

Synchronization of novel hyper-chaotic finance system via adaptive control

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Abstract

In the current studies, we propose a new non-linear hyper-chaotic financial system. Dynamical analysis of this system features dynamic complexity leading to hyper-chaotic behavior. Synchronization has been achieved via an adaptive control technique based on Lyapunov stability theory. The developed control approach ensures synchronization stability

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despite parameter uncertainty and system disturbances. Numerical simulations in MATLAB illustrate the validity of the theoretical strategies.

Subject Classification: 34D06, 93C40, 91B55.

Keywords: Hyper-chaotic financial system, Lyapunov stability theory, Adaptive control, Synchronization.

1. Introduction

Chaotic behavior has been a key focus in non-linear dynamical system investigation during the last few decades. Lorenz [1] significant break through that drawn the attention of the researcher have to visualize the complex system's dynamics. Hyper-chaotic financial model capture complex economic dynamics more effectively than classical economic system. Synchronization, initially introduced by Pecora and Carroll [2], is a fundamental concept in controlling chaotic systems. Synchronization occurs when two or more chaotic systems adjust their trajectories to follow a common dynamic evolution. Various synchronization techniques have been developed to achieve this goal, including Complete synchronization Chen and Guo [3], Generalized synchronization Zheng and Hu [4], projective synchronization Li and Xu [5], Hybrid synchronization Vikash, Khan, and Alam [6]. Among various method adaptive control is particularly effective because it can adjust to uncertainties and parameter variation in real time ensuring reliable synchronization Zhao, Wang, and Zhang [7]. Currently, there has been a growing concern to study innovative hyper-chaotic financial systems. Kareem and Njah [8] that impose more realistic dynamics such as policy delays, nonlinear investments, and feedback mechanisms. Rao and Zhu [9] investigated the non linear chaotic systems using Lyapunov stability theory. A suitable Lyapunov function is constructed to analyse the stability of the controlled system. These contemporary studies underscore the ever-changing nature of control strategies designed to improve the stability and predictability of hyper-chaotic financial systems Al-Azzawi and Al-Talib [10].

This research work is organized as follows: Section 2 addresses the formulation synchronization problem in the chaotic financial system and provides an overview of the dynamical properties of the hyper-chaotic financial system. Section 3 is followed by the identical hyper-chaotic financial system's synchronization using adaptive control. Section 4 covers numerical simulations and discussions. The conclusion is presented in Section 5.

2. Material and Method

Consider a novel financial system:

$$\begin{aligned}\dot{x} &= z + (y - a)x + w^2, \\ \dot{y} &= 1 - byz - x^2, \\ \dot{z} &= -x - czw, \\ \dot{w} &= -0.05xz + rw.\end{aligned}\tag{2.1}$$

The state variables x , y , z and w denotes the interest rate, investment demand, price index and profit margin respectively. Tong et al. [11]. For the parameter values $a = 0.9$, $b = 0.2$, $c = 3.3$, $r = -3$, the system exhibits hyper-chaotic behavior with Lyapunov exponents are $\lambda_1 = 1.0830$, $\lambda_2 = 0.3815$, $\lambda_3 = 0.3510$, $\lambda_4 = -2.6527$ shown in Figure 1. $\sum_{i=1}^4 \lambda_i = -0.8372 \leq 0$. Three positive lyapunov exponents confirm the hyper-chaotic nature of the system, while the negative exponents ensures that the system remains bounded despite its complex and unpredictable behavior.

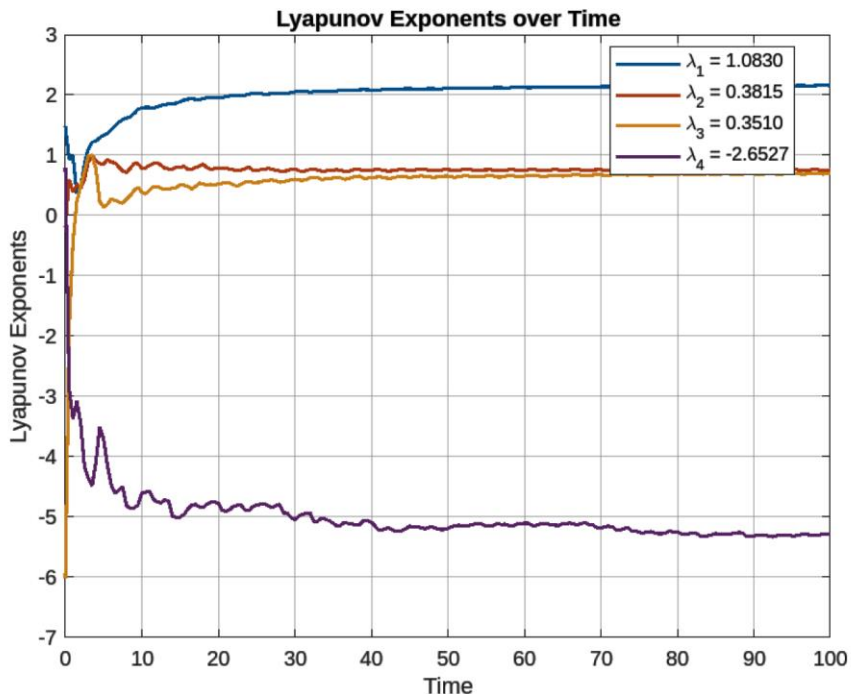


Figure 1
Lyapunov exponents of the financial system (2.1)

2.1 Bifurcation Analysis and Hyper-Chaotic Attractor:

Here the Figure 2 and Figure 3 represent the phase potrait diagrams and bifurcation analysis of four dimentional hyper-chaotic system (2.1).

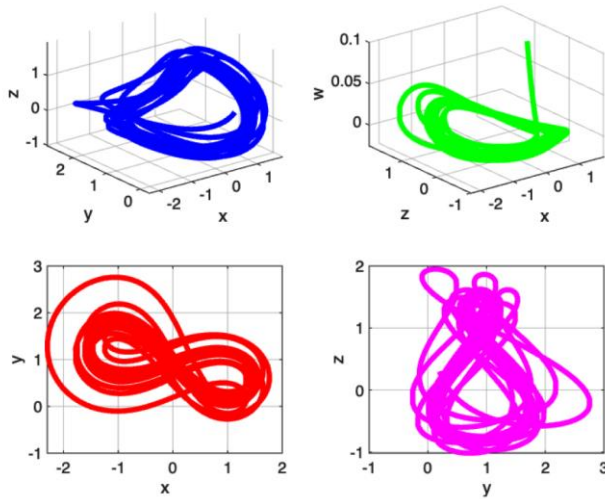


Figure 2

The phase portrait of the financial system (2.1) for $a = 0.9$, $b = 0.2$, $c = 3.3$ and $r = -3$.

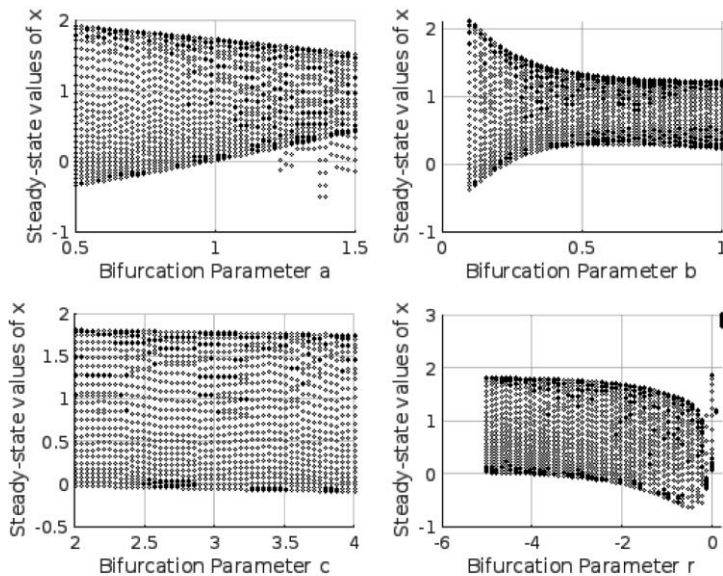


Figure 3

Represent Bifurcation analysis that the system exhibits stable, chaotic and hyper-chaotic behaviors as the parameter vary over the ranges $a \in [0.5, 1.5]$, $b \in [0.1, 1]$, $c \in [2, 4]$ and $r \in [-5, 0]$.

The following set of non-linear differential equation describes the master system:

$$\begin{aligned}\dot{x}_m &= z_m + (y_m - a)x_m + w_m^2, \\ \dot{y}_m &= 1 - by_m z_m - x_m^2, \\ \dot{z}_m &= -x_m - cz_m w_m, \\ \dot{w}_m &= -0.05x_m w_m + r w_m.\end{aligned}\quad (2.2)$$

and it is denoted by the subscript m and the slave system is defined with the subscript s and its dynamics is defined as follows:

$$\begin{aligned}\dot{x}_s &= z_s + (y_s - a)x_s + w_s^2 + u_1, \\ \dot{y}_s &= 1 - by_s z_s - x_s^2 + u_2, \\ \dot{z}_s &= -x_s - cz_s w_s + u_3, \\ \dot{w}_s &= -0.05x_s w_s + r w_s + u_4.\end{aligned}\quad (2.3)$$

Here, the master and slave system possess identical parameters a, b, c and r . However, these parameters are assumed to be unknown during the control designed process and are estimated through adaptive law.

3. Adaptive Synchronization Method

The objective of the adaptive synchronization method presented in this section is to achieve

$$(x_s, y_s, z_s, w_s) \rightarrow (x_m, y_m, z_m, w_m)$$

For this purpose, the following synchronization errors defined:

$$e_1 = x_s - x_m, \quad e_2 = y_s - y_m, \quad e_3 = z_s - z_m, \quad e_4 = w_s - w_m. \quad (3.1)$$

Thus, from (3.1), the error dynamics is:

$$\dot{e}_1 = \dot{x}_s - \dot{x}_m, \quad \dot{e}_2 = \dot{y}_s - \dot{y}_m, \quad \dot{e}_3 = \dot{z}_s - \dot{z}_m, \quad \dot{e}_4 = \dot{w}_s - \dot{w}_m. \quad (3.2)$$

Using (2.2) and (2.3) with (3.2), we have the error system in open-loop is described as follows:

$$\begin{aligned}\dot{e}_1 &= e_3 + (e_2 e_1 + e_2 x_m + e_1 y_m - a e_1) + e_4^2 + 2w_m e_4 + u_1, \\ \dot{e}_2 &= -b(e_2 e_3 + e_2 z_m + e_3 y_m) - e_1^2 - 2x_m e_1 + u_2, \\ \dot{e}_3 &= -e_1 - c(e_3 e_4 + e_3 w_m + e_4 z_m) + u_3, \\ \dot{e}_4 &= -0.05(e_1 e_3 + e_1 z_m + e_3 x_m) + r e_4 + u_4.\end{aligned}\quad (3.3)$$

3.1 Adaptive Control Design

To achieve the target that the slave system follows the master system, i.e.,

$$(x_s, y_s, z_s, w_s) \rightarrow (x_m, y_m, z_m, w_m),$$

the following adaptive controllers u_1, u_2, u_3, u_4 are to designed:

$$\begin{aligned}u_1 &= -e_3 - (e_2 e_1 + e_2 x_m + e_1 y_m - \hat{a} e_1) - e_4^2 - 2w_m e_4 - k_1 e_1, \\ u_2 &= \hat{b}(e_2 e_3 + e_2 z_m + e_3 y_m) + e_1^2 + 2x_m e_1 - k_2 e_2, \\ u_3 &= e_1 + \hat{c}(e_3 e_4 + e_3 w_m + e_4 z_m) - k_3 e_3, \\ u_4 &= 0.05(e_1 e_3 + e_1 z_m + e_3 x_m) - \hat{r} e_4 - k_4 e_4.\end{aligned}\quad (3.4)$$

Here, $\hat{a}, \hat{b}, \hat{c}, \hat{r}$ are the estimated values of the unknown parameter while the gains k_i (for $i = 1, 2, 3, 4$) are greater than zero. Using (3.4) and (3.3) together with the error in the unknown parameters as $\tilde{a} = \hat{a} - a, \tilde{b} = \hat{b} - b, \tilde{c} = \hat{c} - c, \tilde{r} = \hat{r} - r$, the closed-loop error dynamics is obtained as:

$$\begin{aligned}\dot{e}_1 &= \tilde{a} - k_1 e_1, \\ \dot{e}_2 &= \tilde{b}(e_2 e_3 + e_2 z_m + e_3 y_m) - k_2 e_2, \\ \dot{e}_3 &= \tilde{c}(e_3 e_4 + e_3 w_m + e_4 z_m) - k_3 e_3, \\ \dot{e}_4 &= -\tilde{r} e_4 - k_4 e_4.\end{aligned}\quad (3.5)$$

3.2 Stability Analysis and Learning law

To examine the stability of the closed-loop system (3.5), Lyapunov stability theory is applied. The first step is to drive and analyze a Lyapunov function represented in terms of error states e_i (for $i = 1, 2, 3$) and parameters estimation $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{r}$. Based on this formulation, the following function $v(t)$ is presented as:

$$v(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{r}^2) \quad (3.6)$$

To investigate the the stability of the closed-loop system (3.5), the time-derivative of (3.6) is computed along the trajectories of the system (3.5) as follows:

$$\dot{v}(t) = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_4 \dot{e}_4 + \tilde{a} \dot{\tilde{a}} + \tilde{b} \dot{\tilde{b}} + \tilde{c} \dot{\tilde{c}} + \tilde{r} \dot{\tilde{r}} \quad (3.7)$$

$$\begin{aligned}\dot{v}(t) &= \tilde{a}(e_1^2 + \hat{a}) + \tilde{b}(e_2^2 e_3 + e_2 z_m + e_2 e_3 y_m + \hat{b}) \\ &\quad + \tilde{c}(e_3^2 e_4 + e_3^2 w_m + e_3 e_4 z_m + \hat{c}) + \tilde{r}(\hat{r} - e_4^2) \\ &\quad - k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2\end{aligned}\quad (3.8)$$

The unknown parameters are estimated through the following learning laws:

$$\begin{aligned}\hat{a} &= -e_1^2, \\ \hat{b} &= -e_2^2 e_3 - e_2^2 z_m - e_2 e_3 y_m, \\ \hat{c} &= -e_3^2 e_4 - e_3^2 w_m - e_3 e_4 z_m, \\ \hat{r} &= e_4^2.\end{aligned}\quad (3.9)$$

After replacing (3.9) in (3.8), we have

$$\dot{v}(t) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 \quad (3.10)$$

We observe that the equation (3.10) is negative semi-definite, i.e., $\dot{v}(t) \leq 0$. This implies that, system (3.5) is stable in the Lyapunov [12] sense. However, the Barbalat lemma is used to show the asymptotic stability of (3.5).

Lemma (Barbalat's Lemma). *If $g(t)$ is continuous uniformly for $t \geq 0$ and if*

$$\int_0^\infty g(t) dt < \infty,$$

then

$$\lim_{t \rightarrow \infty} g(t) = 0.$$

Corollary: If $e(t) \in L^2 \cap L^\infty$ and $\dot{e}(t)$ is uniformly continuous, then

$$\lim_{t \rightarrow \infty} e(t) = 0. \quad (3.11)$$

Taking

$$k = \min\{k_1, k_2, k_3, k_4\}$$

Integrating both sides, we obtain:

$$\int_0^t \dot{v} dt \leq -k \int_0^t (e_1^2 + e_2^2 + e_3^2 + e_4^2) dt.$$

This implies,

$$\frac{1}{k}(v(0) - v(t)) < \frac{v(0)}{k}.$$

Thus, we obtain:

$$\int_0^t (e_1^2 + e_2^2 + e_3^2 + e_4^2) dt \leq \frac{1}{k} v(0).$$

Taking the limit as $t \rightarrow \infty$:

$$\lim_{t \rightarrow \infty} \int_0^t (e_1^2 + e_2^2 + e_3^2 + e_4^2) dt \leq \frac{1}{k} v(0).$$

Since this holds, we conclude that:

$$e_i \in L_2.$$

Now, since $\dot{v}(t) \leq 0$, integrating both sides gives:

$$\int_0^t \dot{v} dt \leq 0.$$

This leads to:

$$v(t) - v(0) \leq 0 \Rightarrow v(t) \leq v(0) \text{ and } v(t) > 0.$$

Thus, $v(t)$ is bounded. It follows that e_i and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{r}$ are bounded and $e_i \in L^\infty$. Hence, using Lemma 1 $\lim_{t \rightarrow \infty} e_i = 0$.

4. Numerical Simulation

The efficacy and validity of the suggested synchronization technique are demonstrated through numerical simulations Latiff, Othman, and Kumaresan [13]. The parameters of the hyper-chaotic financial system are set as $a = 0.1$, $b = 0.2$, $c = 3.3$, and $r = -3$. The initial states for the master and slave systems are selected as $x(0) = (0.8, 0.2, 0.3, 0.1)$ and $y(0) = (0.12, 0.61, 3.6, 1)$, respectively. The control gains are uniformly set to $k_1 = 1.1367$, $k_2 = 0.93915$, $k_3 = 7.6336$, $k_4 = 1.0181$. Figure 4 illustrates that the synchronization error converges to zero as $t \rightarrow \infty$, confirming successful error stabilization and depict the simulation results of the synchronized state variables of the master and slave systems. Furthermore, Figure 4 demonstrates the convergence of the estimated parameters to their considered values, i.e., $\hat{a} \rightarrow a$, $\hat{b} \rightarrow b$, $\hat{c} \rightarrow c$, $\hat{d} \rightarrow d$.

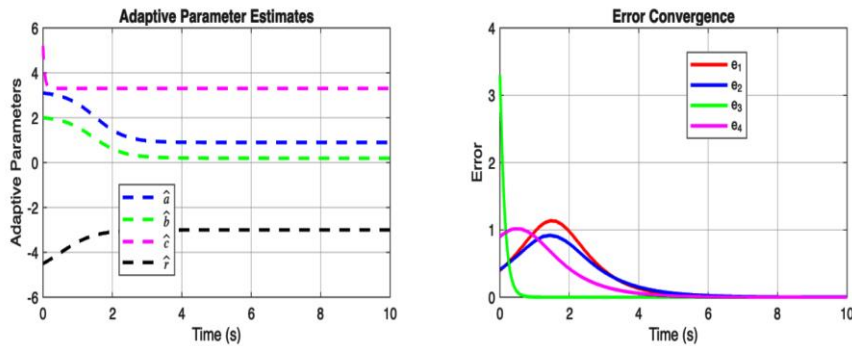


Figure 4

The parameter estimates $\hat{a}, \hat{b}, \hat{c}$ and $\hat{r}$ asymptotically converges to their true values and the synchronization errors converges to 0.

5. Conclusion

For the dynamical analysis of a novel four-dimensional hyperchaotic financial system in this study, bifurcation diagrams and Lyapunov exponents have been used. Lu and Yu [14] groundbreaking work, which presented an adaptive synchronization technique for a hyperchaotic financial system, provides the basis for this investigation. By altering the structure of their system and expanding the control design to account for nonlinear couplings and parameter uncertainty, we have created a more resilient and dynamically rich model that expands the possible uses in financial modeling. The practicality of adaptive techniques in managing chaotic systems with unpredictable dynamics in the actual world is confirmed by successful synchronization.

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Received December, 2025