Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 9, pp. 2687–2697
DOI : 10.47974/JIM-2661

Truncated exponential based Bernoulli Apostol polynomials

Neha Sharma [†]
Chetan Kumar Sharma *
Department of Mathematics
School of Sciences
Noida International University
Greater Noida 201312
Uttar Pradesh
India

Mohammad Shadab [§]
Department of Applied Sciences & Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia
Central University
New Delhi
Delhi 110025
India

Abstract

This study introduces the generating function of a mixed-type polynomial known as the Truncated Exponential-based Apostol-Bernoulli Polynomials of degree β. We also show other important identities and properties, such as the formulae for their partial derivatives, the correlation formula, and the implicit summation formula. A few connections with Stirling numbers are also examined.

Subject Classification: *11B68, 11B73, 11B83, 33C05, 33C10, 33C15.*

Keywords: *Truncated exponentinal polynomial, Apostol euler polynomials, Apostol bernoulli polynomials, Apostol genocchi polynomials, Stirling polynomial of second kind.*

[†] *E-mail:* **nehavasishtha@gmail.com**
* *E-mail:* **cks26april@gmail.com (Corresponding Author)**
[§] *E-mail:* **shadabmohd786@gmail.com**

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1. Introduction and Preliminaries

The Bernoulli Polynomials (Hereafter, Bernoulli Polynomials will be abbreviated as BP.) are useful in both classical analysis and number theory. They are used to approximate differentiable periodic functions in terms of Polynomials and can be found in their integral representation. Numerous mathematical expressions and number theory make use of the Bernoulli numbers [13, 18, 22].

The Truncated Exponential Polynomials (Hereafter, The Truncated Exponential Polynomials will be abbreviated as TEP.) ("E(p)(1)"), their extension, and other recently introduced Polynomials that combine the TEP with other known Polynomials are defined in this study. These have been studied and used in a variety of ways. TE-based Apostol-BP of order β is a mixed-type polynomial whose generating function we define in this article. Additionally, we establish a relationship between the Stirling numbers and the TE-based Apostol-BP of order β, and examine their useful properties and identities, such as implicit summation formulas.

In this paper, we investigate the sets of all the real numbers, complex numbers, and the standard notation for these sets, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{N}_0$.

The series defines the TEP $e_v(p)$ (see [1, 24, 3]).

$$e_v(p) = \sum_{r=0}^{v} \frac{p^r}{r!}, (v \in \mathbb{N}_0). \tag{1.1}$$

These are the initial $v+1$ terms of the Maclaurin series for e^p.

This is a specific example of Schur's theorem (see [23, 24, 5]; see also [9, 12, 8]). The polynomial $E_v(p)$ $(v \in \mathbb{N})$ is irreducible in $\mathbb{Q}[p]$. Any polynomial of the form

$$1 + c_1 p + c_2 \frac{p^2}{2!} + \cdots + c_{v-1} \frac{p^{v-1}}{(v-1)!} + c_v \frac{p^v}{v!}, (v \in \mathbb{N}, c_i \in \mathbb{Z})$$

is irreducible in $\mathbb{Q}[p]$. See, for instance, [2, 25, 31] for additional properties and applications of these Polynomials.

Using the well-known gamma function β, which is defined by (see, for example, [1, 26, 15])

$$\Gamma(z) = \int_0^\infty e^{-r} r^{z-1}\, dt \ (\mathbb{R}(z) > 0) \tag{1.2}$$

in particularly,

$$\Gamma(v+1) = v! = \int_0^\infty e^{-r} r^v\, dt \ (v \in \mathbb{N}_0) \tag{1.3}$$

we find (see,e.g.,[[10]], Equation ((1.2))

$$e_v(p) = \frac{1}{v!} \int_0^\infty e^{-r} (p+r)^v\, dt \ (v \in \mathbb{N}_0) \tag{1.4}$$

The generating function for TEP $e_v(p)$ (see [10]) is given by

$$e^{pr} = \sum_{v=0}^{\infty} e_v(p) \frac{r^v}{v!}, (|r| < 1). \tag{1.5}$$

The generating relation (1.5) can be obtained by applying the Cauchy product to the Maclaurin series of e^{pr} and $\frac{1}{1-r}$. Differentiating both sides of (1.5)

with respect to the variables r and p, respectively, yields the following differential–recursive relations (see [10, 31], Equation (1.5)):

$$\frac{d}{dp}e_v(p) = e_{v-1}(p), (v \in \mathbb{N}), \tag{1.6}$$

and

$$e_{v+1}(p) = \frac{1}{v+1}\left(p - \frac{d}{dp}\right)e_v(p), (v \in \mathbb{N}_0). \tag{1.7}$$

A second-order differential equation may be derived by combining (1.6) and (1.7) (see [10], Equation (1.8)).

$$\left[p\frac{d^2}{dx^2} - (v+p)\frac{d}{dx} + 1\right]e_v(p) = 0, (v \in \mathbb{N}_0). \tag{1.8}$$

The generating function given by

$$\sum_{i=0}^{\infty} B_i(p,q)\frac{r^i}{i!} = e^{xt}e^{q(e^r-1)} \tag{1.9}$$

depict a Bell polynomial with two variables (see [11, 28]).

Changing $p = 0$ yields $B_i(0;q) = B_i(q)$, which is referred to as the classical Bell Polynomial (or exponential polynomial) and is provided by the generating function (see [4, 6, 16, 17]), which is defined as follows:

$$\sum_{i=0}^{\infty} B_i(p,q)\frac{r^i}{i!} = e^{xt}e^{q(e^r-1)}, \tag{1.10}$$

If we take $q = 1$ in (1.2), we obtain $B_i(1) = B_i$, known as the Bell number (see [4, 6, 16, 17]).

$$\sum_{i=0}^{\infty} B_i\frac{r^i}{i!} = e^{e^r-1}, \tag{1.11}$$

The generating functions yield the Euler numbers $E_i(0)$ and the Euler Polynomials $E_i(p)$ (see [14]):

$$e^{pr}\left(\frac{2}{e^r+1}\right) = \sum_{i=0}^{\infty} E_i(p)\frac{r^i}{i!}, (|r| < \pi). \tag{1.12}$$

The numbers $E_i(0) = E_i$ are obtained by taking $p = 0$ in (1.12):

$$\frac{2}{e^r+1} = \sum_{i=0}^{\infty} E_i\frac{r^i}{i!}, (|r| < \pi). \tag{1.13}$$

Dattoli, Lorenzutta, and Cesarano [7], Choudhary and Srivastava [27] introduced BP and the Bernoulli numbers through the generating function

$$e^{pr}\left(\frac{r}{e^r-1}\right) = \sum_{i=0}^{\infty} B_i(p)\frac{r^i}{i!}, (|r| < 2\pi). \tag{1.14}$$

The Bernoulli numbers $B_i(0) = B_i$ are obtained by setting $p = 0$ in (1.14):

$$\frac{r}{e^r-1} = \sum_{i=0}^{\infty} B_i\frac{r^i}{i!}, (|r| < 2\pi). \tag{1.15}$$

The Genocchi Polynomials $G_i(p)$ and the Genocchi numbers $G_i = G_i(0)$ are defined by (see [7, 26])

$$e^{pr}\left(\frac{2r}{e^r+1}\right) = \sum_{i=0}^{\infty} G_i(p)\frac{r^i}{i!}, (|r| < \pi). \tag{1.16}$$

Taking $p = 0$ in (1.16) yields the Genocchi numbers:

$$\frac{2r}{e^r+1} = \sum_{i=0}^{\infty} G_i \frac{r^i}{i!}, \quad (|r| < \pi). \tag{1.17}$$

The Euler Polynomials of order β are defined by the generating function (see [26, 30]):

$$\sum_{i=0}^{\infty} E_i^{(\beta)}(p) \frac{r^i}{i!} = e^{pr} \left(\frac{2}{e^r+1}\right)^{\beta}, \quad (|r| < \pi;\ 1^{\beta} := 1). \tag{1.18}$$

Taking $p = 0$ in (1.18) gives the Euler numbers of order β:

$$\sum_{i=0}^{\infty} E_i^{(\beta)} \frac{r^i}{i!} = \left(\frac{2}{e^r+1}\right)^{\beta}. \tag{1.19}$$

BP of order β are defined by the generating function (see [26, 30]):

$$\sum_{i=0}^{\infty} B_i^{(\beta)}(p) \frac{r^i}{i!} = e^{pr} \left(\frac{r}{e^r-1}\right)^{\beta}, \quad (|r| < 2\pi;\ 1^{\beta} := 1). \tag{1.20}$$

Taking $p = 0$ in (1.20) gives the Bernoulli numbers of order β:

$$\sum_{i=0}^{\infty} B_i^{(\beta)} \frac{r^i}{i!} = \left(\frac{r}{e^r-1}\right)^{\beta}. \tag{1.21}$$

The generating function of the Genocchi Polynomials of order β (see [26, 30]) is as follows:

$$\sum_{i=0}^{\infty} G_i^{(\beta)}(p \frac{r^i}{i!} = e^{xt} \left(\frac{2r}{e^r+1}\right)^{\beta}, \quad (|r| < \pi;\ 1^{\beta} := 1) \tag{1.22}$$

If we take $p = 0$ in (1.22), i.e., $G_i^{(\beta)}(0) = G_i^{(\beta)}$, they are called the Genocchi numbers, defined as follows:

$$\sum_{i=0}^{\infty} G_i^{(\beta)} \frac{r^i}{i!} = \left(\frac{2r}{e^r+1}\right)^{\beta}, \tag{1.23}$$

The generating function of the generalized Apostol-BP$B_i^{(\beta)}(p, \beta)$ of order $\beta \in \mathbb{C}$ (see [8, 21, 29]) is as follows:

$$e^{xt} \left(\frac{r}{\mu e^r-1}\right)^{\beta} = \sum_{i=0}^{\infty} \frac{r^i}{i!} B_i^{(\beta)}(p; \mu), \quad (|r + \log\mu| < 2\pi;\ 1^{\beta} := 1). \tag{1.24}$$

with

$$B_i^{(\beta)}(p; 1) := B_i^{(\beta)}(p), \tag{1.25}$$

and

$$B_i^{(\beta)}(0; \mu) := B_i^{(\beta)}(\mu), \tag{1.26}$$

which are known as the Apostol-Bernoulli numbers $B_i^{(\beta)}(\mu)$ of order β.

The generating function of the generalized Apostol-Euler Polynomials $E_i^{(\beta)}(p, \beta)$ of order $\beta \in \mathbb{C}$ (see [8, 19, 29]) is as follows:

$$e^{xt} \left(\frac{2}{\mu e^r+1}\right)^{\beta} = \sum_{i=0}^{\infty} \frac{r^i}{i!} E_i^{(\beta)}(p; \mu), \quad (|r + \log\mu| < \pi;\ 1^{\beta} := 1), \tag{1.27}$$

where

$$E_i^{(\beta)}(p;1) := E_i^{(\beta)}(p) \tag{1.28}$$

and

$$E_i^{(\beta)}(0;\mu) := E_i^{(\beta)}(\mu) \tag{1.29}$$

They are known as the Apostol-Euler numbers $E_i^{(\beta)}(\mu)$ of order β.

The generating function of the generalized Apostol-Genocchi polynomial $G_i(p,\mu)$ of order $\beta \in \mathbb{C}$ (see [8, 20, 29]) is as follows:

$$\{e^{xt}\}\left(\frac{2r}{\mu e^r+1}\right)^{\beta} = \sum_{i=0}^{\infty} \frac{r^i}{i!} G_i^{(\beta)}(p;\mu), \quad (r+\log\mu < \pi;\ 1^{\beta} := 1), \tag{1.30}$$

with

$$G_i^{(\beta)}(p;1) = G_i^{(\beta)}(p), \tag{1.31}$$

and

$$G_i^{(\beta)}(0;\mu) = G_i^{(\beta)}(\mu), \tag{1.32}$$

They are known as the Apostol-Genocchi numbers $G_i^{(\beta)}(\mu)$ of order β.

2. New Apostol-Euler Polynomials Using the Truncated Exponential Function

The generating function for a new family of truncated exponential-based Apostol-Euler Polynomials of order β is introduced in this section. Furthermore, we obtain correlation formulas for the truncated exponential-based Euler-Apostol Polynomials of order β.

Definition 2.1: The truncated exponential-based Apostol-Euler Polynomials of order $\beta \in \mathbb{C}$ are defined by the generating function

$$\frac{e^{2pr}}{1-qr}\left(\frac{2}{\mu e^r+1}\right)^{\beta} = \sum_{i=0}^{\infty}{}_{\mathrm{e}} E_i^{\beta}(p,q;\mu)\ \frac{r^i}{i!}, \quad (\,|\,r+\log\mu\,|\,<\pi;\ 1^{\beta} := 1\,). \tag{2.1}$$

The truncated exponential-based Apostol-Euler Polynomials of order β are obtained by substituting $p=0$ and $q=1$ into (2.1). Then

$$\sum_{i=0}^{\infty}{}_{\mathrm{e}} E_i^{\beta}(0,1;\mu)\ \frac{r^i}{i!} = \frac{1}{1-r}\left(\frac{2}{\mu e^r+1}\right)^{\beta}. \tag{2.2}$$

When $\beta = 0$ in (2.1), the Polynomials reduce to the truncated exponential Polynomials:

$$\frac{e^{2pr}}{1-qr} = \sum_{i=0}^{\infty}{}_{\mathrm{e}} E_i^{(0)}(p,q;\mu)\ \frac{r^i}{i!}. \tag{2.3}$$

Remark 2.2: The well-known Apostol-Euler Polynomials $E_i^{\beta}(p)$ are recovered from (2.1) by choosing $q=0$ and $\mu=1$ (see [25, 29]):

$$\sum_{i=0}^{\infty}{}_{\mathrm{e}} E_i^{\beta}(p,0;1)\ \frac{r^i}{i!} = e^{pr}\left(\frac{2}{e^r+1}\right)^{\beta} = \sum_{i=0}^{\infty} E_i^{\beta}(p)\ \frac{r^i}{i!}. \tag{2.4}$$

Remark 2.3: In (2.1), the truncated exponential-based Apostol-Euler Polynomials ${}_{\mathrm{e}}E_i^{\beta}(p,q;\mu)$ reduce to the classical Euler Polynomials $E_i(p)$ when we select $q=0$, $\beta=1$, and $\mu=1$:

$$\sum_{i=0}^{\infty} {}_{\mathrm{e}}E_i^{(1)}(p,0;1)\,\frac{r^i}{i!} = e^{pr}\left(\frac{2}{e^r+1}\right) = \sum_{i=0}^{\infty} E_i(p)\,\frac{r^i}{i!}. \tag{2.5}$$

Remark 2.4: When $\mu=1$, the generating relation (2.1) reduces to the truncated exponential-based Euler Polynomials of order β:

$$\sum_{i=0}^{\infty} {}_{\mathrm{e}}E_i^{\beta}(p,q)\,\frac{r^i}{i!} = \frac{e^{2pr}}{1-qr}\left(\frac{2}{e^r+1}\right)^{\beta}. \tag{2.6}$$

Theorem 1: *The following formula holds for* $\beta\in\mathbb{C}$ *and* $i\in\mathbb{N}$.

$${}_{\mathrm{e}}E_i^{\beta}(p,q;\mu) = \sum_{j=0}^{i}\binom{i}{j}E_j^{\beta}(p;\mu)e_{i-j}(q). \tag{2.7}$$

Proof: By applying relation (2.1), we get: a

$$\begin{aligned}\sum_{i\geq 0} {}_{\mathrm{e}}E_i^{\beta}(p,q;\mu)\frac{r^i}{i!} &= \left(\frac{2}{\mu e^r+1}\right)^{\beta}\cdot\frac{e^{2pr}}{1-qr}.\\ &= \left(\frac{2}{\mu e^r+1}\right)^{\beta} e^{pr}\cdot\frac{e^{pr}}{1-qr}.\\ &= \left(\sum_{j\geq 0} E_j^{\beta}(p;\mu)\frac{r^j}{j!}\right)\cdot\left(\sum_{i\geq 0} E_i(q)\frac{r^i}{i!}\right).\end{aligned} \tag{2.8}$$

Using the series rearrangement, we obtain

$$= \sum_{i\geq 0}\left(\sum_{j=0}^{i}\binom{i}{j}E_j^{\beta}(p;\mu)E_{i-j}(q)\right)\frac{r^i}{i!}. \tag{2.9}$$

We acquire the desired result (2.7) after applying the series rearrangement and simplifying.

Theorem 2: *Consider* $\beta\in\mathbb{C}$ *and* $i\in\mathbb{N}$, *Satisfy the following relation :*

$${}_{\mathrm{e}}E_i^{\beta}(p,q;\mu) = \sum_{j=0}^{\infty}\binom{i}{j}E_j^{\beta}(\mu)E_{i-j}(p,q) \tag{2.10}$$

Proof: Using relation (2.1), we obtain

$$\begin{aligned}{}_{\mathrm{e}}E_i^{\beta}(p,q;\mu)\frac{r^i}{i!} &= \left(\frac{2}{\mu e^r+1}\right)^{\beta}\cdot\frac{e^{2pr}}{(1-qr)}.\\ &= \left\{\left(\frac{2}{\mu e^r-1}\right)^{\beta}\right\}\cdot\left\{\frac{e^{2pr}}{(1-qr)}\right\}.\\ &= \left\{\sum_{j\geq 0} E_j^{\beta}(\mu)\frac{r^j}{j!}\right\}\cdot\left\{\sum_{i\geq 0} E_i(p,q)\frac{r^i}{i!}\right\}.\end{aligned} \tag{2.11}$$

With the series rearrangement, we get

$$= \sum_{i\geq 0}\left\{\sum_{j=0}^{\infty}\binom{i}{j}E_j^{\beta}(\mu)E_{i-j}(q)\right\}\frac{r^i}{i!}. \tag{2.12}$$

We get the desired outcome after using the series rearrangement and simplifying. (2.10).

Theorem 3: *The following formula is valid for* $\beta \in \mathbb{C}$ *and* $i \in \mathbb{N}$*:*

$$ {}_{e}E_i^{\beta}(p,q;\mu) = \sum_{j=0}^{i} \binom{i}{j} {}_{e}E_j^{\beta}(q;\mu) \cdot 2p^{i-j}. \tag{2.13}$$

Proof: Using relation (2.1), we obtain

$$ {}_{e}E_i^{\beta}(p,q;\mu) \cdot \frac{r^i}{i!} = \left(\frac{2}{\mu e^r+1}\right)^{\beta} \cdot \frac{e^{2pr}}{(1-qr)}.$$

$$ = \left\{\left(\frac{2}{\mu e^r+1}\right)^{\beta}\right\}\left\{\frac{1}{(1-qr)}\right\} \cdot e^{2pr}.$$

$$ = \left\{\sum_{j\geq 0}{}_{e} E_j^{\beta}(q;\mu)\frac{r^j}{j!}\right\} \cdot \left\{\sum_{i\geq 0} \frac{2pr^i}{i!}\right\}.$$

$$ = \left\{\sum_{i\geq 0} \sum_{j\geq 0}{}_{e} E_j^{\beta}(q;\mu)\frac{r^{i+j}}{j!} \cdot \frac{p^i}{i!}\right\}. \tag{2.14}$$

Using the series rearrangement, we get

$$ = \sum_{i\geq 0} \left\{\sum_{j=0}^{i} \binom{i}{j} {}_{e}E_j^{\beta}(q;\mu) \cdot 2p^{i-j} \cdot q^j\right\}\frac{r^i}{i!}. \tag{2.15}$$

The result is obtained with series rearrangement simplification. (2.13).

Theorem 4: *For* $\beta \in \mathbb{C}$ *and* $i \in \mathbb{N}$*, the following formula is true:*

$$ {}_{e}E_i^{\beta}(p+q,z;\mu) = \sum_{j=0}^{i} \binom{i}{j} E_j^{\beta}(p;\mu)E_{i-j}(q,z). \tag{2.16}$$

Proof: Using relation (2.1), we obtain

$$ {}_{e}E_i^{\beta}(p+q,z;\mu) = \left(\frac{r}{\mu e^r-1}\right)^{\beta} \cdot \frac{e^{(2p+q)r}}{(1-zt)}.$$

$$ = \left\{\left(\frac{r}{\mu e^r-1}\right)^{\beta} \cdot\right\}\left\{\frac{e^{(2p+q)r}}{(1-zt)}\right\}.$$

$$ = \left\{\left(\frac{r}{\mu e^r-1}\right)^{\beta} \cdot e^{2pr}\right\}\left\{\frac{e^{2qr}}{(1-zt)}\right\}.$$

$$ = \left\{\sum_{i\geq 0} B_j^{\beta}(p;\mu)\frac{r^j}{j!}\right\} \cdot \left\{\sum_{i\geq 0} E_i(q;z)\frac{r^i}{i!}\right\}. \tag{2.17}$$

Following the series rearrangement and simplification, we get the necessary (2.16).

3. Implicit Summation Formulae

Deriving practical identities, such as implicit summation formulas for the truncated exponential-based Apostol-Euler Polynomials of order β, is the focus of this section.

Theorem 5: *The following identity holds for any* $i \in \mathbb{N}$ *and* $\beta_1, \beta_2 \in \mathbb{C}$*:*

$$ {}_{\mathrm{e}}E_i^{(\beta_1+\beta_2)}(p_1+p_2,q;\mu) = \sum_{j=0}^{i} \binom{i}{j} {}_{\mathrm{e}}E_j^{(\beta_1)}(p_1,q;\mu)\, {}_{\mathrm{e}}E_{i-j}^{(\beta_2)}(p_2,q;\mu). \quad (3.1) $$

Proof: Using the identity substituted into relation (2.1), we have

$$ \left(\frac{2}{\mu e^r+1}\right)^{\beta_1+\beta_2} \frac{e^{2(p_1+p_2)r}}{1-qr} = \left(\frac{2}{\mu e^r+1}\right)^{\beta_1} \frac{e^{2p_1 r}}{1-qr} \cdot \left(\frac{2}{\mu e^r+1}\right)^{\beta_2} \frac{e^{2p_2 r}}{1-qr}. $$

Using the generating functions gives

$$ \left(\sum_{j\geq 0} {}_{\mathrm{e}}E_j^{(\beta_1)}(p_1,q;\mu)\frac{r^j}{j!}\right) \cdot \left(\sum_{k\geq 0} {}_{\mathrm{e}}E_k^{(\beta_2)}(p_2,q;\mu)\frac{r^k}{k!}\right). $$

Applying the Cauchy product, we obtain

$$ \sum_{i\geq 0} \left\{ \sum_{j=0}^{i} \binom{i}{j} {}_{\mathrm{e}}E_j^{(\beta_1)}(p_1,q;\mu)\, {}_{\mathrm{e}}E_{i-j}^{(\beta_2)}(p_2,q;\mu) \right\} \frac{r^i}{i!}. \quad (3.2) $$

Comparing coefficients of $r^i/i!$ on both sides yields the desired identity (3.1).

Theorem 6: *The following identity holds for any* $j, l \in \mathbb{N}$ *and* $\beta_1, \beta_2 \in \mathbb{C}$*:*

$$ {}_{\mathrm{e}}E_{j+l}^{\beta}(p,q;\mu) = \sum_{i=0}^{j} \sum_{m=0}^{l} \binom{j}{i}\binom{l}{m} (p-z)^{i+m}\, {}_{\mathrm{e}}E_{j+l-i-m}^{\beta}(z,q;\mu). \quad (3.3) $$

Proof: For series manipulation, we use the well-known identity:

$$ \sum_{N=0}^{\infty} f(N)\frac{(p+q)^N}{N!} = \sum_{i,m=0}^{\infty} f(i+m)\,\frac{p^i}{i!}\,\frac{q^m}{m!}. \quad (3.4) $$

Replacing r in relation (2.1) by $(r+v)$ yields:

$$ \left(\frac{2}{\mu e^{r+v}+1}\right)^{\beta} e^{\,q(e^{r+v}-1)} = e^{-2p(r+v)} \sum_{j,l=0}^{\infty} {}_{\mathrm{e}}E_{j+l}^{\beta}(p,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!}. \quad (3.5) $$

Similarly, replacing p by z gives:

$$ \left(\frac{2}{\mu e^{r+v}+1}\right)^{\beta} e^{\,q(e^{r+v}-1)} = e^{-2z(r+v)} \sum_{j,l=0}^{\infty} {}_{\mathrm{e}}E_{j+l}^{\beta}(z,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!}. \quad (3.6) $$

Equating the right-hand sides of (3.5) and (3.6), we obtain:

$$ \sum_{j,l=0}^{\infty} {}_{\mathrm{e}}E_{j+l}^{\beta}(p,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!} = e^{\,2(p-z)(r+v)} \sum_{j,l=0}^{\infty} {}_{\mathrm{e}}E_{j+l}^{\beta}(z,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!}. \quad (3.7) $$

Now expand the exponential:

$$ e^{\,2(p-z)(r+v)} = \sum_{i,m=0}^{\infty} (p-z)^{i+m}\,\frac{r^i}{i!}\,\frac{v^m}{m!}. \quad (3.8) $$

Using the Cauchy product of series, we get:

$$ \sum_{j,l=0}^{\infty} {}_{\mathrm{e}}E_{j+l}^{\beta}(p,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!} = $$

$$ \sum_{j,l=0}^{\infty} \sum_{i=0}^{j} \sum_{m=0}^{l} \binom{j}{i}\binom{l}{m} (p-z)^{i+m}\, {}_{\mathrm{e}}E_{j+l-i-m}^{\beta}(z,q;\mu)\frac{r^j}{j!}\frac{v^l}{l!}. \quad (3.9) $$

Matching coefficients of $r^j v^l$ proves the identity in (3.3).

4. Conclusion

Motivated by the many applications in the mathematical sciences, we define TE-based Apostol-BP of order β in this paper and investigate their many valuable identities, such as correlation formulae, implicit summation formulas, and derivative formulas. The highly general and specialized results yield numerous new and acknowledged identities using fundamental and unified Polynomials given by other authors.

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Received January, 2026