Q} x = Tr(\mathbf{Q}X)$, where $Tr(\cdot)$ indicates the trace of a matrix. Thus, the SDR of the model (7) can be expressed as

$$\begin{array}{ll} \min\limits_{x,X} & Tr(\mathbf{Q}X) - \lambda \mu^T x \\ \text{s.t} & e^T x = 1, \\ & -M y_i \leq x_i \leq M y_i, \\ & \sum_{i=1}^{n} y_i \leq k, \\ & diag(Y) \leq y, \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succcurlyeq 0, \begin{pmatrix} 1 & y^T \\ y^T & Y \end{pmatrix} \succeq 0. \end{array} \tag{10}$$

Proposition 1: Suppose there is an interior point in the problem's feasible set (9). Then, the optimal values of the model (9) and the SDR model (10) are equal.

Proof: See ([40]).

3.1 *A direct Semidefinite Programming Relaxation*

In this subsection, we use the direct semidefinite relaxation approach of d'Asperemont et al. [32], Kim et al. [33] and Lee et al. [34] to the cardinality constraint portfolio optimization model without converting to MIQP first.

We have the following proposition:

Proposition 2: The CCPO model (6) is equivalent to the following model (11).

$$\begin{array}{lll}\min\limits_{x,X} & Tr(\mathbf{Q}X) - \lambda\mu^T x & \\ \text{s.t.} & Tr(ee^T X) = 1, & \\ & card(X) \le k^2, & (11) \\ & Rank(X) = 1, & \\ & X \in S_n^+. & \end{array}$$

where the set S_n^+ denotes positive semidefinite (psd) matrices of order n and $Rank(X)$ refers rank of the matrix X.

Proof: Since a positive semidefinite matrix with rank-one has a unique eigenvector, the feasible solution $X \in S_n^+$ can be expressed as $X = xx^T$. Also, the constraint $e^T x = 1$ is equivalent to $Tr(ee^T X) = 1$ and $card(x) \le k$ implies $card(X) \le k^2$. Since, $x^T\mathbf{Q}x = Tr(\mathbf{Q}X)$, the proof is done.

3.2 *Convexifying cardinality constraint*

Due to the presence of the cardinality constraint and the rank-one constraint, model (11) remains non-convex. To address this, we apply the direct semidefinite relaxation method proposed by d'Aspremont et al. [32]. For any $x \in R^n$, we know $\|x\|_1 \le \sqrt{n}\|x\|_2$, and this inequality can be tightened to $\|x\|_1 \le \sqrt{k}\|x\|_2$ if $card(x) \le k$. Consequently, we can relax the cardinality constraint into a weaker but convex constraint as $e^T|X|e \le kTr(X)$, where $|\cdot|$ represents the element-wise absolute value operator [34].

3.3 *Convexifying rank-one constraint:*

In most of the research work, the authors dropped the rank-one constraint to convert the model into a convex problem. However, there may be sometimes produce a significant discrepancy between the original problem's optimal values and the relaxed problem's when the rank-one constraint is removed. Thus, we prefer to convexify the rank-one constraint by using the Frobenius norm $\|X\|_F$. The following Lemma justifies it.

Lemma 1: *Let*

$$A = \{X \in S_n : X \succcurlyeq 0, X = xx^T\}.$$

Then we have the inequality $\|X\|_F \le n$.

Proof: For $X \in A$, $\|X\|_F^2 = Tr(XX^T) = \|\lambda(X)\|_2^2 \le \|\lambda(X)\|_1^2$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote l_1 and l_2 norm of a vector respectively and $\|\lambda(X)\|$ is the vector of eigenvalues of X. Since $X \succcurlyeq 0$, then $\lambda(X) \ge 0 \Rightarrow \|\lambda(X)\|_1 = Tr(X)$. But,

$$\begin{array}{ll} & Tr(X) = \sum_{i=1}^n x_i^2 \le \sum_i^n 1 = n \\ \Rightarrow & \|\lambda(X)\|_1 \le n \\ \Rightarrow & \|X\|_F \le \|\lambda(X)\|_1 \le n. \end{array}$$

Alternatively, the straight forward proof is as follows. For $X = xx^T$,

$$\|X\|_F = \sqrt{Tr(XX^T)} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n X_{ij}^2} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i^2 x_j^2}$$
$$\leq \sqrt{\sum_{i=1}^n \sum_{j=1}^n 1} = \sqrt{n^2} = n.$$

Thus, $Rank(X) = 1$ can be relaxed as a convex constraint $\|X\|_F \leq n$. But, in most of the real-world scenario n is large and $\|X\|_F$ lies in $(0,1)$ which yields $\|X\|_F \ll n$. Therefore, to make a balance between the difference between $\|X\|_F$ and n, we choose a $\alpha \in (0,1)$ such that $\|X\|_F \leq \alpha n$. Thus, the equivalent semidefinite relaxation of the cardinality constraint mean-variance portfolio model is:

$$\begin{array}{ll} \min\limits_{x} & Tr(\mathbf{Q}X) - \lambda\mu^T x \\ \text{s.t.} & Tr(ee^T X) = 1, \\ & e^T|X|e \leq kTr(X), \\ & \|X\|_F \leq \alpha n, \alpha \in (0,1), \\ & X \in S_n^+. \end{array} \tag{1}$$

Proposition 3: The optimization model (11) and relaxed optimization model (12) are equivalent for some $\alpha \in (0,1)$.

Proof: The proof is straightforward.

Further, the SDR (12) can be presented as:

$$\begin{array}{ll} \min\limits_{x,X} & Tr(\mathbf{Q}X) - \lambda\mu^T x - \beta(\alpha n - \|X\|_F) \\ \text{s.t.} & Tr(ee^T X) = 1, \\ & e^T|X|e \leq kTr(X), \\ & X \in S_n^+. \end{array} \tag{13}$$

where $\beta > 0$ is a scalar parameter. Note, one can always find a β such that problems (12) and (13) are equivalent. If β approaches zero the model (13) reduces to the MV model with a relaxed cardinality constrained model and a dropped rank-one constraint adopted by most of the researchers. The following lemma establishes the tightness of the optimal bound over (12).

Lemma 2: *If* $f(X) = Tr(\boldsymbol{Q}X) - \lambda\mu^T x$ *and* $f_\beta(X) = Tr(\boldsymbol{Q}X) - \lambda\mu^T x - \beta(\alpha n - \|X\|_F)$ *with* $\alpha \in (0,1)$, *then* $f_\beta(X^*) \leq f(X^*)$ *for an optimal solution* X^*.

Proof:

$$f_\beta(X^*) - f(X^*) = \beta(\|X^*\|_F - \alpha n) \leq 0,$$

for a positive β.

$$f_\beta(X^*) \leq f(X^*).$$

3.4 *Extended CCPO*

In this subsection, we introduce a portfolio optimization model that has minimal buy-in thresholds, maximal hold requirements, and the cardinality constraint. The model can be written as follows:

$$\begin{array}{ll} \min\limits_{x} & x^T\mathbf{Q}x - \lambda\mu^T x \\ \text{s.t} & e^T x = 1, \\ & card(x) \leq k, \\ & x_i \geq l_i, \forall i \in card(x), \\ & 0 \leq x_i \leq u_i, \forall i = 1, \dots, n. \end{array} \tag{14}$$

In this case, u_i is the maximum hold level and l_i is the minimum buy-in threshold in i-th asset; that is, the constraint $0 \leq x_i \leq u_i, \forall i = 1, \dots, n,$ prohibits short sales and limits the maximum amount held for each asset, while the constraint $x_i \geq l_i, \forall i \in card(x)$ establishes the minimum threshold for the chosen i-th asset. Also, by using the relaxation approach of [11, 24], the CCPO model (14) is equivalent to a nonlinear mixed integer programming problem described as follows:

$$\begin{array}{ll} \min\limits_{x} & x^T\mathbf{Q}x - \lambda\mu^T x \\ \text{s.t} & e^T x = 1, \\ & l_i(1-y_i) \leq x_i \leq u_i(1-y_i), \\ & \sum_{i=1}^{n} y_i \geq n-k, \forall i, \\ & x \circ y = 0, \\ & y \in \{0,1\}^n. \end{array} \tag{15}$$

where $y \in \{0,1\}^n$ are the slack variables and $x \circ y = 0$ is the nonlinear complementary constraint relates x and y. Here $\circ$ denotes the the element-wise multiplication. We replace the cardinality constraint $card(x) \leq k$ by $e^T y \geq n-k$ as it is required that y has at least $n-k$ non-zero elements. It implies that x has at most k non-zero elements. i.e. $x \circ y = 0$ is defined as follows

$$x \circ y = 0 \Leftrightarrow \begin{cases} x = 0, & \text{when } y = 1, \\ l_i \leq x_i \leq u_i, & \text{when } y = 0. \end{cases}$$

Now, the SDR of the model (15) can be obtained by the following strategy. As $y_i \in \{0,1\}$ equivalently written as $y_i^2 = y_i$ and hence $diag(yy^T) = y$, thus the the constraint $y_i^2 = y_i$ can be replaced by $diag(Y) = y$, where $Y = yy^T$. Also, the constraint $x \circ y = 0$ represents diagonal of the matrix xy^T being zero. Thus SDR requires the relaxation of $X = xx^T$ by $X \succcurlyeq xx^T$ and $Y = yy^T$ by $Y \succcurlyeq yy^T$.

Thus, by using Schur complement, for

$$X \succeq xx^T \text{ and } Y \succcurlyeq yy^T, \exists W \in S^n s.t. \bar{X} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \begin{pmatrix} 1 & x^T & y^T \end{pmatrix} = \begin{pmatrix} 1 & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{pmatrix} \succcurlyeq 0.$$

Hence, the SDR model of (15) is given as

$$\begin{array}{ll} \min\limits_{x} & Tr(\mathbf{Q}X) - \lambda\mu^T x \\ s.t & e^T x = 1, \\ & l_i(1-y_i) \leq x_i \leq u_i(1-y_i), \\ & \sum_{i=1}^{n} y_i \geq n-k, \forall i, \\ & diag(W) = 0, \\ & diag(Y) = y, \\ & \bar{X} \succcurlyeq 0. \end{array} \tag{16}$$

where

$$\bar{X} = \begin{bmatrix} 1 & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in S_{2n+1}^{+}.$$

4. Numerical results

This section presents a numerical experiment to evaluate the performance of the different proposed models discussed in Section 3. We select the mixed integer quadratic portfolio selection model (7) and reformulation models (10), (13) and (16) for the experiments on real world financial data. We used two real market data sets. The test set named `port' was taken from OR-Library which is built from weekly price data to estimate the mean return μ and covariance matrix $\mathbf{Q}$. This set of data instances is openly accessible at: http://people.brunel.ac.uk/mastjjb/jeb/orlib/portinfo.html. The second test set has been taken from Kenneth French's 49 industry portfolios introduced by Fama and French [41]. Here we have chosen the daily returns for a 46 year period from 1975 to 2020 and estimated the expected return and the covariance from historical data on an annual basis. All the numerical tests are implemented on CVX [42, 43] in MATLAB R2021b and run on a PC(2GHz, 4 GB RAM) with an Intel Core i3 CPU. Here, all proposed models are solved by Mosek [44].

Initially, we set the values of unknown parameters used to construct these portfolio models. In this connection, we set the risk preference parameter $\lambda = 0.001$ which is crucial for the investment. In the MIQP model, the parameter M restricts the maximum weight invested in a particular asset, which also acts as an upper and lower bound restriction for x, we consider the maximum weight taken by any asset to be 0.425. Each instance of l_i and u_i in the SDP-model (16) is uniformly picked from the intervals $[0.04, 0.06]$ and $[0.375, 0.425]$, respectively. In the SDP model (13), the unknown parameters $\alpha = 0.001$ and $\beta = 0.01$ are the tuning parameters taken to achieve the optimal solution. Here we take four different values for the cardinality parameter $k = \{5,10,15,20\}$ and show the performance of different models in Table 1 and Table 2.

Table 1 summarizes the average performance of the proposed models by computing the annual mean return and covariance using the daily historical return data of Kenneth French's 49 industry portfolios. We solved all the models individually and fund the average objective value, average mean return and average risk for 46 years range period.

Table 1

Average performance measure of proposed portfolio model with different value of k for 49 industry portfolios data

k	MIQP MODEL (7)			SDP MODEL (10)			SDP MODEL (13)			SDP MODEL (16)		
	obj	Return	Risk	obj	Return	Risk	obj	Return	Risk	obj	Return	Risk
5	3.85E-01	3.98E-02	3.85E-01	2.82E-01	3.59E-02	2.82E-01	2.79E-01	4.97E-02	2.82E-01	9.86E-01	5.22E-02	8.01E-01
10	2.95E-01	3.64E-02	2.95E-01	2.28E-01	3.47E-02	2.28E-01	2.26E-01	4.90E-02	2.38E-01	8.89E-01	5.15E-02	7.53E-01
15	2.62E-01	3.39E-02	2.62E-01	2.23E-01	3.52E-02	2.23E-01	2.07E-01	4.80E-02	2.25E-01	8.32E-01	5.09E-02	7.14E-01
20	2.44E-01	3.40E-02	2.44E-01	2.23E-01	3.52E-02	2.23E-01	1.99E-01	4.79E-02	2.22E-01	8.72E-01	5.13E-02	7.49E-01

Table 2 summarizes the comparison results of different proposed models in the sense of objective value, expected return, and expected risk for four different real life instances, with the number of variables $n = 31,85,89,98$. We solve these instances with different cardinality k. The results clearly show that the proposed SDP models provide better optimal solutions as compared to the general MIQP model. Also, these models provided better return with a reasonable risk amount.

Table 2

Performance measure of proposed portfolio model with different value of k

Inst	n	k	MIQP MODEL (7)			SDP MODEL (10)			SDP MODEL (13)			SDP MODEL (16)		
			obj	Return	Risk	obj	Return	Risk	obj	Return	Risk	obj	Return	Risk
port 1	31	5	6.48E-04	2.15E-03	6.50E-04	5.04E-04	2.71E-03	5.06E-04	1.93E-04	2.98E-03	6.29E-04	1.12E-03	3.30E-03	9.11E-04
		10	5.53E-04	2.84E-03	5.56E-04	4.94E-04	2.67E-03	4.97E-04	1.18E-04	2.93E-03	5.84E-04	9.49E-04	3.12E-03	7.85E-04
		15	5.16E-04	2.47E-03	5.18E-04	4.95E-04	2.67E-03	4.97E-04	9.29E-05	2.94E-03	5.68E-04	8.73E-04	3.10E-03	7.65E-04
		20	5.02E-04	2.65E-03	5.04E-04	4.94E-04	2.67E-03	4.97E-04	8.74E-05	3.00E-03	5.60E-04	8.46E-04	3.13E-03	7.84E-04
port 2	85	5	1.82E-04	1.63E-03	1.84E-04	9.93E-05	2.37E-03	1.02E-04	1.78E-04	2.25E-03	1.38E-04	3.23E-04	1.49E-03	2.61E-04
		10	1.46E-04	2.17E-03	1.48E-04	9.67E-05	2.38E-03	9.90E-05	1.18E-04	2.18E-03	1.39E-04	2.79E-04	1.56E-03	2.26E-04
		15	1.36E-04	2.25E-03	1.38E-04	9.67E-05	2.38E-03	9.90E-05	1.02E-04	2.17E-03	1.34E-04	2.54E-04	1.64E-03	2.08E-04
		20	1.35E-04	2.14E-03	1.37E-04	9.67E-05	2.38E-03	9.90E-05	9.43E-05	2.16E-03	1.31E-04	2.41E-04	1.67E-03	2.00E-04
port 3	89	5	2.36E-04	2.62E-03	2.38E-04	1.42E-04	1.77E-03	1.43E-04	2.30E-04	2.56E-03	2.00E-04	3.36E-04	2.65E-03	3.00E-04
		10	2.04E-04	2.42E-03	2.06E-04	1.32E-04	1.41E-03	1.34E-04	1.69E-04	2.47E-03	1.94E-04	3.07E-04	2.51E-03	2.69E-04
		15	1.98E-04	2.45E-03	2.00E-04	1.32E-04	1.41E-03	1.34E-04	1.50E-04	2.38E-03	1.85E-04	2.89E-04	2.48E-03	2.53E-04
		20	1.96E-04	2.42E-03	1.99E-04	1.32E-04	1.41E-03	1.34E-04	1.41E-04	2.31E-03	1.80E-04	2.79E-04	2.49E-03	2.51E-04
port 4	98	5	1.71E-04	1.52E-03	1.72E-04	7.97E-05	1.45E-03	8.12E-05	1.35E-04	2.01E-03	1.24E-04	2.48E-04	2.87E-03	2.06E-04
		10	1.31E-04	1.75E-03	1.33E-04	7.63E-05	1.45E-03	7.77E-05	7.86E-05	1.93E-03	1.17E-04	2.19E-04	2.73E-03	1.85E-04
		15	1.22E-04	1.94E-03	1.24E-04	7.63E-05	1.44E-03	7.77E-05	6.13E-05	1.87E-03	1.12E-04	2.03E-04	2.65E-03	1.75E-04
		20	1.20E-04	2.05E-03	1.23E-04	7.63E-05	1.45E-03	7.77E-05	5.35E-05	1.84E-03	1.10E-04	1.94E-04	2.60E-03	1.71E-04

Table 3 summarizes the average performance of the proposed models by computing the CPU time (in sec) for four different real life instances with the number of variables $n = 31,85,89,98$ and from the daily historical return data taken from Kenneth French's 49 industry portfolios. We solved all the models individually and record the time taken by the different proposed models.

Table 3

Performance comparison of proposed SDP models over MIQP with respect to CPU Time (in sec).

Instances	n	k	CPU Time in sec			
			MIQP MODEL (7)	SDP MODEL (10)	SDP MODEL (13)	SDP MODEL (16)
port 1	31	5	1.75	0.43	0.26	0.41
		10	1.78	0.44	0.23	0.23
		15	1.26	0.41	0.23	0.23
		20	1.27	0.41	0.25	0.25
port 2	85	5	22.43	2.72	20.53	6.59
		10	391.91	0.75	19.05	6.07
		15	1253.62	0.71	20.46	6.62
		20	326.78	0.67	17.49	6.13
port 3	89	5	40.94	1.072	22.08	8.20
		10	10.32	0.74	23.88	6.81
		15	81.75	0.84	22.01	6.80
		20	30.14	0.69	22.65	6.80
port 4	98	5	959.87	10.38	42.64	12.45
		10	3860.34	0.98	38.94	11.66
		15	57.34	1.11	37.86	11.77
		20	273.26	0.85	39.86	12.88
avg CPU time			457.10	1.45	20.53	6.49
49-Industry	49	5	4.54	0.51	2.03	1.03
		10	228.25	0.50	2.10	0.95
		15	1276.60	0.49	2.03	0.96
		20	257.48	0.49	2.08	0.99
avg CPU time			441.72	0.50	2.06	0.98

4.1 *Error analysis*

Considering the MIQP model as the base model, in Table 4 we report the percentage relative deviation of the objective value, relative improvement in mean return, and risk for different SDP models with the MIQP model. We compute the the relative deviation by the formula

$$\left(\frac{V_{SDP} - V_{MIQP}}{V_{MIQP}}\right) \times 100,$$

where V_{SDP} and V_{MIQP} stand for objective value, mean return, and risk of different SDP models and MIQP models, respectively. Note that the negative sign in the relative deviation column indicates the better objective value. Similarly, the negative sign in the relative improvement return column indicates poor return and a negative sign in the relative improvement risk column indicates less risk compared to the MIQP model.

One can observe that in SDP MODEL (10) and SDP MODEL (13), the objective value and variance are better than the MIQP MODEL (7) in all the problems with both the data sets, while SDP MODEL (16) shows better performance against MIQP, but in this model the relative percentage deviation is higher than the other two SDP models and the risk is higher.

Table 4

Percentage relative error among models considering MIQP as base model.

Instances	n	k	Relative deviation			Relative improvement in return			Relative improvement in risk		
			MODEL (10)	MODEL (13)	MODEL (16)	MODEL (10)	MODEL (13)	MODEL (16)	MODEL (10)	MODEL (13)	MODEL (16)
port 1	31	5	-22.28	-70.18	72.94	26.13	38.63	53.84	-22.12	-3.23	40.08
		10	-10.67	-78.75	71.55	-6.02	2.92	9.72	-10.65	5.06	41.14
		15	-4.15	-82.01	69.10	8.04	18.96	25.50	-4.13	9.60	47.62
		20	-1.48	-82.58	68.54	0.92	13.52	18.36	-1.47	11.00	55.45
port 2	85	5	-45.47	-2.38	77.41	45.05	38.12	-8.69	-44.66	-24.61	42.27
		10	-33.78	-19.35	90.82	9.37	0.24	-28.02	-33.15	-6.05	52.40
		15	-29.02	-25.18	86.56	5.67	-3.40	-27.11	-28.49	-3.13	49.85
		20	-28.38	-30.09	78.29	10.86	0.71	-22.16	-27.77	-4.42	45.91
port 3	89	5	-39.91	-2.42	42.61	-32.45	-2.38	0.94	-39.83	-16.24	25.91
		10	-35.11	-16.83	50.64	-41.74	2.22	3.70	-35.20	-5.69	30.40
		15	-33.12	-23.99	46.15	-42.33	-2.81	1.46	-33.26	-7.72	26.70
		20	-32.74	-28.45	42.01	-41.83	-4.65	2.92	-32.86	-9.61	26.38
port 4	98	5	-53.24	-20.56	45.52	-4.57	32.16	89.21	-52.82	-27.76	19.67
		10	-41.90	-40.16	66.74	-17.31	10.26	56.05	-41.58	-12.40	38.90
		15	-37.47	-49.78	66.73	-25.73	-3.83	36.13	-37.30	-9.57	41.37
		20	-36.70	-55.63	61.03	-29.64	-10.60	26.49	-36.58	-10.29	39.26
49 industry	49	5	-26.89	-27.71	155.80	-9.82	25.08	31.22	-26.89	-26.90	107.82
		10	-22.59	-23.38	201.53	-4.79	34.38	41.47	-22.59	-19.16	155.29
		15	-14.64	-20.87	218.13	3.80	41.43	50.18	-14.64	-14.09	172.86
		20	-8.38	-18.30	257.86	3.45	40.83	50.85	-8.38	-8.71	207.24

4.2 *Performance profile*

To compare among the proposed models, we examine the performance of the MIQP model (7) and reformulation models (10), (13) and (16) taking into consideration the objective values, mean return, risk, and CPU time of all the SDP models and MIQP model using the profile of Dolan and Moré [45].

Define $P = \{p: p$ is the portfolio problems of the port data and 49 industry data for different value of $k\}$, and $S = \{s: s$ is the different model such as (7), (10), (13) and (16)$\}$.

The performance ratio provided as follows is used to compare the model s's performance on issue p with the best performance of any other models on this problem:

$$r_{p,s} = \frac{t_{p,s}}{min\{t_{p,s}: s \in S\}}$$

where $t_{p,s}$ is the computing time required to solve problem p by model s. The performance of the model s can be ascertained using the formula.

$$P_s(\tau) = \frac{1}{n_p} size\{p \in P: r_{p,s} \leq \tau\}$$

where n_p denote the number of problems. $P_s(\tau)$ represents the probability that the performance ratio, $r_{p,s}$, of model s on a given problem lies within a factor of $\tau \in R$ of the best possible performance ratio for that problem.

Albeit, the above measure is used for CPU time, the same can be used for other measures like objective value, mean return and risk obtained by different the model.

The Figures 1a, 1b, 1c and 1d represent the performance of different models with respect to objective value, mean return, risk, and CPU time respectively.

From the performance profile, we record the poor performance of different models such as MIQP model for CPU time, model (16) with respect to risk, model (10) with respect to return, and model (16) with respect to objective value. Similarly, model (10) is best with respect to CPU time and risk, model (16) is best in return and model (13) is best with respect to objective values.

Thus, if the investor seeks low risk with a high return, the proposed model (13) is performing well whereas model (16) provides high risk with high return.

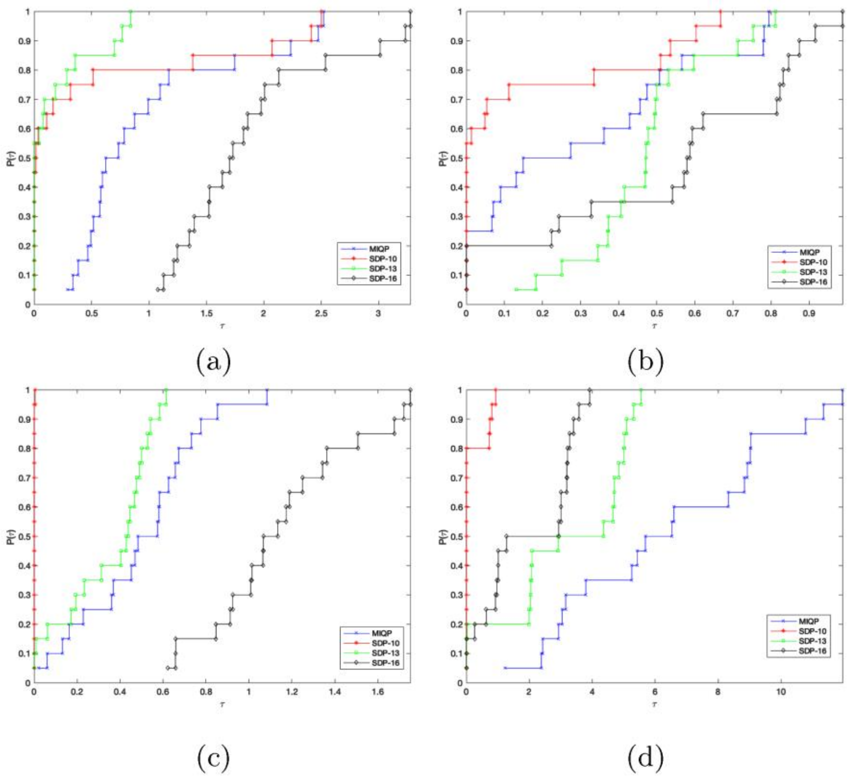

Figure 1
Dolan-Moré performance profile

5. Conclusion

In this work, we present risk preference portfolio optimization model with a cardinality constraint. In general, to solve the cardinality constraint portfolio optimization problem, one can convert the problem into the MIQP-model and solve it with the help of a global solver to obtain the optimal solution of the NP-hard problem, even for small cardinality values. We developed two relaxed semidefinite models for the MIQP model and a direct semidefinite relaxation model for the CCPO model with a relaxed rank-one constraint. The computational study confirms that the proposed SDP models provide a tight solution to the CCPO model.

Compliance with Ethical Standards

- This article does not contain any studies involving animals performed by any of the authors.
- This article does not contain any studies involving human participants performed by any of the authors.

Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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