

Novel results on transformation identities for bilateral basic hypergeometric series

Ranu Singh [†]

Nidhi Sahni ^{*}

Department of Mathematics & Data Science

Sharda School of Engineering & Science

Sharda University

Greater Noida 201310

Uttar Pradesh

India

Abstract

The present work develops a systematic approach to construct some transformation formulas and identities for bilateral basic hypergeometric series. By employing the bilateral Bailey transformation given by Andrews and Warnaar [1] together with a collection of established summation formulae, we derive several identities of the type ${}_r\Psi_r$, and examine particular limiting cases. The identities obtained here both subsume classical results in the theory of bilateral q -series and constitute a meaningful extension of the Bailey transform machinery to the bilateral setting.

Subject Classification (2020): 33D15, 33A30.

Keywords: Basic hypergeometric series, Bilateral series, Bailey transform, Identities, Transformation, Summation formula.

1. Introduction, Notations and Definitions

Bailey established a simple and useful transform in 1949, which is midway to the development of these series identities. Later, Andrews and Warnaar [1], established two bilateral versions of Bailey's transform. The bilateral transforms, symmetric bilateral Bailey transform and the asymmetric bilateral Bailey transform were defined to prove certain false theta identities found in Ramanujan's Lost Notebook. Other researchers, like Joshi and Vyas [6], established the extended Bailey's transform for basic hypergeometric series, whose different proof was provided independently by Denis, Singh, and Singh [2,

[†] E-mail: 2024104857.ranu@dr.sharda.ac.in

^{*} E-mail: nidhi.sahni@sharda.ac.in (Corresponding Author)

3, 4], they deduced certain transformation formulae by using it. Singh, Singh, and Yadav [7], Singh, Singh, and Yadav [8], Singh and Yadav [9], Srivastava et al. [10], Verma and Jain [11, 12], Yadav, Yadav, and Mishra [13], Zala and Prajapati [14], Nagar and Mishra [15], Lather and Nagar [16] also established an bilateral Bailey's transform for basic hypergeometric series. The paper motivated upon this history by using these bilateral Bailey transforms and known summation formulas to derive new transformation identities for bilateral basic hypergeometric series.

The paper centre of attention on establishing novel results on transformation identities for bilateral basic hypergeometric series. It utilizes the bilateral Bailey transform developed by Andrews and Warnaar [2] and certain known summation formulae to achieve its results. The established identities generalize classical bilateral identities and extend the scope of the Bailey transform.

Bailey, during 1949, discovered a simple but very useful transform known as the *Bailey transform* that make every relevant infinite series absolutely convergent.

Bailey in 1949 introduced the concept of a *Bailey pair* (α_n, β_n) relative to (l, q) The classical *Bailey transform* representation

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{(m;q)_r (n;q)_r}{(lq/m;q)_r (lq/n;q)_r} \left(\frac{lq}{mn}\right)^n \beta_n \\ &= \frac{(lq/m;q)_{\infty} (lq/n;q)_{\infty}}{(lq;q)_{\infty} (lq/mn;q)_{\infty}} \sum_{r=0}^{\infty} \frac{(m;q)_r (n;q)_r}{(lq/m;q)_r (lq/n;q)_r} \left(\frac{lq}{mn}\right)^r \alpha_r. \end{aligned}$$

In order to prove certain false theta identities found in the “*Lost Notebook*” of Ramanujan, Andrews and Warnaar [1], defined two bilateral versions of Bailey’s transform: Symmetric bilateral Bailey transform and Asymmetric bilateral Bailey transform

which make every relevant infinite series absolutely convergent.

We also make use of the following known summation [5] in our analysis:

$$\begin{aligned} & \sum_{k=-m}^n \frac{[lspq;pq]_k [mp/sq;p/q]_k}{[ls;pq]_k [m/s;p/q]_k} \frac{[l,m;p]_k [n,ls^2/mn;q]_k}{[lsp/n,mnp/s;p]_k [sq,lsq/m;q]_k} q^k \\ &= \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \\ & \times \left\{ \frac{[lp,mp;p]_n \left[nq, \frac{ls^2}{mn}q;q\right]_n}{\left[\frac{ls}{n}, \frac{mnp}{s};p\right]_n \left[sq, \frac{ls}{m}q;q\right]_n} \frac{\left[\frac{n}{ls'} \frac{s}{mn'};p\right]_{m+1} \left[\frac{1}{ls'} \frac{m}{ls};q\right]_{m+1}}{\left[\frac{1}{l'} \frac{1}{m'};p\right]_{m+1} \left[\frac{1}{n'} \frac{mn}{ls'^2};q\right]_{m+1}} \right\}. \end{aligned} \quad (1.1)$$

[Gasper and Rahman [5]; AppII (13.8.2) p. 298]

Let $m = n$, then

$$\begin{aligned} & \sum_{k=-n}^n \frac{[lspq;pq]_k \left[\frac{mp}{s} \frac{p}{q}\right]_k}{[ls;pq]_k \left[\frac{m}{s}, \frac{p}{q}\right]_k} \frac{[l,m;p]_k [n,ls^2/mn;q]_k}{[lsp/n,mnp/s;p]_k [sq,lsq/m;q]_k} q^k \\ &= \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \\ & \times \frac{[lp,mp;p]_n \left[nq, \frac{ls^2}{mn}q;q\right]_n}{\left[\frac{ls}{c}, \frac{mnp}{d};p\right]_n \left[sq, \frac{ls}{m}q;q\right]_n} \frac{\left[\frac{n}{ls'} \frac{s}{mn'};p\right]_{n+1} \left[\frac{1}{ls'} \frac{m}{ls};q\right]_{n+1}}{\left[\frac{1}{l'} \frac{1}{m'};p\right]_{n+1} \left[\frac{1}{n'} \frac{mn}{ls'^2};q\right]_{n+1}}. \end{aligned} \quad (1.2)$$

$$\begin{aligned}
& \sum_{k=-m}^n \frac{[\beta; p]_k [n; q]_k [y; p]_k \left[\frac{\beta y n, pP}{s^2, q} \right]_k \left[\frac{yP, P}{sq, q} \right]_k \left[\frac{\beta p, p}{sq, q} \right]_k \left[\frac{\beta y n, pP; pP}{s} \right]_k}{[sq; q]_k \left[\frac{\beta np, p}{s} \right]_k \left[\frac{nyP, P}{d} \right]_k \left[\frac{\beta y pP, pP}{s, q, q} \right]_k \left[\frac{y, P}{s, q} \right]_k \left[\frac{\beta, p}{s, q} \right]_k \left[\frac{\beta ny, pP}{d} \right]_k} q^k \\
&= \frac{s(1-\beta)(1-n)(1-y)(s^2-\beta ny)}{(s-\beta ny)(s-y)(s-\beta)(n-s)} \\
&\times \left[\frac{\left[\frac{s}{\beta n}, p \right]_{m+1} \left[\frac{1}{s}, q \right]_{m+1} \left[\frac{s}{ny}, p \right]_{m+1} \left[\frac{s}{\beta y}, q \right]_{m+1}}{\left[\frac{1}{\beta}, p \right]_{m+1} \left[\frac{1}{n}, q \right]_{m+1} \left[\frac{1}{y}, p \right]_{m+1} \left[\frac{s^2, pP}{\beta ny, q} \right]_{m+1}} \right. \\
&\quad \left. - \frac{[\beta p; p]_n [nq; q]_n [yP; P]_n \left[\frac{\beta ny, pP, pP}{s^2, q, q} \right]_n}{\left[\frac{\beta np}{s}, p \right]_n [sq; q]_n \left[\frac{nyP}{s}, p \right]_n \left[\frac{\beta y pP, pP}{sq, q} \right]_n} \right].
\end{aligned} \tag{1.3}$$

[Verma and Jain [12], 12A), p. 85]

Let $m = n$, then

$$\begin{aligned}
& \sum_{k=-n}^n \frac{[\beta; p]_k [n; q]_k [y; p]_k \left[\frac{\beta y n, pP}{s^2, q} \right]_k \left[\frac{yP, P}{sq, q} \right]_k \left[\frac{\beta p, p}{sq, q} \right]_k \left[\frac{\beta y n, pP; pP}{s} \right]_k}{[sq; q]_k \left[\frac{\beta np, p}{d} \right]_k \left[\frac{nyP, P}{d} \right]_k \left[\frac{\beta y pP, pP}{s, q, q} \right]_k \left[\frac{y, P}{s, q} \right]_k \left[\frac{\beta, p}{s, q} \right]_k \left[\frac{\beta ny, pP}{s} \right]_k} q^k \\
&= \frac{s(1-\beta)(1-n)(1-y)(s^2-\beta ny)}{(s-\beta ny)(s-y)(s-\beta)(n-s)} \\
&\times \left[\frac{\left[\frac{s}{\beta n}, p \right]_{n+1} \left[\frac{1}{s}, q \right]_{n+1} \left[\frac{s}{ny}, p \right]_{n+1} \left[\frac{d}{\beta y}, q \right]_{n+1}}{\left[\frac{1}{\beta}, p \right]_{n+1} \left[\frac{1}{n}, q \right]_{n+1} \left[\frac{1}{y}, p \right]_{n+1} \left[\frac{s^2, pP}{\beta ny, q} \right]_{n+1}} \right. \\
&\quad \left. - \frac{[\beta p; p]_n [nq; q]_n [yP; P]_n \left[\frac{\beta ny pP, pP}{s^2, q, q} \right]_n}{\left[\frac{\beta np}{d}, p \right]_n [sq; q]_n \left[\frac{nyP}{s}, p \right]_n \left[\frac{\beta y pP, pP}{sq, q} \right]_n} \right].
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
& \sum_{k=-m}^n \frac{[lspqPQ; pqpQ]_k \left[\frac{spQ, pQ}{npq, pQ} \right]_k \left[\frac{mpP, pP}{sqQ, qQ} \right]_k}{[ls; pqpQ]_k \left[\frac{s, pQ}{n, pQ} \right]_k \left[\frac{m, pP}{s, qQ} \right]_k \left[\frac{ls, pQ}{mn, pQ} \right]_k \left[\frac{sqPQ, qPQ}{p, p} \right]_k} \\
&\times \frac{\left[\frac{ls pQ, pQ}{mnqP, pQ} \right]_k [l; p^2]_k [n; q^2]_k [m; p^2]_k \left[\frac{ls^2, Q^2}{mn, Q^2} \right]_k}{\left[\frac{ls pQ, pPQ}{nq, q} \right]_k \left[\frac{ls pQ, pQ}{mP, P} \right]_k \left[\frac{mn p q P, p q P}{sqQ, Q} \right]_k} q^{2k} \\
&= \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \\
&\times \left\{ \frac{[lp^2; p^2]_n [nq^2; q^2]_n [mp^2; p^2]_n \left[\frac{ls^2, Q^2}{mn, Q^2} \right]_n}{\left[\frac{sqPQ, qPQ}{p, p} \right]_n \left[\frac{ls pPQ, pPQ}{n, q} \right]_n \left[\frac{ls p q Q, p q Q}{m, P} \right]_n \left[\frac{mn p q P, p q Q}{s, Q} \right]_n} \right. \\
&\quad \left. \times \frac{\left[\frac{n, pPQ}{ls, q} \right]_{m+1} \left[\frac{1, qPQ}{s, p} \right]_{m+1} \left[\frac{m, p q Q}{ls, P} \right]_{m+1} \left[\frac{s, p q P}{mn, Q} \right]_{m+1}}{\left[\frac{1}{l}, p^2 \right]_{m+1} \left[\frac{1}{n}, q^2 \right]_{m+1} \left[\frac{1}{m}, p^2 \right]_{m+1} \left[\frac{mn}{ls^2, q^2} \right]_{m+1}} \right\}
\end{aligned} \tag{1.5}$$

Let $m = n$, then

$$\begin{aligned}
&= \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \\
&\times \left\{ \frac{[lp^2:p^2]_n [nq^2:q^2]_n [mP^2:P^2]_n \left[\frac{ls^2}{mn}Q^2:Q^2\right]_n}{\left[\frac{sqPQ}{p}:\frac{qPQ}{p}\right]_n \left[\frac{lsPQ}{n}:\frac{pPQ}{q}\right]_n \left[\frac{lsPQ}{m}:\frac{pPQ}{P}\right]_n \left[\frac{mnPQ}{s}:\frac{pPQ}{Q}\right]_n} \right. \\
&\times \left. \frac{\left[\frac{n}{ls}:\frac{pPQ}{q}\right]_{n+1} \left[\frac{1}{s}:\frac{qPQ}{p}\right]_{n+1} \left[\frac{m}{ls}:\frac{pPQ}{P}\right]_{n+1} \left[\frac{s}{mn}:\frac{pPQ}{Q}\right]_{n+1}}{\left[\frac{1}{l}:\frac{p^2}{p^2}\right]_{n+1} \left[\frac{1}{n}:\frac{q^2}{q^2}\right]_{n+1} \left[\frac{1}{m}:\frac{P^2}{P^2}\right]_{n+1} \left[\frac{mn}{ls^2}:\frac{q^2}{Q^2}\right]_{n+1}} \right\} \quad (1.6)
\end{aligned}$$

2. Main Results

In this section we establish our results.

2.1 Choosing suitable α_r , u_r, v_r and δ_r in Bailey transform to get bilateral transform ${}_6\Psi_6$

$$\alpha_r = \frac{[lspq;pq]_r \left[\frac{mp}{sq}, \frac{p}{q}\right]_r [l,m;p]_r \left[n, \frac{ls^2}{mn}, q\right]_r q^r}{[ls;pq]_r \left[\frac{m}{s}, \frac{p}{q}\right]_r \left[\frac{ls}{n}, \frac{mn}{s}, p;p\right]_r \left[sq, \frac{ls}{m}, q\right]_r}.$$

and $u_r = v_r = 1$, $\delta_r = z^r$ in Bailey Transform and using (1.2) we get,

$$\begin{aligned}
\beta_n &= \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \left\{ \frac{[lp, mp;p]_n \left[nq, \frac{ls^2}{mn}, q\right]_n}{\left[\frac{ls}{n}, p; \frac{mn}{s}, p;p\right]_n \left[sq, \frac{ls}{m}, q\right]_n} \right\} \\
&\times \frac{(ls-n)(mn-s)(s-1)(ls-m)}{(1-ls)(s-m)(s-n)(mn-ls)} \left[\frac{\left[\frac{np}{ls}, \frac{sp}{mn}, p\right]_n \left[\frac{q}{s}, \frac{mq}{ls}, q\right]_n}{\left[\frac{p}{l}, \frac{p}{m}, p\right]_n \left[\frac{q}{n}, \frac{mnq}{ls}, q\right]_n} \right]. \quad (2.1)
\end{aligned}$$

and

$$\gamma_n = \frac{z^n}{1-z}. \quad (2.2)$$

Putting this value of β_n, γ_n in Asymmetric Bilateral Bailey Transform we obtain,

$$\begin{aligned}
&{}_6\Psi_6 \left[\begin{matrix} lspq, \frac{mp}{sq}, l, m, n, \frac{ls^2}{mn}; pq, \frac{p}{q}, p, q \\ ls, \frac{m}{s}, \frac{ls}{n}, p, \frac{mn}{s}, p, sq, \frac{lsq}{m} \end{matrix} ; zq \right] \\
&= M_5 {}_5\Phi_4 \left[\begin{matrix} lp, mp, p, nq, \frac{ls^2q}{mn}; p, q \\ \frac{ls}{n}, p, \frac{mn}{s}, p, sq, \frac{lsq}{m} \end{matrix} \right] - L_5 {}_5\Phi_4 \left[\begin{matrix} \frac{np}{ls}, \frac{sp}{mn}, p, \frac{q}{s}, \frac{mq}{ls}; m, q \\ \frac{p}{l}, \frac{p}{m}, \frac{q}{n}, \frac{mnq}{ls^2} \end{matrix} ; z \right]. \quad (2.3)
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \frac{(1-z)s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)}, \\
L_1 &= \frac{(1-z)(ls-n)(s-1)(ls-m)}{(1-ls)(s-m)(s-n)(mn-ls)}.
\end{aligned}$$

2.2 Taking suitable α_r , u_r, v_r and δ_r in Bailey transform to get bilateral transform ${}_7\Psi_7$

$$\alpha_r = \frac{[\beta;p]_k [n;q]_k [y;p]_k \left[\frac{\beta y n}{s^2}; \frac{pP}{q}\right]_k \left[\frac{yP}{sq}; \frac{p}{q}\right]_k \left[\frac{\beta p}{sq}; \frac{p}{q}\right]_k \left[\frac{\beta y n}{d}; \frac{pP;pP}{q}\right]_k}{[dq;q]_k \left[\frac{\beta cp}{d}; p\right]_k \left[\frac{nyP}{d}; p\right]_k \left[\frac{\beta y pP}{sq}; \frac{pP}{q}\right]_k \left[\frac{yP}{sq}; \frac{p}{q}\right]_k \left[\frac{\beta p}{sq}; \frac{p}{q}\right]_k \left[\frac{\beta ny}{s}; \frac{pP}{q}\right]_k} q^k.$$

and $u_r = v_r = 1$, $\delta_r = z^r$ in symmetric Bilateral Bailey Transform and using Asymmetric Bilateral Bailey Transform we get,

$$\begin{aligned} \beta_n &= \frac{(\beta n - s)(s - 1)(ny - s)(\beta y - s)}{(s - \beta ny)(s - y)(s - \beta)(n - s)} \times \left\{ \frac{\left[\frac{sp}{\beta n}; p\right]_n \left[\frac{q}{s}; q\right]_n \left[\frac{sp}{ny}; p\right]_n \left[\frac{spP}{\beta y q}; \frac{pP}{q}\right]_n}{\left[\frac{p}{\beta}; p\right]_n \left[\frac{q}{n}; q\right]_n \left[\frac{p}{y}; p\right]_n \left[\frac{s^2 pP}{\beta ny}; \frac{pP}{q}\right]_n} \right. \\ &\quad \left. - \frac{s(1 - \beta)(1 - n)(1 - y)(s^2 - \beta ny)}{(s - \beta ny)(s - y)(s - \beta)(n - s)} \times \frac{[\beta p; p]_n [nq; q]_n [yP; p]_n \left[\frac{\beta ny pP}{s^2 q}; \frac{pP}{q}\right]_n}{\left[\frac{\beta np}{d}; p\right]_n [sq; q]_n \left[\frac{nyP}{s}; p\right]_n \left[\frac{\beta y pP}{sq}; \frac{pP}{q}\right]_n} \right\}. \end{aligned} \quad (2.4)$$

and

$$\gamma_n = \frac{z^n}{1 - z}. \quad (2.5)$$

Putting this value of β_n, γ_n in Asymmetric Bilateral Bailey Transform we get,

$$\begin{aligned} {}_7\Psi_7 &\left[\begin{matrix} \beta, ny, \frac{\beta ny}{s^2}, \frac{\beta ny pP}{s}, \frac{yP}{sq}, \frac{\beta p}{sq}; p, q, P, \frac{pP}{q}, pP, \frac{p}{q}, \frac{p}{q}; qz \end{matrix} \right] \\ &= M_3 {}_5\Phi_4 \left[\begin{matrix} \frac{sp}{\beta n}, p, \frac{sP}{ny}, \frac{spP}{\beta y q}; p, q, P, \frac{pP}{q}; z \end{matrix} \right] \\ &\quad - L_3 {}_5\Phi_4 \left[\begin{matrix} \beta p, p, nq, yP, \frac{\beta y pP}{s^2 q}, p, q, P, \frac{pP}{q}; z \end{matrix} \right] \\ &\quad \left[\begin{matrix} \frac{\beta np}{s}, sq, \frac{nyP}{s}, \frac{\beta y pP}{sq}, \frac{\beta ny}{s}, \frac{y}{s}, \frac{\beta}{s} \end{matrix} \right] \\ &\quad \left[\begin{matrix} \frac{p}{\beta}, \frac{q}{n}, \frac{p}{y}, \frac{s^2 pP}{\beta ny q} \end{matrix} \right] \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} M_3 &= \frac{(\beta n - s)(1 - z)(s - 1)(ny - s)(\beta y - s)}{(s - \beta ny)(s - y)(s - \beta)(n - s)}, \\ L_3 &= \frac{(1 - z)s(1 - \beta)(1 - n)(1 - y)(s^2 - \beta ny)}{(s - \beta ny)(s - y)(s - \beta)(n - s)}. \end{aligned}$$

2.3 Putting suitable α_r , u_r, v_r and δ_r in Bailey transform to get bilateral transform ${}_8\Psi_8$

$$\begin{aligned} \alpha_r &= \frac{[lspqPQ;pqPQ]_r \left[\frac{spQ}{npq}; \frac{pP}{pq}\right]_r \left[\frac{mpP}{sqQ}; \frac{pP}{qQ}\right]_r}{[ls;pqPQ]_r \left[\frac{s}{n}; \frac{pQ}{pq}\right]_r \left[\frac{m}{s}; \frac{pP}{qQ}\right]_r \left[\frac{ls}{mn}; \frac{pQ}{pq}\right]_r \left[\frac{sqPQ}{p}; \frac{qPQ}{p}\right]_r} \\ &\quad \times \frac{\left[\frac{lsPQ}{mnqP}; \frac{pP}{pq}\right]_r [l;p^2]_r [n;q^2]_r [m;p^2]_r \left[\frac{l^2}{mn}; Q^2\right]_r}{\left[\frac{lspPQ}{nq}; \frac{pPQ}{q}\right]_r \left[\frac{lspqQ}{mP}; \frac{pqQ}{P}\right]_r \left[\frac{mnpqP}{sqQ}; \frac{pqP}{Q}\right]_r} q^{2r}. \end{aligned}$$

and $u_r = v_r = 1$, $\delta_r = z^r$ in symmetric Bilateral Bailey Transform and using (1.6) we get,

$$\beta_r = \frac{s(1-l)(1-m)(1-n)(mn-ls^2)}{(1-ls)(s-m)(s-n)(mn-ls)} \times \left\{ \frac{[lp^2:p^2]_n [nq^2:q^2]_n [mP^2:P^2]_n \left[\frac{ls^2}{mn} Q^2:Q^2 \right]_n}{\left[\frac{sqPQ}{p} : \frac{qPQ}{p} \right]_n \left[\frac{lspPQ}{n} : \frac{pPQ}{q} \right]_n \left[\frac{lsPQ}{m} : \frac{pqQ}{P} \right]_n \left[\frac{mnPQ}{s} : \frac{pqQ}{Q} \right]_n} \right\} \\ \times \left\{ \frac{(s-1)(ls-n)(ls-m)(mn-s)s}{(1-ls)(s-n)(s-m)(mn-ls)} \times \frac{\left[\frac{qPQ}{sq} : \frac{pPQ}{q} \right]_r \left[\frac{mpqQ}{lsP} : \frac{qPQ}{p} \right]_{r+1} \left[\frac{npPQ}{lsq} : \frac{pqQ}{P} \right]_r \left[\frac{dpqP}{bcQ} : \frac{pqP}{Q} \right]_r}{\left[\frac{p^2}{l} : p^2 \right]_r \left[\frac{q^2}{n} : q^2 \right]_r \left[\frac{P^2}{m} : P^2 \right]_r \left[\frac{mnQ^2}{ls^2} : Q^2 \right]_r} \right\}. \quad (2.7)$$

and

$$\gamma_n = \frac{z^n}{1-z}. \quad (2.8)$$

Putting these value of β_n, γ_n in Asymmetric Bilateral Bailey Transform we set the following transformation

$${}_8\Psi_8 \left[\begin{matrix} l, m, n, \frac{ls^2}{mn}, -, -, lspqPQ, -, \frac{sPQ}{npq}, \frac{mpP}{sqQ}, \frac{lsPQ}{mnqp}; p^2, q^2, P^2, Q^2 \\ -, -, -, \frac{sqPQ}{p}, \frac{lspPQ}{nq}, \frac{mnpqP}{sQ}, ls, \frac{lspqQ}{mP}, \frac{s}{n}, \frac{m}{s}, \frac{ls}{mn}, \\ \frac{qPQ}{p}, \frac{pPQ}{q}, \frac{pqQ}{P}, pqPQ, pq \frac{Q}{P}, \frac{PQ}{pq}; zq^2 \end{matrix} \right] \\ = M_5 {}_5\Phi_4 \left[\begin{matrix} p^2, -, -, \frac{qPQ}{sp}, \frac{npPQ}{lsq}, \frac{mnpQ}{lsp}, \frac{spqP}{mnQ}; p^2, q^2, P^2, Q^2, \frac{pPQ}{p}, \frac{pqQ}{p}, \frac{pqQ}{P}; z \\ \frac{sqPQ}{bp}, \frac{lspPQ}{nq}, \frac{lspQ}{bp}, \frac{mnpqP}{sQ}, -, -, -, -, - \end{matrix} \right] \\ - L_5 {}_5\Phi_4 \left[\begin{matrix} p^2, -, -, \frac{qPQ}{sp}, \frac{npPQ}{lsq}, \frac{mpqQ}{lsp}, \frac{spqP}{mnQ}; p^2, q^2, P^2, Q^2, \frac{qPQ}{p}, \frac{pPQ}{q}, \frac{pqQ}{P}; z \\ \frac{p^2}{l}, \frac{q^2}{n}, \frac{P^2}{m}, \frac{mnQ^2}{ls^2}, -, -, -, -, - \end{matrix} \right]. \quad (2.9)$$

where

$$M_5 = \frac{s(1-l)(1-m)(1-n)(mn-ls^2)(1-z)}{(1-ls)(s-m)(s-n)(mn-ls)}, \\ L_5 = \frac{(1-z)(s-1)(ls-n)(ls-m)(mn-s)}{(1-ls)(s-n)(s-m)(mn-ls)}.$$

3. Special cases

3.1 On taking $s = 1$ in (2.3) we get

$${}_6\Phi_5 \left[\begin{matrix} lpq, l, m, n, \frac{l}{mn}, \frac{mp}{q}; pq, p, q, \frac{p}{q}; zq \end{matrix} \right] = (1-z)$$

$${}_5\Phi_4 \left[\begin{matrix} lp, mp, p, np, \frac{lq}{mn}, \frac{mp}{q}; pq, p, q, \frac{p}{q}; zq \\ \frac{l}{n}p, mnq, q, \frac{lq}{m} \end{matrix} \right] \quad (3.1)$$

3.2 On taking $p = q$ in (3.1) we get

$${}_6\Phi_5 \left[\begin{matrix} l, q\sqrt{l}, -q\sqrt{l}, m, n, \frac{l}{mn}; q \\ \sqrt{l}, -\sqrt{l}, \frac{lq}{n}, mnq, \frac{lq}{m} \end{matrix} ; zq \right] = (1-z) {}_4\Phi_3 \left[\begin{matrix} lq, mq, nq, \frac{lq}{mn}; q \\ \frac{lq}{n}, mnq, \frac{lq}{m} \end{matrix} ; z \right] \quad (3.2)$$

3.3 On taking $s = 1$ in (2.6) we get

$$\begin{aligned} & {}_7\Phi_6 \left[\begin{matrix} \beta, n, y, \beta ny, \beta nyp, \frac{\beta yP}{q}, \frac{\beta p}{q}; p, q, P, \frac{pP}{q}, Pp, \frac{p}{q}, \frac{p}{q}; zq \\ \beta np, -nyP, \beta y \frac{pP}{q}, \beta ny, y, \beta \end{matrix} \right] \\ &= (1-z) {}_5\Phi_4 \left[\begin{matrix} \beta p, p, nq, yP, \beta y \frac{pP}{q}, p, q, P, \frac{pP}{q}; z \\ \beta np, q, nyP, \beta y \end{matrix} \right] \end{aligned} \quad (3.3)$$

3.4 On taking $p = q = P$ in (3.3) we get

$${}_6\Phi_5 \left[\begin{matrix} \beta ny, q\sqrt{\beta ny}, -q\sqrt{\beta ny}, \beta, n, y; q \\ \sqrt{\beta ny}, -\sqrt{\beta ny}, \beta nq, nyq, \beta yq \end{matrix} ; zq \right] = (1-z) {}_4\Phi_3 \left[\begin{matrix} \beta p, cq, \beta yq; qz \\ \beta nq, nyq, \beta y \end{matrix} \right] \quad (3.4)$$

3.5 On taking $s = 1$ in (2.9) we get

$$\begin{aligned} & {}_8\Phi_7 \left[\begin{matrix} l, n, m, \frac{l}{mn}, lpqPQ, \frac{PQ}{npq}, \frac{mpP}{qQ}, \frac{lpQ}{mnqP}; p^2, q^2, P^2, Q^2, \frac{qPQ}{p}, \frac{pPQ}{q}, \frac{pqQ}{P}, pqPQ, \frac{pqP}{Q}, \frac{PQ}{pq}, \frac{pP}{qQ}, \frac{qQ}{pP}; z \\ - , - , - , \frac{lpPQ}{nq}, \frac{lpqQ}{mP}, a, \frac{mnpqP}{Q}, \frac{1}{n}, \frac{1}{m}, \frac{l}{mn} \end{matrix} \right] \\ &= (1-z) {}_5\Phi_4 \left[\begin{matrix} lp^2, p^2, cq^2, bP^2, \frac{lQ^2}{mn}; p^2, q^2, Q^2, \frac{qPQ}{p}, \frac{pPQ}{q}, \frac{pqQ}{P}, \frac{pqP}{Q}; z \\ \frac{qPQ}{mp}, \frac{lpPQ}{nq}, \frac{lpqQ}{mP}, \frac{pqP}{Q} \end{matrix} \right] \end{aligned} \quad (3.5)$$

3.6 On taking $p = q = P = Q$ in (3.5) we get

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} l, q^2\sqrt{l}, -q^2\sqrt{l}, m, n, \frac{l}{mn}; q^2 \\ \sqrt{l}, -\sqrt{l}, \frac{lq^2}{n}, mnq^2, \frac{lq^2}{m} \end{matrix} ; zq^2 \right] = (1-z) \\ & {}_4\Phi_3 \left[\begin{matrix} lq^2, mq^2, nq^2, \frac{lq^2}{mn}; q^2 \\ \frac{lq^2}{n}, mnq^2, \frac{lq^2}{m} \end{matrix} ; z \right] \end{aligned} \quad (3.6)$$

4. Conclusion

Using bilateral Bailey transforms and classical summation theorems, we have derived several new transformation identities for bilateral basic

hypergeometric series of the form ${}_r\psi_r$. These results generalize known q-series identities and provide frameworks for deriving many further bilateral summation theorems.

Acknowledgment. The author is thankful to the referee for his valuable suggestions.

References

- [1] G. E. Andrews and S. O. Warnaar, "The Bailey transform and false theta functions," *Ramanujan J.*, vol. 14, no. 1, pp. 173–188 (2007), doi: 10.1007/s11139-006-9006-4.
- [2] R. Y. Denis, S. N. Singh, and S. P. Singh, "On certain summations and transformations involving basic hypergeometric functions," *Bull. Calcutta Math. Soc.*, vol. 98, no. 6, pp. 567–578 (2007).
- [3] R. Y. Denis, S. N. Singh, and S. P. Singh, "On extended Bailey's transform," *South East Asian J. Math. Math. Sci.*, vol. 8, no. 1, pp. 59–66 (2009).
- [4] R. Y. Denis, S. N. Singh, and S. P. Singh, "On bilateral Bailey transform," *South East Asian J. Math. Math. Sci.*, vol. 9, no. 2, pp. 59–67 (2011).
- [5] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed. Cambridge, U.K.: Cambridge University Press (2004).
- [6] C. M. Joshi and Y. Vyas, "Extensions of Bailey's transform and applications," *Int. J. Math. Math. Sci.*, vol. 2005, pp. 1909–1923 (2005), doi: 10.1155/IJMMS.2005.1909.
- [7] S. N. Singh, S. P. Singh, and V. Yadav, "On Bailey's transform and expansion of basic hypergeometric functions—II," *South East Asian J. Math. Math. Sci.*, vol. 11, no. 2, pp. 37–46 (2015).
- [8] S. P. Singh, S. N. Singh, and V. Yadav, "A note on symmetric bilateral Bailey transform," *South East Asian J. Math. Math. Sci.*, vol. 15, no. 1, pp. 89–96 (2019).
- [9] S. P. Singh and V. Yadav, "On certain transformation formulas involving basic hypergeometric series," *Proc. Natl. Acad. Sci. India Sect. A Phys. Sci.*, vol. 90, no. 4, pp. 623–627 (2020), doi: 10.1007/s40010-019-00609-4.
- [10] H. M. Srivastava, S. N. Singh, S. P. Singh, and V. Yadav, "Certain derived WP-Bailey pairs and transformation formulas for q-hyper-

- geometric series," *Filomat*, vol. 31, no. 14, pp. 4619–4628 (2017), doi: 10.2298/FIL1714619S.
- [11] A. Verma and V. K. Jain, "Some summation formulae of basic hypergeometric series," *Indian J. Pure Appl. Math.*, vol. 11, no. 8, pp. 1021–1038 (1980).
- [12] A. Verma and V. K. Jain, "Certain transformations of basic hypergeometric series and their applications," *Pacific J. Math.*, vol. 101, no. 2, pp. 333–349 (1982).
- [13] J. Yadav, V. Yadav, and M. Mishra, "On some application of Bailey's transform in generalized basic hypergeometric transformations," *South East Asian J. Math. Math. Sci.*, vol. 17, no. 2, pp. 75–82 (2021).
- [14] B. V. Zala and J. C. Prajapati, "Certain properties of the Wright function," *J. Interdiscip. Math.*, vol. 29, no. 4, pp. 909–925 (2026), doi: 10.47974/JIM-2380.
- [15] H. Nagar and S. Mishra, "Application of Marichev-Saigo-Maeda fractional differential and integral operators on composition of two special functions," *J. Interdiscip. Math.*, vol. 27, no. 8, pp. 1869–1874 (2024).
- [16] S. Lather and H. Nagar, "A study of fractional kinetic equations associated with k-incomplete second Appell hypergeometric matrix functions," *J. Interdiscip. Math.*, vol. 29, no. 8, pp. 2421–2430 (2026), doi: 10.47974/JIM-2627.

Received January, 2026