Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 9, pp. 2673–2686
DOI : 10.47974/JIM-2659

Hyperchaotic analysis and complete and anti-synchronizations in identical hyperchaotic financial systems with active control strategies

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Abstract

The dynamical properties of a recently developed four-dimensional hyperchaotic financial system are examined in this work. The model's complexity is investigated using Lyapunov exponents, Poincaré sections, and bifurcation analysis, all of which together validate the existence of persistently hyperchaotic behaviour over a wide range of parameter values. Nonlinear active control strategies are developed to accomplish both complete synchronization and anti-synchronization between two similar systems that are started from different initial states in order to govern the dynamics of the system. Lyapunov stability

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theory is used to rigorously establish the stability of the suggested controllers, ensuring that the synchronization error would converge globally exponentially. Numerical simulations show the control methods' resilience and effectiveness while also confirming the theoretical findings. The outcomes of this study contribute to a deeper understanding of stability enhancement in financial dynamical systems and may be extended to applications such as secure communication and other complex nonlinear systems.

Subject Classification (2020): *34D05, 34C23, 37D45, 93C10.*

Keywords: *Chaos, Hyperchaotic, Lyapunov exponents, Synchronization, Lyapunov stability.*

1. Introduction

Following the pioneering work of Carroll and Pecora [20] on the synchronization of chaotic systems [11, 12], extensive research has been devoted to developing effective synchronization schemes for both chaotic and hyperchaotic systems, including identical and non-identical configurations. Yassen [10] employed an active control strategy to investigate chaos synchronization between two different chaotic systems. Saaban et al. [1] examined both linear and nonlinear active control approaches for synchronizing a newly proposed chaotic model. In addition, [14] applied three different synchronization techniques to study the behaviour of identical [15, 16] as well as non-identical hyperchaotic systems, successfully achieving synchronization under various configurations. Al-Obeidi and Al-Azzawi [18], and Al-Azzawi and Al-Hayali [19] extensively studied synchronization and control of hyperchaotic systems.

The literature reports the presence of chaotic dynamics in several financial systems [6, 5], demonstrating that the interaction among multiple economic variables can produce highly complex and difficult-to-interpret behaviours. The extraordinary sensitivity of chaotic systems to initial conditions even a slight change in the initial state can result in radically divergent system trajectories is one of their basic features. Despite this sensitivity, global boundedness a feature that has been thoroughly studied across a variety of nonlinear dynamical models is frequently present in chaotic systems. The study of chaos and how to regulate it has garnered significant interest in a number of fields, including biology [9], engineering [4], finance [7, 8], and physics [13, 17]. Within this framework, synchronization in hyperchaotic systems is an especially interesting occurrence, in which two or more coupled systems adapt their dynamics to evolve in unison, even though each system is inherently unpredictable [3]. Agrawal, Srivastava, and Das [2] showed the effectiveness of active control strategies are in achieving synchronization in hyperchaotic systems. Although the initial conditions may vary, synchronization is usually studied under similar system dynamics and the mathematical structure is the same in identical systems. Non-identical dynamical systems, on the other hand, show comparable structure behaviour but have different initial states or parameter values. Examining synchronization in both situations offers important information on the resilience and flexibility needed for workable synchronization techniques.

This is how the rest of the paper is structured. In Section 2, the concept of chaotic synchronization is discussed together with the mathematical formulation of the suggested model. The control principles for the hyperchaotic financial system are developed in Section 3 in order to ensure total synchronization amongst systems that were initially started under various situations. In Section 4, numerical simulations and a thorough mathematical analysis are used to confirm the efficacy of the suggested control technique. The method for achieving anti-synchronization between systems with different initial states is expanded in Section 5. Section 6 uses simulation studies and analytical findings to further assess the control schemes' performance and stability. Section 7 concludes with a summary of the key conclusions and closing thoughts.

2. Model Description

The following describes a new 4-D hyperchaotic financial system [17]:

$$\begin{aligned} \dot{x} &= z^2 + (y - a')x + w \\ \dot{y} &= 1 - b'y - x^2 \\ \dot{z} &= -xy - c'z \\ \dot{w} &= -0.05xz^2 - d'w \end{aligned} \tag{1}$$

The system specifies the temporal evolution of four state variables: average profit margins (w), price index (z), investment demand (y), and interest rate (x). Differentiation with respect to time t is shown by the dot, and $a' \geq 0$ denotes amount saved, $b' \geq 0$ the cost per investment, $c' \geq 0$ is the elasticity, and $d' \geq 0$ positive parameter. System (1) has complex dynamic behaviour. The chaotic attractor corresponding to system (1) is shown in Figure 1.

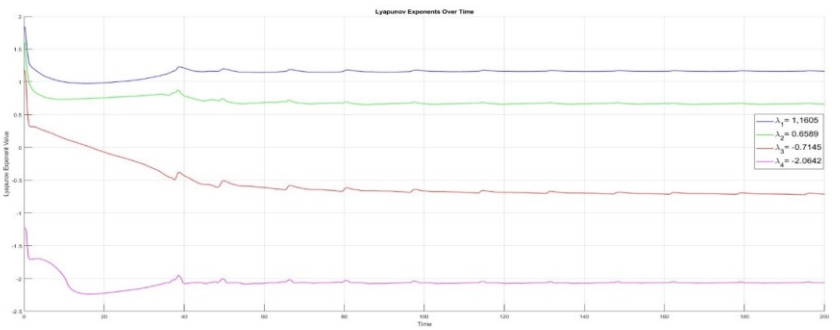


Figure 1
Lyapunov exponents Over Time

Figure 1 for $a' = 0.9$, $b' = 0.2$, $c' = 1.5$, and $d' = 0.17$, the system's trajectory forms a chaotic attractor with considered parameters in the system (1) has Lyapunov exponents are 1.1605, 0.6589, -0.7145, and -2.0642, according to our computation of the Lyapunov exponent for the system witnesses the two positive Lyapunov exponents, the system is classified as hyperchaotic.

Phase Portraits Analysis

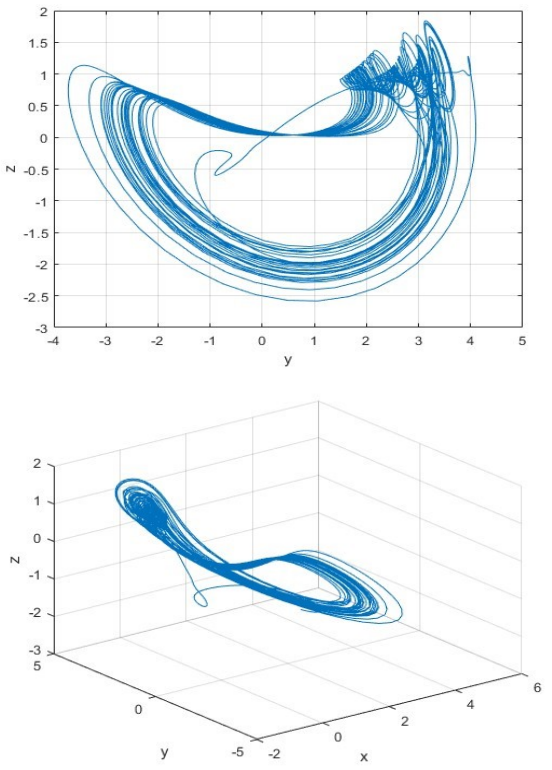


Figure 2
Phase portrait projections

Figure 2 illustrates the phase portrait projections of system (1) when $a' = 0.9, b' = 0.2, c' = 1.5, and\ d' = 0.17$ in the x-y-z space, and projected on the y-z plane,

Lyapunov exponent spectrum

In dynamical systems, the Lyapunov exponent spectrum describes how the distance between two initially close trajectories changes over time. Each exponent corresponds to the average rate of separation or convergence in a specific direction within the phase space. A positive value indicates that the trajectories move apart, signifying chaotic dynamics, while a negative value shows they approach each other, leading to a stable point or periodic motion. An exponent of zero typically reflects neutral stability, as seen in certain conservative systems. By examining all of the Lyapunov exponents together, one may assess the system's overall stability and understand the traits of its attractors.

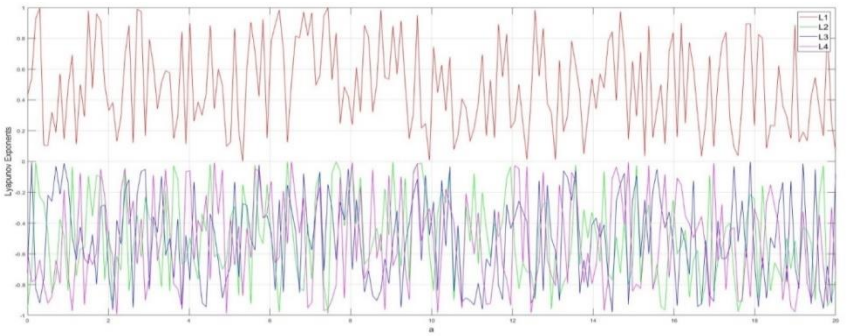


Figure 3
Lyapunov exponents

Figure 3 shows the Lyapunov exponent spectrum of system (1), where L1 represents the largest Lyapunov exponent in all cases. (a) represented versus parameter a'.

Bifurcation Analysis

Bifurcation analysis as parameter a increases from 0.5 to 2, the system transitions from stable behavior to hyperchaotic oscillations in the interest rate x. For values of parameter b' ranging from 0.1 to 0.5, the system exhibits very large fluctuations in the steady state of x reaching magnitudes on the order 10^{12} suggesting instability in the system. As parameter c' increases from 1 to 2, the system shows hyperchaotic behavior initially, but the fluctuations in x reduce as c' increases. Parameter d', ranging from 0.1 to 0.3, leads to hyperchaotic behaviour for the most of its range, with multiple steady-state value of x. The systems remains hyperchaotic.

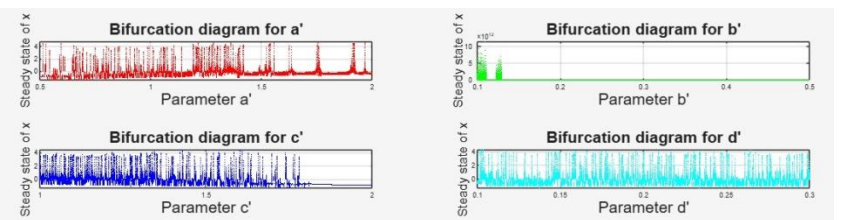


Figure 4
Bifurcation Diagram for a', Bifurcation Diagram for b', Bifurcation Diagram for c', Bifurcation Diagram for d'

The bifurcation behaviour corresponding to different parameters is illustrated in Figure 4 parameter are a', Chaotic Behaviour with Increasing Parameter a', Bifurcation for Parameter b', Extreme Oscillations and Instability with Parameter b'. Bifurcation for Parameter c', Transition to Stability with

Increasing Parameter c', Bifurcation for Parameter d', Chaotic Dynamics with Varying Parameter d'.

Equilibrium and Stability: Equations can be solved to determine system (1)'s equilibrium:

$$z^2 + (y - a')x + w = 0$$
$$1 - x^2 - b'y = 0$$
$$-xy - c'z = 0$$
$$-0.05xz^2 - d'w = 0$$

system (1) has two non-trivial equilibrium points E_1= (0.9, 0.7, -0.4, -0.05), and E_2= (2.6, 5, -9.06, -3.53). Therefore, E_1 and E_2 are unstable equilibrium points.

Poincare Section

Hyperchaotic systems, known for their high sensitivity to initial conditions and complex evolution, play a vital role in modelling nonlinear dynamics across science and engineering. Financial systems with such behaviour can more realistically reflect the irregular nature of economic variables. This work examines a recently introduced hyperchaotic financial model involving four interacting factors: profit margin, price index, interest rate, and investment demand. To understand its internal dynamics, we apply the Poincare section method, which simplifies continuous trajectories into discrete intersections with a chosen surface. The Poincare section of the system (1) on taking $x_2 = 0$.

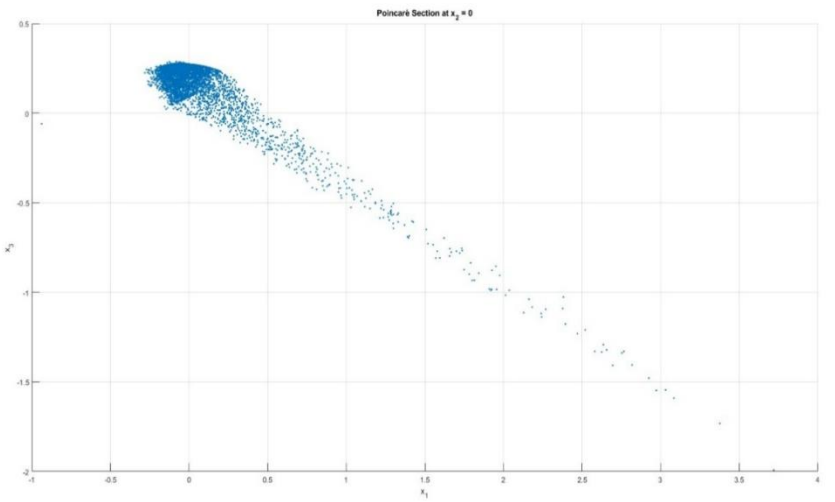


Figure 5
Poincare section at x_2=0

The Poincare section of system (1) is shown in Figure 5 for the section when x_2=0

Model Description

In a 4-D hyperchaotic financial system, the slave/response system has the subscript (3) while the master/drive system has the subscript (2). Corresponding ordinary differential equations with four positive real parameters have been given.

The non-linear differential equations are given by

$$\left\{\begin{array}{c} \dot{x_1} = {x_3}^2 + x_1(x_2 - a') + x_4 \\ \dot{x_2} = 1 - {x_1}^2 - b'x_2 \\ \dot{x_3} = -x_1x_2 - c'x_3 \\ \dot{x_4} = -0.05x_1{x_3}^2 - d'x_4 \end{array}\right\} \quad (2)$$

Where $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = w$ the average profit margins, price index, investment demand, and interest rate and $a', b', c' \; and \; d'$ are positive parameters.

The slave system is described as:

$$\left\{\begin{array}{c} \dot{y_1} = {y_3}^2 + y_1(y_2 - a') + y_4 + u_1 \\ \dot{y_2} = 1 - {y_1}^2 + u_2 - b'y_2 \\ \dot{y_3} = -y_1y_2 - c'y_3 + u_3 \\ \dot{y_4} = -0.05y_1{y_3}^2 - d'y_4 + u_4 \end{array}\right\} \quad (3)$$

Where $y_1 = x, y_2 = y, y_3 = z$ and $y_4 = w$ the average profit margins, price index, investment demand, and interest rate and $a', b', c' \; and \; d'$ are positive parameters and $u_1, u_2, u_3 \; and \; u_4$ are control features to be designed.

3. The active control method is used to fully synchronize two identical new 4-D hyperchaotic financial systems.

To provide synchronization between systems (2) and (3). The errors in synchronization $e_i, i = 1..4$ are defined by $e_i = y_i - x_i$. From (2) and (3),

The following is how the error dynamics are obtained:

$$\begin{array}{c} \dot{e_1} = {y_3}^2 + (y_2 - a')y_1 + y_4 + u_1 - {x_3}^2 - (x_2 - a')x_1 - x_4 \\ \dot{e_2} = 1 - b'y_2 - {y_1}^2 + u_2 - 1 + b'x_2 + {x_1}^2 \\ \dot{e_3} = -y_1y_2 - c'y_3 + u_3 + x_2x_1 + c'x_3 \\ \dot{e_4} = -0.05y_1{y_3}^2 - d'y_4 + u_4 + 0.05x_1{x_3}^2 + d'x_4 \end{array} \quad (4)$$

The controllers are given as follows:

$$\begin{array}{c} u_1 = -{y_3}^2 - y_2y_1 + {x_3}^2 + x_2x_1 - e_4 - k_1e_1 \\ u_2 = {y_1}^2 - {x_1}^2 - k_2e_2 \\ u_3 = y_1y_2 - x_1x_2 - k_3e_3 \\ u_4 = 0.05y_1{y_3}^2 - 0.05x_1{x_3}^2 - k_4e_4 \end{array} \quad (5)$$

Where $k_1, k_2, k_3 \; and \; k_4$ are positive gains constants Substituting (5) into (4), we get

$$\left.\begin{aligned}\dot{e}_1 &= -(a' + k_1)e_1\\ \dot{e}_2 &= -(b' + k_2)e_2\\ \dot{e}_3 &= -(c' + k_3)e_3\\ \dot{e}_4 &= -(d' + k_4)e_4\end{aligned}\right\} \tag{6}$$

Let us examine the Lyapunov function, which is defined as follows:

$$V(e) = \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2 + {e_4}^2). \tag{7}$$

Differentiating (12) along the trajectories (11) we get

$$\dot{V}(e) = -k_1{e_1}^2 - k_2{e_2}^2 - k_3{e_3}^2 - k_4{e_4}^2. \tag{8}$$

The derivative of the Lyapunov function is negative definite. The error dynamics are consequently regarded as asymptotically stable and eventually converge exponentially to zero according to "Lyapunov stability theory" [12]. Consequently, there is complete synchronization between the slave system (3) and the master system (2). Here, the same system (1) uses a nonlinear active control law to achieve global and exponential synchronization.

4. Numerical Simulations

For simulations, our New 4-D Hyperchaotic Financial System's the parameters are chosen as $a' = 0.9, b' = 0.2, c' = 1.5\ and\ d' = 0.17$. The slave and master system's initial conditions are selected as $x(0) = (3, 1, -2, -3)$ $and\ y(0) = (5, 3, -6, -3)$, respectively. Equation (5) defines the control functions that are used to produce complete synchronization, while equations (2) and (3) characterize the systems that are simulated. Figure 6: (a) displays the $x_1\ and\ y_1$, (b) displays the $x_2\ and\ y_2$, (c) displays the $x_3\ and\ y_3$, and (d) displays the $x_4\ and\ y_4$ displays the simulation findings for the two identical hyperchaotic financial systems that are displays in the absence of controller.

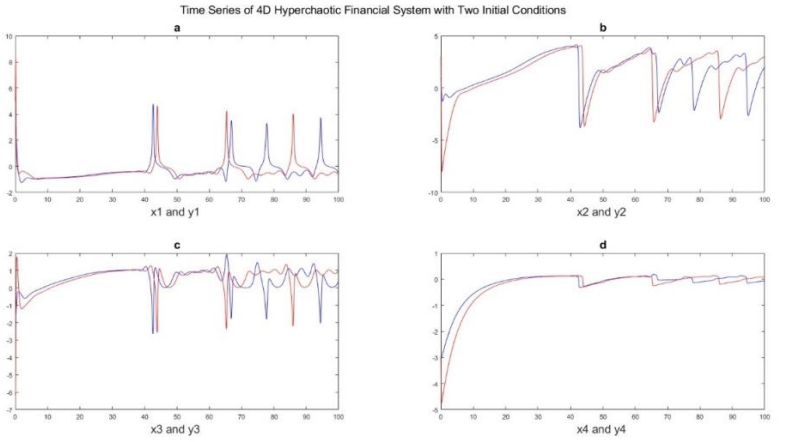


Figure 6
Time Series of 4D Hyperchaotic Financial System with Two Initial Conditions

The time series behaviour of master and slave systems without control is shown in Figure 6 of states (a) displays the x_1 *and* y_1, (b) displays the x_2 *and* y_2, (c) displays the x_3 *and* y_3, and (d) displays the x_4 *and* y_4 between master (2) and slave system (3) in colour red for x_i and in colour blue for y_i.

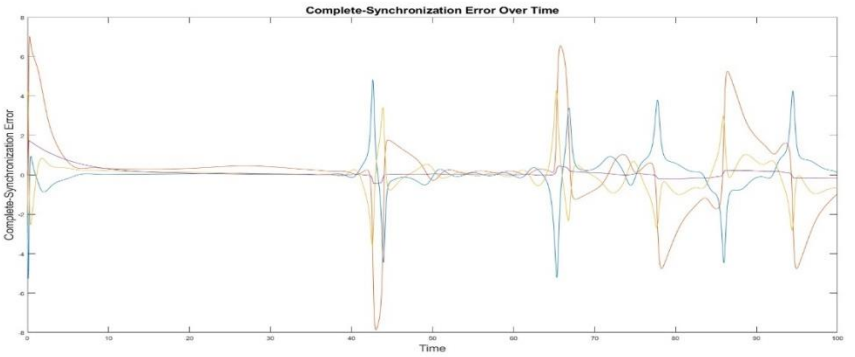


Figure 7
Complete-Synchronzation Error Over Time

Figure 7 shows the behaviour of system (4) during complete synchronization when $u_1 = u_2 = u_3 = u_4 = 0$

For the initial condition in the error vector set as (e_1, e_2, e_3, e_4) = (2, 2, -4, 0), and selecting $k_1 = k_2 = k_3 = k_4 = 1$ Figure 8 shows the time series analysis of the synchronization errors after the active control has complete synchronized the system. Here we observe that t=4 the system gets synchronized

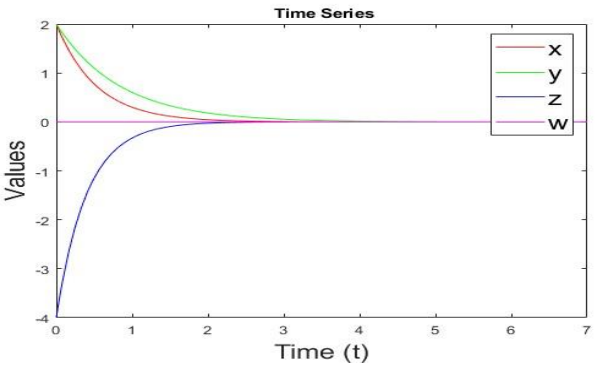


Figure 8
Time Series

Analysis of the errors in time series e_1, e_2, e_3 *and* e_4 versus time t when the systems (4) and (5) are completely synchronized shown in Figure 8.

5. Using the active control strategy, two identical new 4-D hyperchaotic financial systems are ant-synchronized.

To discuss anti-synchronization, we first define the synchronization errors.. We initially define synchronization errors.

$$e_i, i = 1, 2, 3, 4 \; by \; e_i = y_i + x_i.$$

From (2) and (3), the error dynamics is described as:

$$\begin{aligned} \dot{e_1} &= {y_3}^2 + (y_2 - a')y_1 + y_4 + u_1 + {x_3}^2 + (x_2 - a')x_1 + x_4 \\ \dot{e_2} &= 1 - b'y_2 - {y_1}^2 + u_2 + 1 - b'x_2 - {x_1}^2 \\ \dot{e_3} &= -y_1y_2 - c'y_3 + u_3 - x_2x_1 - c'x_3 \\ \dot{e_4} &= -0.05y_1{y_3}^2 - d'y_4 + u_4 - 0.05x_1{x_3}^2 - d'x_4 \end{aligned} \tag{9}$$

The nonlinear controllers are selected as:

$$\begin{aligned} u_1 &= -{y_3}^2 - y_2y_1 - {x_3}^2 - x_2x_1 - e_4 - k_1e_1 \\ u_2 &= {y_1}^2 + {x_1}^2 - k_2e_2 - 2 \\ u_3 &= y_1y_2 + x_1x_2 - k_3e_3 \\ u_4 &= 0.05y_1{y_3}^2 + 0.05x_1{x_3}^2 - k_4e_4 \end{aligned} \tag{10}$$

Where $k_1, k_2, k_3 \; and \; k_4$ are positive gains constants. Using (10) and (9), we get

$$\begin{aligned} \dot{e}_1 &= -(a' + k_1)e_1 \\ \dot{e}_2 &= -(b' + k_2)e_2 \\ \dot{e}_3 &= -(c' + k_3)e_3 \\ \dot{e}_4 &= -(d' + k_4)e_4 \end{aligned} \tag{11}$$

Let's define the Lyapunov function as:

$$V(e) = \frac{1}{2}({e_1}^2 + {e_2}^2 + {e_3}^2 + {e_4}^2). \tag{12}$$

Differentiating (12) along the trajectories (11) we get

$$\dot{V}(e) = -k_1{e_1}^2 - k_2{e_2}^2 - k_3{e_3}^2 - k_4{e_4}^2. \tag{13}$$

It has a clear negative function. Thus, “Lyapunov stability theory” [12] ensures that the error dynamics are asymptotically stable, converging exponentially to zero over the evolution of times Thus, anti-synchronization between the slave system (3) and the master system (2) is achieved. Consequently, to the further established that the identical system (1) achieves exponential and global synchronization through the utilization of a nonlinear active control law.

6. Numerical Simulations

For simulations, the New 4-D Hyperchaotic Financial System's parameters are chosen as $a' = 0.9, b' = 0.2, c' = 1.5 \; and \; d' = 0.17$. The initial condition for the master and slave system are taken as $x(0) = (3, 1, -2, -3) \; and \; y(0) = (5, 3, -6, -3)$, respectively. Equation (10) defines the control functions to be used to produce anti-synchronization, while equations (2) and (3) characterize the systems that have been simulated. Figure 6 : (a) displays the $x_1 \; and \; y_1$, (b)

displays the $x_2\ and\ y_2$, (c) displays the $x_3\ and\ y_3$, and (d) displays the $x_4\ and\ y_4$ shows the simulation findings for the two identical hyperchaotic financial systems that are without active control.

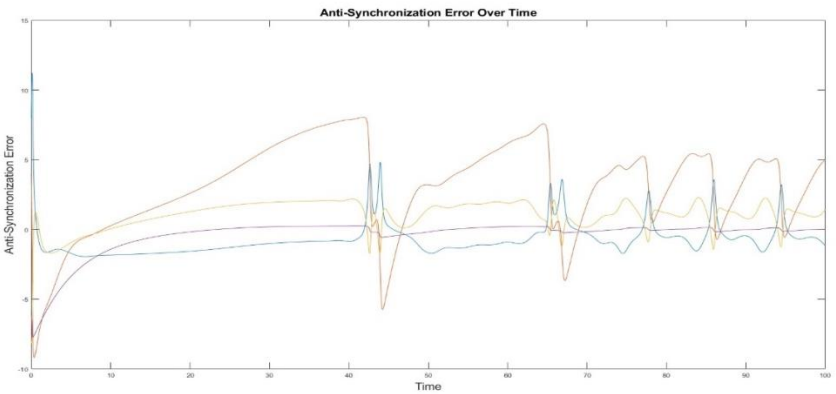


Figure 9
Anti-Synchronization Error Over Time

Figure 9 shows the behaviour of system (6) during anti-synchronization when $u_1 = u_2 = u_3 = u_4 = 0$

Subsequently, with initial condition for the error vector set as (e_1, e_2, e_3, e_4) = (8, 4, -8, -3), and selecting $k_1 = k_2 = k_3 = k_4 = 1$ Figure 10 shows the time series analysis of the synchronization errors after the active control has anti synchronized the system. Hence, it is observed that the system becomes synchronized at approximately t = 5.

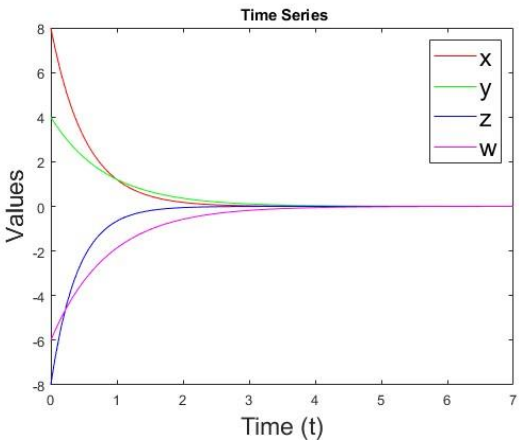


Figure 10
Time Series

Analysis of the errors in time series e_1, e_2, e_3 *and* e_4 versus time t when the systems (4) and (5) are anti-synchronized shown in Figure 10.

7. Conclusion

Using the active control technique and "Lyapunov stability theory" the synchronization dynamics of identical hyperchaotic financial systems. Strategies have been employed to achieving synchronization between two such financial systems with diverse initial conditions, substantiating the reliability of the control law through mathematical proofs. Furthermore, computer research has been conducted to confirm the efficiency and flexibility of the proposed control systems. Thus, the methods presented herein offer a means to achieve synchronous chaotic systems, it may provide a significant contribution to this field given the inherent unpredictability and risk associated with chaotic behaviour in financial systems. By achieving synchronization, we mitigate risk, thereby enhances the predictability within financial systems. In summary, this study sheds light on how hyperchaotic financial systems manage risk.

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Received December, 2025