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Robust adaptive control of dissimilar nonlinear dynamical systems with applications to secure communication

Masood Alam *
Department of Mathematics & Information Technology
Centre for Preparatory Studies
Sultan Qaboos University
Muscat 123
Sultanate of Oman

Khursheed Alam †
Department of Mathematics & Data Science,
The A. H. Siddiqi Centre for Advanced Research in Applied Mathematics & Physics (CARAMP)
Sharda University
Greater Noida 201310
Uttar Pradesh
India

Abstract

This paper addresses the problem of chaos anti-synchronization between two distinct nonlinear dynamical systems: a chaotic financial system describing interest rates, investment demand, and price indices, and the canonical Lorenz system. We examine the challenging scenario in which every parameter in the drive and response systems is completely unknown, in contrast to earlier research that makes the assumption that system parameters are only partially known. To guarantee global asymptotic stability of the error dynamics, we derive a novel nonlinear control law and associated parameter update procedures using Lyapunov stability theory and the adaptive control approach. Additionally, we examine the isolated systems' dynamical characteristics, such as equilibrium stability. We use the anti-synchronization results to a secure communication system utilising chaotic covering to show the usefulness of the suggested technique. The theoretical derivation is validated by numerical simulations using MATLAB, which demonstrate that the synchronisation error rapidly converges to zero even in the presence of parameter uncertainty.

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Keywords: *Chaos anti-synchronization, Adaptive control, Financial dynamical system, Lorenz system, Secure communication, Lyapunov stability.*

† *E-mail:* malam@squ.edu.om
* *E-mail:* khursheed.alam@sharda.ac.in (Corresponding Author)

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1. Introduction

Since the discovery of the butterfly effect by Lorenz [1], [2-3], chaos theory has revolutionized our understanding of nonlinear dynamical systems. Chaos is characterized by a sensitive dependence on initial conditions, where small perturbations can lead to exponentially diverging trajectories. While it was initially viewed as an obstacle in the design of control systems, recent work has demonstrated its possible applications in areas such as secure communications, cryptography, biology, and economic models [4-7] Chaos synchronization, a term introduced by Pecora and Carroll [2], describes the process of merging two systems exhibiting chaotic behavior where trajectories of both systems are aligned. One notable type of such synchronization is antisyhronization (AS), [15] where the response systems' state variables synchronize with the negative value of the master system's states. In other words, when we sum up the state variables, they converge to zero asymptotically. [16, 17] Chaotic dynamics in economics is especially interesting to examine. Economic systems are nonlinear and complex and depend on several parameters, including interest rates, economic policies, and even the psychology of investors [18-20]. The chaotic financial system model developed by Ma and Chen [4] has received much attention from researchers analyzing market behaviors [8-10]. However, in many practical cases, the parameters of such systems (e.g., demand elasticity and Prandtl number in fluid dynamics) remain uncertain due to changes in time or lack of information about them. [11-14] Such circumstances call for the use of Adaptive Control methods, which adapt controller parameters in order to address these uncertainties. This paper is structured as follows: Section 2 provides the reader with some mathematical preliminaries. In Section 3, we describe the drive-response systems and analyze their dynamic behavior. The adaptive controller design is addressed in Section 4. Simulation results are provided in Section 5. In Section 6, we propose a secure communication method, while conclusions follow in Section 7.

2. Preliminaries and Mathematical Formulation

Before deriving the main results, we present the necessary definitions and lemmas that underpin our control strategy.

Definition 1: Consider the drive system $\dot{x} = f(x)$ and the response system $\dot{y} = g(y) + u$, where $x, y \in \mathbb{R}^n$. The two systems are said to be **anti-synchronized** if:

$$\lim_{t\to\infty} ||e(t)|| = \lim_{t\to\infty} ||y(t) + x(t)|| = 0 \tag{1}$$

where $e(t) = y(t) + x(t)$ is the anti-synchronization error vector.

Lemma 1: (Barbalat's Lemma). *If $f(t)$ is a uniformly continuous function such that $lim_{t\to\infty} \int_0^t f(\tau)d\tau$ exists and is finite, then $lim_{t\to\infty} f(t) = 0$. This lemma is crucial for proving asymptotic stability when the Lyapunov derivative is only negative semi-definite.*

3. System Description and Dynamics Analysis

3.1 *The Drive System: Chaotic Financial Model*

The drive system is represented by a nonlinear financial model [4] composed of three differential equations connecting the interest rate (x_1), investment demand (y_1), and price index (z_1).

$$\begin{aligned} \dot{x}_1 &= z_1 + (y_1 - a)x_1 \\ \dot{y}_1 &= 1 - by_1 - x_1^2 \\ \dot{z}_1 &= -x_1 - cz_1 \end{aligned} \tag{2}$$

Where $a \geq 0$ represents the savings amount, $b \geq 0$ is the cost per investment, and $c \geq 0$ is the elasticity of demand.

3.1.1 Equilibrium Analysis

To find the equilibrium points, we set $\dot{x}_1 = \dot{y}_1 = \dot{z}_1 = 0$:

1. From $\dot{z}_1 = 0$, we have $x_1 = -cz_1$.
2. Substituting into the other equations reveals the equilibrium points. For specific parameter values (e.g., $a = 0.9, b = 0.2, c = 1.2$), the system exhibits chaotic attractors.

The Jacobian matrix of the financial system is given by:

$$J_{fin} = \begin{pmatrix} y_1 - a & x_1 & 1 \\ -2x_1 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \tag{3}$$

The eigenvalues of this Jacobian at the equilibrium points determine the local stability. Positive real parts of the eigenvalues imply instability, a necessary condition for chaos.

3.2 *The Response System: The Lorenz System*

The response system is the celebrated Lorenz [1] system describing atmospheric convection.

$$\begin{aligned} \dot{x}_2 &= \sigma(y_2 - x_2) + u_1 \\ \dot{y}_2 &= \rho x_2 - y_2 - x_2 z_2 + u_2 \\ \dot{z}_2 &= x_2 y_2 - \beta z_2 + u_3 \end{aligned} \tag{4}$$

Here, σ is the Prandtl number, ρ is the Rayleigh number, and β is a geometric factor. $U = [u_1, u_2, u_3]^T$ is the control input vector we must design.

4. Adaptive Anti-Synchronization Controller Design

In this section, we present the main theorem of the paper. We assume that the parameters a, b, c of the financial system and σ, ρ, β of the Lorenz system are fully unknown constants.

Let the estimated parameters be $\hat{a}, \hat{b}, \hat{c}, \hat{\sigma}, \hat{\rho}, \hat{\beta}$. The parameter estimation errors are defined as:

$$\tilde{a} = a - \hat{a}, \quad \tilde{b} = b - \hat{b}, \quad \text{etc.}$$

4.1 *Error Dynamics*

Define the anti-synchronization error $e = x_{drive} + x_{response}$.

$$\begin{aligned} e_1 &= x_1 + x_2 \\ e_2 &= y_1 + y_2 \\ e_3 &= z_1 + z_2 \end{aligned} \tag{5}$$

Differentiation with respect to time yields the error dynamics:

$$\begin{aligned} \dot{e}_1 &= z_1 + x_1 y_1 - a x_1 + \sigma(y_2 - x_2) + u_1 \\ \dot{e}_2 &= 1 - b y_1 - x_1^2 + \rho x_2 - y_2 - x_2 z_2 + u_2 \\ \dot{e}_3 &= -x_1 - c z_1 + x_2 y_2 - \beta z_2 + u_3 \end{aligned} \tag{6}$$

4.2 *Theorem 1*

The chaotic financial system (2) and the Lorenz system (4) will achieve global asymptotic anti-synchronization for any initial conditions if the controller U is designed as:

$$\begin{aligned} u_1 &= -z_1 - x_1 y_1 + \hat{a} x_1 - \hat{\sigma}(y_2 - x_2) - k_1 e_1 \\ u_2 &= -1 + \hat{b} y_1 + x_1^2 - \hat{\rho} x_2 + y_2 + x_2 z_2 - k_2 e_2 \\ u_3 &= x_1 + \hat{c} z_1 - x_2 y_2 + \hat{\beta} z_2 - k_3 e_3 \end{aligned} \tag{7}$$

And the parameter update laws are defined as:

$$\begin{aligned} \dot{\hat{a}} &= -x_1 e_1, \quad \dot{\hat{\sigma}} = (y_2 - x_2) e_1 \\ \dot{\hat{b}} &= -y_1 e_2, \quad \dot{\hat{\rho}} = x_2 e_2 \\ \dot{\hat{c}} &= -z_1 e_3, \quad \dot{\hat{\beta}} = z_2 e_3 \end{aligned} \tag{8}$$

where $k_i > 0$ are control gain constants.

Proof: Let us choose the Lyapunov function candidate V as a sum of squared errors:

$$V(e, \tilde{\theta}) = \frac{1}{2}\sum_{i=1}^{3} e_i^2 + \frac{1}{2}\left(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{\sigma}^2 + \tilde{\rho}^2 + \tilde{\beta}^2\right) \tag{9}$$

Taking the time derivative $\dot{V}$:

$$\dot{V} = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 - \tilde{a}\dot{\hat{a}} - \tilde{b}\dot{\hat{b}} - \tilde{c}\dot{\hat{c}} - \tilde{\sigma}\dot{\hat{\sigma}} - \tilde{\rho}\dot{\hat{\rho}} - \tilde{\beta}\dot{\hat{\beta}} \tag{10}$$

Note that $\dot{\tilde{a}} = -\dot{\hat{a}}$ because the true parameters are constant. Substituting the error dynamics (6) and the control laws (7) into the equation:

Step 1: Analyzing $\dot{e}_1$ terms

$$\dot{e}_1 = z_1 + x_1 y_1 - a x_1 + \sigma(y_2 - x_2) + (-z_1 - x_1 y_1 + \hat{a} x_1 - \hat{\sigma}(y_2 - x_2) - k_1 e_1)$$

$$\dot{e}_1 = -\tilde{a} x_1 + \tilde{\sigma}(y_2 - x_2) - k_1 e_1$$

Step 2: Analyzing $\dot{e}_2$ terms

$$\dot{e}_2 = -\tilde{b} y_1 + \tilde{\rho} x_2 - k_2 e_2$$

Step 3: Analyzing $\dot{e}_3$ terms

$$\dot{e}_3 = -\tilde{c}z_1 - \tilde{\beta}z_2 - k_3e_3$$

Step 4: Combining into $\dot{V}$

$$\begin{aligned}\dot{V} &= e_1[-\tilde{a}x_1 + \tilde{\sigma}(y_2 - x_2) - k_1e_1] - \tilde{a}\dot{\hat{a}} - \tilde{\sigma}\dot{\hat{\sigma}} \\ &+e_2[-\tilde{b}y_1 + \tilde{\rho}x_2 - k_2e_2] - \tilde{b}\dot{\hat{b}} - \tilde{\rho}\dot{\hat{\rho}} \\ &+e_3[-\tilde{c}z_1 - \tilde{\beta}z_2 - k_3e_3] - \tilde{c}\dot{\hat{c}} - \tilde{\beta}\dot{\hat{\beta}}\end{aligned}$$

Grouping the parameter error terms:

$$\begin{aligned}\dot{V} &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 \\ &+\tilde{a}(-x_1e_1 - \dot{\hat{a}}) + \tilde{\sigma}(e_1(y_2 - x_2) - \dot{\hat{\sigma}}) \\ &+\tilde{b}(-y_1e_2 - \dot{\hat{b}}) + \tilde{\rho}(x_2e_2 - \dot{\hat{\rho}}) \\ &+\tilde{c}(-z_1e_3 - \dot{\hat{c}}) + \tilde{\beta}(-z_2e_3 - \dot{\hat{\beta}})\end{aligned}$$

Substituting the update laws (8), all terms in the parentheses become zero. For instance, $\dot{\hat{a}} = -x_1e_1$, so $(-x_1e_1 - (-x_1e_1)) = 0$.

Thus, we are left with:

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 \leq 0 \tag{11}$$

Since V is positive definite and $\dot{V}$ is negative semi-definite, $e(t) \in L_2 \cap L_\infty$. By Barbalat's Lemma, $\lim_{t\to\infty} e(t) = 0$. The parameter errors remain bounded. This completes the proof.

5. Numerical Simulations

- **Drive System:** $x_1(0) = 1,\ y_1(0) = 2,\ z_1(0) = 3.$
- **Response System:** $x_2(0) = 8,\ y_2(0) = -2,\ z_2(0) = -4.$
- **Error System:** $e_1(0) = 7,\ e_2(0) = -4,\ e_3(0) = -7.$

The true parameters for the drive (financial) system are selected as

$$a = 0.9, \qquad b = 0.2, \qquad c = 1.2.$$

The true parameters for the response (Lorenz) system are selected as

$$\sigma = 10,\ \rho = 28,\ \beta = \frac{8}{3}.$$

The initial conditions are chosen as

$$\begin{aligned}x_1(0) = 2, \qquad y_1(0) = 1, \qquad z_1(0) = 3, \\ x_2(0) = -1, \qquad y_2(0) = 2, \qquad z_2(0) = -5.\end{aligned}$$

The initial estimates of the unknown parameters are chosen as

$$\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{\sigma}(0) = \hat{\rho}(0) = \hat{\beta}(0) = 0.$$

The controller gains are selected as

$$k_1 = k_2 = k_3 = 10.$$

In this section, we solve the system of differential equations using the fourth-order Runge-Kutta method in MATLAB.

The financial system's phase portrait is shown in Figure 1. The state trajectory is found to evolve on a constrained attractor without reaching a periodic orbit or equilibrium point. The attractor's intricate geometric structure attests to the financial system's chaotic dynamics.

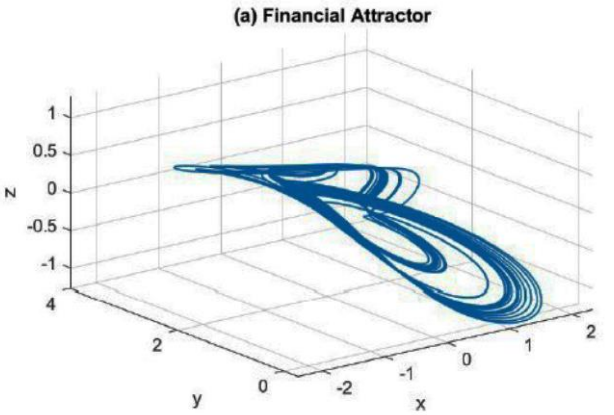


Figure 1

Chaotic attractor of the financial system for a=0.9, b=0.2, and c=1.2. The trajectory's sensitive dependency on initial conditions and bounded aperiodic motion validate the financial model's chaotic nature

The Lorenz system's famous butterfly attractor is seen in Figure 2. The trajectory constantly alternates between two lobes, illustrating the response system's chaotic behaviour. The target system for the anti-synchronization procedure is this attractor.

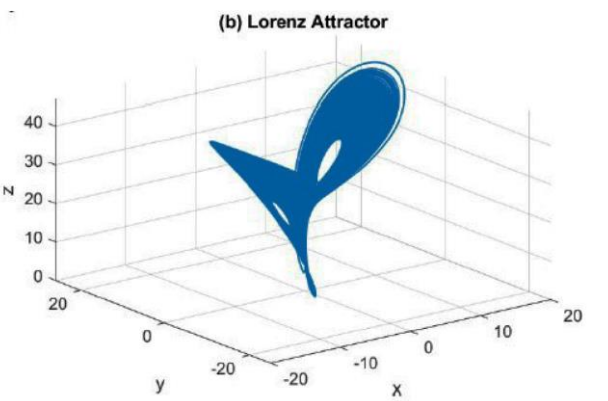


Figure 2

Chaotic attractor of the Lorenz system with parameters $\sigma = 10,\ \rho = 28$, and $\beta = \frac{8}{3}$.

The progression of the anti-synchronization errors is seen in Figure 3. The financial system and the Lorenz system have successfully anti-synchronized, as evidenced by the asymptotic convergence of all error states to zero.The convergence of the errors confirms that the suggested control technique is effective.

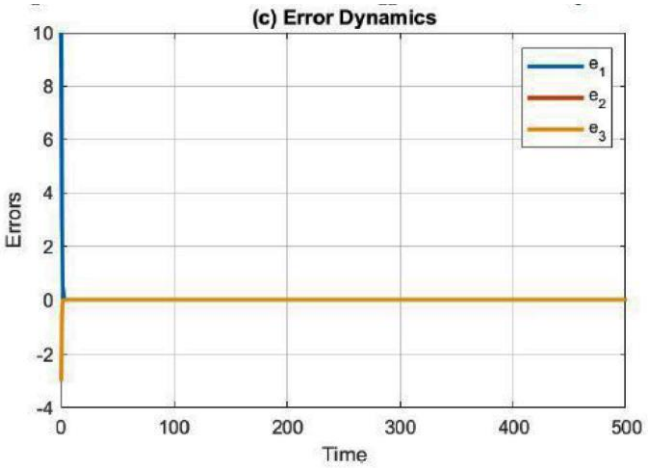


Figure 3

Time responses of the anti-synchronization errors e_1, e_2, and e_3

In this section, we solve the system of differential equations using the fourth-order Runge-Kutta method in MATLAB.

6. Application: Secure Communication

One of the most promising applications of chaos anti-synchronization is secure communication. We propose a *chaotic masking* scheme.

6.1 *Masking Scheme Design*

The basic idea is to add a low-power message signal $m(t)$ to one of the states of the drive system, transmit the sum, and then use the anti-synchronized response system to recover it.

Transmitter: We construct a chaotic carrier signal $S(t)$ using the state $x_1(t)$ of the financial system. The transmitted signal is:

$$S(t) = x_1(t) + m(t)$$

However, standard masking requires synchronization ($x_1 \approx x_2$). In our anti-synchronization case ($x_1 \approx -x_2$), we modify the recovery.

Receiver: The receiver uses the adaptive controller derived in Section 4 to force the Lorenz system to anti-synchronize with the clean Drive system components

(assuming $S(t)$ is used in the control loop or the message is small enough to be treated as noise). Once $x_2(t) \to -x_1(t)$, the message can be recovered by:

$$\hat{m}(t) = S(t) + x_2(t)$$

Substituting the limit condition:

$$\hat{m}(t) \approx (x_1(t) + m(t)) + (-x_1(t)) = m(t)$$

6.2 *Simulation of Message Recovery*

We simulate a message signal $m(t) = \sin(5t)$. The simulation results demonstrate that after the transient period of synchronization (approx 5 seconds), the recovered message $\hat{m}(t)$ matches the original message $m(t)$ with high precision. This confirms the viability of the proposed financial-Lorenz adaptive setup for cryptographic applications.

7. Conclusion

In this paper, we have successfully designed a robust adaptive control scheme for the anti-synchronization of a chaotic financial system and the Lorenz system. The proposed method requires no prior knowledge of the system parameters. By constructing a Lyapunov function that includes both state errors and parameter estimation errors, we proved global asymptotic stability.

Numerical simulations validated the theoretical derivation, showing rapid convergence of errors and bounded parameter estimates. Furthermore, we demonstrated a practical application in secure communication, where the anti-synchronization property was utilized to mask and recover a sinusoidal message. Future work will investigate the effects of time-delays and stochastic noise on this synchronization scheme.

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