

Fractional order disturbance observer based adaptive sliding mode hybrid synchronization of Tuberculosis system

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Abstract

This paper explores chaotic behaviour in a fractional-order Tuberculosis model, utilizing Caputo derivatives to capture essential memory and hereditary effects in disease transmission. To address real-world uncertainties and external perturbations, the paper uses a robust fractional-order adaptive sliding mode control strategy within a master-slave synchronization framework. A fractional-order disturbance observer is integrated to estimate unknown bounded disturbances, while hybrid projective synchronization allows for flexible scaling between systems. Validated through MATLAB simulations, the results prove that this fractional approach outperforms integer-order models in accuracy and stability. This framework offers significant potential for advancing disease forecasting, epidemic control, and secure communication systems.

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1. Introduction

Chaos refers to the qualitative study of complex, aperiodic behavior in deterministic non-linear systems. These systems are defined by a high sensitivity to initial conditions generally referred to as the "Butterfly Effect" by Lorenz [1]. Beyond theory, chaos has applications in different disciplines [2, 3, 4]. In biomathematics, chaotic oscillations naturally occur in ecological and epidemic systems due to non-linear interactions. Mathematical modeling, specifically the SEIR model, is important for understanding diseases like TB which is a severe health threat in resource-limited areas caused by *Mycobacterium tuberculosis* [5]. While traditional integer-order models provide insights, they often ignore the chaos and uncertainties (e.g., unreported cases or environmental fluctuations) found in real-world environments. Consequently, fractional-order differential equations have gained recognition for their ability to capture the "memory effects" inherent in biological systems [6, 7]. Chaos synchronization, matching the trajectories of multiple systems, is essential for predicting and stabilizing the chaotic dynamics [8]. Various techniques exist, such as the state observer [9], sliding mode [10], etc. Fractional-order disturbance observers (FODO) are used to estimate unknown perturbations in real time. For TB modelling, combining FODO with adaptive sliding mode control ensures robust synchronization and finite-time convergence without needing prior knowledge of disturbance bounds. Motivated by these observations, this paper focus on developing a FODO based adaptive sliding mode control scheme for the synchronization of a chaotic fractional-order Tuberculosis system. Section 2 include description of the TB model, section 3 develops the FODO design, section 4 include adaptive sliding mode synchronization, section 5 and 6 are numerical simulations and conclusion respectively.

2. Description of the Tuberculosis model

We consider a fractional-order Tuberculosis model to describe the non-linear transmission dynamics of Tuberculosis. The following set of fractional order differential equations modified from [11] with same assumptions forms the foundation of this paper:

$$\begin{aligned} D^q z_1(t) &= a - az_1 - bz_1z_3 \\ D^q z_2(t) &= bz_1z_3 - az_2 - cz_2 \\ D^q z_3(t) &= cz_2 - az_3 - dz_3 \\ D^q z_4(t) &= dz_3 - az_4 \end{aligned} \tag{1}$$

Where, z_1, z_2, z_3 and z_4 refers to susceptible, exposed, infected and recovered classes respectively. a, b, c and d are birth-death rate, contact rate, disease rate and recovery rate respectively and all of them are non-negative. Also, $0 < q \leq 1$ is the order of derivative. Figure 1 represents 3-dimensional phase portraits of (1) showing the trajectories of the state variables in system. The fractional order taken is $q = 0.97$ which is a key parameter that governs the system's dynamics and its memory characteristics. Assumed initial conditions are $[1, 0.05, 0.02, 0.001]$ and the parameters are $a = 0.06, b = 15, c = 0.96, d =$

6.5. In all the trajectories of Figure 1, it exhibits a clear spiralling pattern that converge to a stable equilibrium point indicating the presence of strange attractor.

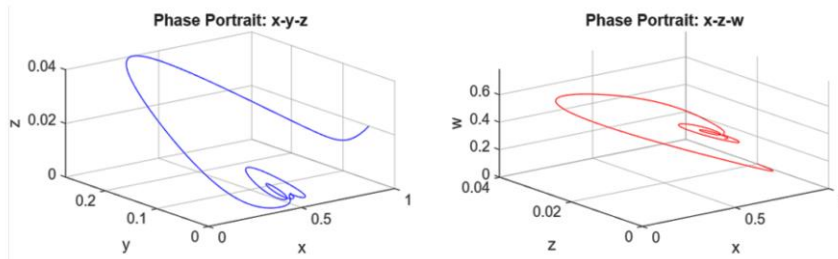


Figure 1
Phase Portraits of TB model in x-y-z and x-z-w planes respectively

Now, the Figure 2 provides a time-series plot for lyapunov exponents of (1). The lyapunov exponents are a critical measure for characterizing the stability and chaotic behaviour of a system. Assumed parameter values are: $a = 0.0012, b = 0.2568, c = 0.148, d = 0.104$, fractional order as $q = 0.92$ and initial conditions as $[1, 0.7, 0.3, 0.1]$. After that, according to Figure 2, the values for the lyapunov exponents are: $\lambda_1 = -0.0233, \lambda_2 = 0.4398, \lambda_3 = -0.9654, \lambda_4 = -6.5264$. The presence of a single positive lyapunov exponent, that is, $\lambda_2 = 0.4398$ is a key indicator of chaos. The overall trend of the exponents reaching a stable value over time indicates that the systems chaotic properties are persistent and well-defined.

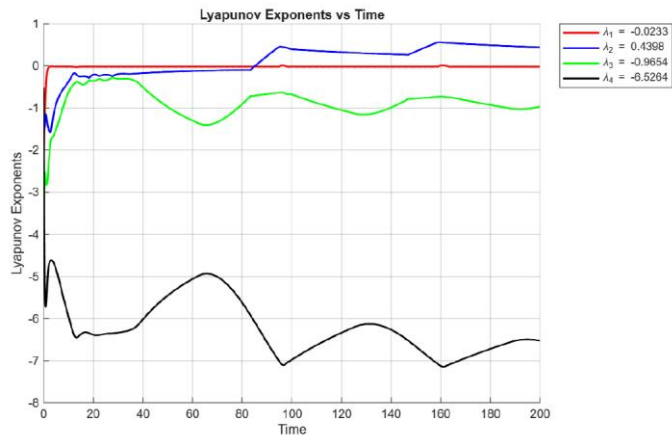


Figure 2
Lyapunov exponents VS time graph of the TB model

To study synchronization and control of the fractional Tuberculosis system (1), the corresponding slave system of (1) is as follows:

$$\begin{aligned}
D^q w_1(t) &= a - aw_1 - bw_1w_3 + d_1(t) + u_1(t) \\
D^q w_2(t) &= bw_1w_3 - aw_2 - cw_2 + d_2(t) + u_2(t) \\
D^q w_3(t) &= cw_2 - aw_3 - dw_3 + d_3(t) + u_3(t) \\
D^q w_4(t) &= dw_3 - aw_4 + d_4(t) + u_4(t)
\end{aligned} \tag{2}$$

Where, w_1, w_2, w_3 and w_4 refers to susceptible, exposed, infected and recovered classes respectively. a, b, c and d are birth-death rate, contact rate, disease rate and recovery rate respectively and all of them are non-negative. $d_1(t), d_2(t), d_3(t)$ and $d_4(t)$ are unknown bounded disturbances and $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ are the control inputs that are to be designed later. The goal is to design appropriate control laws $u_i(t)$ so that the slave system follows the master system properly, that is, $\lim_{t \rightarrow \infty} \|w_i(t) - z_i(t)\| = 0$ where $i = 1, 2, 3, 4$.

3. Fractional Order Disturbance Observer Design

In this section, a non-linear FODO is developed to estimate the unknown disturbances acting on the fractional-order Tuberculosis model (1). Let us define an auxillary state variable $\psi_i(t)$ for unknown disturbance since these disturbances cannot be measured directly [6]:

$$\psi_i(t) = d_i(t) - \lambda_i w_i(t), i = 1, 2, 3, 4 \tag{3}$$

where, $\lambda_i > 0$ that are to be determined.

Taking the fractional derivative on both sides of (3), we get:

$$D^q \psi_i(t) = D^q d_i(t) - \lambda_i D^q w_i(t), i = 1, 2, 3, 4 \tag{4}$$

Substituting (2) and using (3) in (4), we get:

$$\begin{aligned}
D^q \psi_1(t) &= D^q d_1(t) - \lambda_1(a - aw_1 - bw_1w_3 + \psi_1 + \lambda_1 u_1) \\
D^q \psi_2(t) &= D^q d_2(t) - \lambda_2(bw_1w_3 - aw_2 - cw_2 + \psi_2 + \lambda_2 u_2) \\
D^q \psi_3(t) &= D^q d_3(t) - \lambda_3(cw_2 - aw_3 - dw_3 + \psi_3 + \lambda_3 u_3) \\
D^q \psi_4(t) &= D^q d_4(t) - \lambda_4(dw_3 - aw_4 + \psi_4 + \lambda_4 u_4)
\end{aligned} \tag{5}$$

Now, the objective is to estimate $\psi_i(t), i = 1, 2, 3, 4$ using an observer equation of the same fractional order and thereby obtaining an estimate $\hat{\psi}_i(t), i = 1, 2, 3, 4$ from which $\hat{d}_i(t), i = 1, 2, 3, 4$ can be reconstructed as:

$$\hat{d}_i(t) = \hat{\psi}_i(t) + \lambda_i w_i(t), i = 1, 2, 3, 4 \tag{6}$$

So, estimated value of $\psi_i(t), i = 1, 2, 3, 4$ is:

$$\begin{aligned}
D^q \hat{\psi}_1(t) &= -\lambda_1(a - aw_1 - bw_1w_3 + \lambda_1 w_1) - \lambda_1 \hat{\psi}_1(t) - \lambda_1 u_1 \\
D^q \hat{\psi}_2(t) &= -\lambda_2(bw_1w_3 - aw_2 - cw_2 + \lambda_2 w_2) - \lambda_2 \hat{\psi}_2(t) - \lambda_2 u_2 \\
D^q \hat{\psi}_3(t) &= -\lambda_3(cw_2 - aw_3 - dw_3 + \lambda_3 w_3) - \lambda_3 \hat{\psi}_3(t) - \lambda_3 u_3 \\
D^q \hat{\psi}_4(t) &= -\lambda_4(dw_3 - aw_4 + \lambda_4 w_4) - \lambda_4 \hat{\psi}_4(t) - \lambda_4 u_4
\end{aligned} \tag{7}$$

Define estimation of the errors:

$$\tilde{d}_i(t) = d_i(t) - \hat{d}_i(t), i = 1, 2, 3, 4 \tag{8}$$

$$\text{and, } \tilde{\psi}_i(t) = \psi_i(t) - \hat{\psi}_i(t), i = 1, 2, 3, 4 \tag{9}$$

Now, using equations (3), (6), (8) and (9), we get:

$$\tilde{\psi}_i(t) = \tilde{d}_i(t), i = 1, 2, 3, 4 \quad (10)$$

Now, the fractional derivative of (9):

$$D^q \tilde{\psi}_i(t) = D^q \psi_i(t) - D^q \tilde{\psi}_i(t), i = 1, 2, 3, 4 \quad (11)$$

Substituting equations (5), (7) and (9) in equation (11), we get:

$$D^q \tilde{\psi}_i(t) = D^q d_i(t) - \lambda_i \tilde{\psi}_i(t), i = 1, 2, 3, 4. \quad (12)$$

Now, based on the above calculations and observations, to analyse the stability of $\tilde{d}_i(t)$, we will consider the Lyapunov function as:

$$V_{d_i}(t) = \frac{1}{2} \tilde{d}_i^2(t) = \frac{1}{2} \tilde{\psi}_i^2(t) \quad i = 1, 2, 3, 4 \quad (13)$$

Taking derivative of (13) and applying Lemma 1 [6]:

$$D^q V_{d_i}(t) \leq \tilde{\psi}_i(t) D^q \tilde{\psi}_i(t) \quad (14)$$

Substituting equation (12) in (14) and using assumption [6]:

$$D^q V_{d_i}(t) \leq \tilde{\psi}_i(t) (D^q d_i(t) - \lambda_i \tilde{\psi}_i(t)) = -\lambda_i \tilde{\psi}_i^2(t) + \tilde{\psi}_i(t) \xi_i \quad (15)$$

Now, we will use the inequality

$$(\tilde{\psi}_i - \xi_i)^2 = \tilde{\psi}_i^2 + \xi_i^2 - 2\tilde{\psi}_i \xi_i \geq 0$$

So,

$$2\tilde{\psi}_i \xi_i \leq \tilde{\psi}_i^2 + \xi_i^2 \Rightarrow \tilde{\psi}_i \xi_i \leq \frac{1}{2} \tilde{\psi}_i^2 + \frac{1}{2} \xi_i^2 \quad (16)$$

Substituting (16) in (15), we get:

$$\begin{aligned} D^q V_{d_i}(t) &\leq -\lambda_i \tilde{\psi}_i^2(t) + \frac{1}{2} \tilde{\psi}_i^2(t) + \frac{1}{2} \xi_i^2 \\ &= -(\lambda_i - \frac{1}{2}) \tilde{\psi}_i^2(t) + \frac{1}{2} \xi_i^2 \\ &= -b_0 V_{d_i}(t) + b_1 \end{aligned} \quad (17)$$

where, $b_0 = 2\lambda_i - 1, b_1 = \frac{1}{2} \xi_i^2$

Using Lemma 2 [6], we get:-

$$|V_{d_i}(t)| \leq \frac{\xi_i^2}{2\lambda_i - 1} = \frac{\xi_i^2}{2\lambda_i - 0.5}$$

So, the non-linear FODO control gain λ_i should be chosen such that $\lambda_i > 0.5$ so that the estimated error is guaranteed to be bounded.

Also, since from (13) :- $V_{d_i}(t) = \frac{1}{2} \tilde{d}_i^2(t)$

Hence,

$$|\tilde{d}_i(t)| \leq \sqrt{\frac{\xi_i^2}{(\lambda_i - 0.5)}} \quad (18)$$

Therefore, we can see from equations (18), $\tilde{d}_i(t)$ is bounded above. Thus, for the external disturbances $d_i(t), i = 1, 2, 3, 4$, the disturbance estimation error

satisfies $|\tilde{d}_i(t)| \leq k_i, k > 0$ is unknown constant and is bounded. So, the proposed FODO accurately reconstructs unknown disturbances in the slave system of Tuberculosis (2) using only measurable state variables.

4. Synchronization of Tuberculosis model via Adaptive Sliding Mode Control Scheme

Now, we design an adaptive sliding mode synchronization controller using the disturbance estimates produced by the FODO to produce bounded hybrid projective synchronization between the master (1) and slave (2) systems of Tuberculosis system. Design the synchronization error for each compartment as:

$$e_i(t) = w_i(t) - \eta_i z_i(t), i = 1, 2, 3, 4 \quad (19)$$

where, $\eta_i, s, i = 1, 2, 3, 4$ are constant scaling parameters.

The fractional derivative of (19) is:

$$D^q e_i(t) = D^q w_i(t) - \eta_i D^q z_i(t), i = 1, 2, 3, 4 \quad (20)$$

Substituting (1) and (2) in equation (20) and using (19):

$$\begin{aligned} D^q e_1(t) &= (a - aw_1 - bw_1w_3 + d_1(t) + u_1(t)) - \eta_1(a - az_1 - bz_1z_3) \\ D^q e_2(t) &= (bw_1w_3 - aw_2 - cw_2 + d_2(t) + u_2(t)) - \eta_2(bz_1z_3 - az_2 - cz_2) \\ D^q e_3(t) &= (cw_2 - aw_3 - dw_3 + d_3(t) + u_3(t)) - \eta_3(cz_2 - az_3 - dz_3) \\ D^q e_4(t) &= (dw_3 - aw_4 + d_4(t) + u_4(t)) - \eta_4(dz_3 - az_4) \end{aligned} \quad (21)$$

Next, we will introduce a simple sliding surface to study the stability defined as:

$$s_i(t) = e_i(t), i = 1, 2, 3, 4 \quad (22)$$

The fractional derivative of the above equation is:

$$D^q s_i(t) = D^q e_i(t), i = 1, 2, 3, 4 \quad (23)$$

Now, the control input designed using the adaptive sliding mode:

$$\begin{aligned} u_1(t) &= ae_1 - a(1 - \eta_1) + bw_1w_3 + b\eta_1z_1z_3 - \beta_1s_1 - \hat{k}_1\text{sign}(s_1(t)) - \hat{d}_1(t) \\ u_2(t) &= ae_2 + ce_2 - bw_1w_3 + b\eta_2z_1z_3 - \beta_2s_2 - \hat{k}_2\text{sign}(s_2(t)) - \hat{d}_2(t) \\ u_3(t) &= -ce_2 + ae_3 + de_3 - c(\eta_2 - \eta_3)z_2 - \beta_3s_3 - \hat{k}_3\text{sign}(s_3(t)) - \hat{d}_3(t) \\ u_4(t) &= -de_3 + ae_4 - d(\eta_3 - \eta_4)z_3 - \beta_4s_4 - \hat{k}_4\text{sign}(s_4(t)) - \hat{d}_4(t) \end{aligned} \quad (24)$$

here, $\text{sign}(\cdot)$ refers to the signum function and $\beta'_i s > 0, i = 1, 2, 3, 4$ are constants.

So, the estimated values of $\hat{k}_i$ are as follows:

$$\begin{aligned} D^q \hat{k}_1 &= \gamma_1(|s_1(t)| - \hat{k}_1) \\ D^q \hat{k}_2 &= \gamma_2(|s_2(t)| - \hat{k}_2) \\ D^q \hat{k}_3 &= \gamma_3(|s_3(t)| - \hat{k}_3) \\ D^q \hat{k}_4 &= \gamma_4(|s_4(t)| - \hat{k}_4) \end{aligned} \quad (25)$$

where, $\gamma_i (i = 1, 2, 3, 4) > 0$ are constants that have been designed.

Now, substituting equation (22) in (21) and using (5), we get:

$$\begin{aligned} D^q e_1(t) &= -\beta_1 s_1 - \hat{k}_1 \text{sign}(s_1(t)) - \tilde{d}_1(t) \\ D^q e_2(t) &= -\beta_2 s_2 - \hat{k}_2 \text{sign}(s_2(t)) - \tilde{d}_2(t) \\ D^q e_3(t) &= -\beta_3 s_3 - \hat{k}_3 \text{sign}(s_3(t)) - \tilde{d}_3(t) \\ D^q e_4(t) &= -\beta_4 s_4 - \hat{k}_4 \text{sign}(s_4(t)) - \tilde{d}_4(t) \end{aligned} \quad (26)$$

Therefore, $s_i(t), i = 1, 2, 3, 4$ will be bounded and stable if the controllers are designed as (22) for (21). For that purpose, we use:

$$|s_i(t)| \leq C \quad (27)$$

$C > 0$ is some constant which is not known.

Using equations (22) and (27), we get:

$$|e_i(t)| \leq C \quad (28)$$

where, $C > 0$ is some unknown constant.

From the above equations we can clearly see that the boundedness of the sliding surface $s_i(t), i = 1, 2, 3, 4$ implies the boundedness of the synchronization error $e_i(t), i = 1, 2, 3, 4$.

The hybrid projective synchronization of Tuberculosis system with unknown disturbances using the non-linear FODO based adaptive sliding mode control method can be summarized using the theorem below.

Theorem 1: *An adaptive sliding control method (24), (25) is used to stabilize the hybrid projective synchronization error system (21), where $0 < q_i < 1$. This method ensures that the error $e(t)$ is ultimately bounded and stable by approximating external unknown bounded disturbances with a non-linear FODO (5), (7) and driving the system states onto a predefined sliding surface (22).*

Proof: First, we will design the Lyapunov function $V(t)$ to prove the convergence of the hybrid projective synchronization error $e(t)$ which is as follows:

$$V(t) = \sum_{i=1}^4 \frac{1}{2} s_i^2(t) + \sum_{i=1}^4 \frac{1}{2} \tilde{d}_i^2(t) + \sum_{i=1}^4 \frac{1}{2} \left(\frac{1}{\sqrt{\gamma_i}} (\hat{k}_i - k_i) \right)^2 \quad (29)$$

The proof is same as in [6] by taking $i = 1, 2, 3, 4$.

From [6], we get that:

$$Ps(t)P \leq \sqrt{\frac{2(\sum_{i=1}^4 \xi_i^2 + \sum_{i=1}^4 k_i^2)}{b_2}} \quad (30)$$

And, from the above inequality, we can clearly see that the synchronization systems $e(t)$ and $s(t)$ will be bounded as $t \rightarrow \infty$. Hence, according to lemma 2 [6], the error dynamical systems (21) is bounded as well as stable. Therefore, the bounded hybrid projective synchronization between the master system (1) and the slave system (2) has been successfully achieved.

5. Numerical Simulations

All the simulations are conducted using MATLAB for systems of order $0 < q < 1$. The parameter values chosen for the simulations: $a = 0.0012, b = 0.2568, c = 0.148, d = 0.104$, fractional order is taken to be as $q = 0.95$ and disturbances are: $d_1(t) = 0.02\sin(t), d_2(t) = 6 + \cos(t), d_3(t) = \sin(t), d_4(t) = \cos(0.1t)$. And, the initial conditions are taken as: $x_0(t) = [0.5, 0.3, 0.2, 0.1], y_0(t) = [0.8, 0.5, 0.4, 0.3]$.

Figure 3 (a) is the time waveform diagram of synchronization of the state variables using controllers (24) and (25). The systems converge within a short time-frame ($t < 2$) in most cases. Now, Figure 3 (b) shows synchronization of error functions that converge rapidly at $t \approx 0.2$ which highlights the high-efficiency of the synchronization method (24), (25) and (26) used for the fractional TB system (1). The simulation is observed over a time interval of $t = 0 - 0.9$. For 3(b), the parameter values are: $a = 1, b = 2, c = 1.5, d = 0.8$

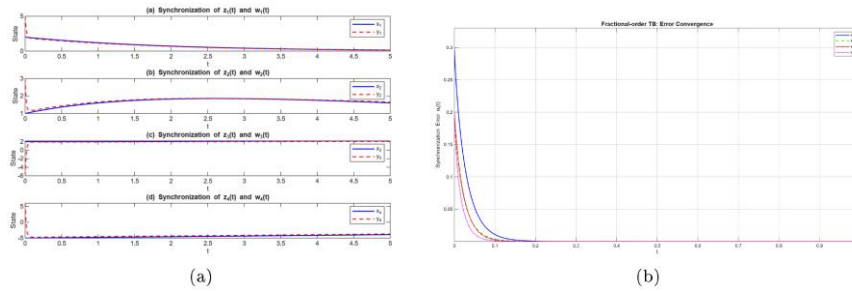
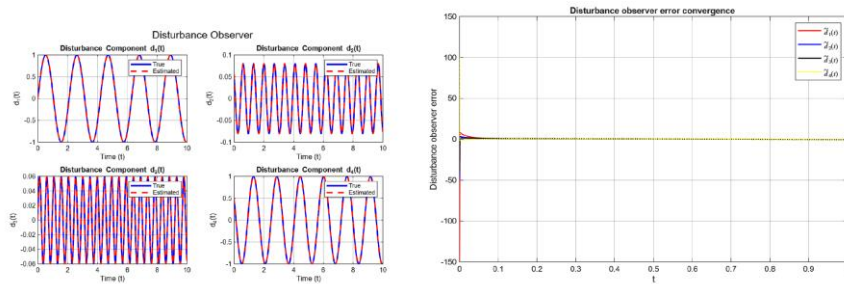


Figure 3

(a) Time waveform graphs for synchronization of $z_i(t)$ and $w_i(t)$ where $i = 1, 2, 3, 4$ (b) Error synchronization graphs of Tuberculosis master (1) and slave (2) systems.

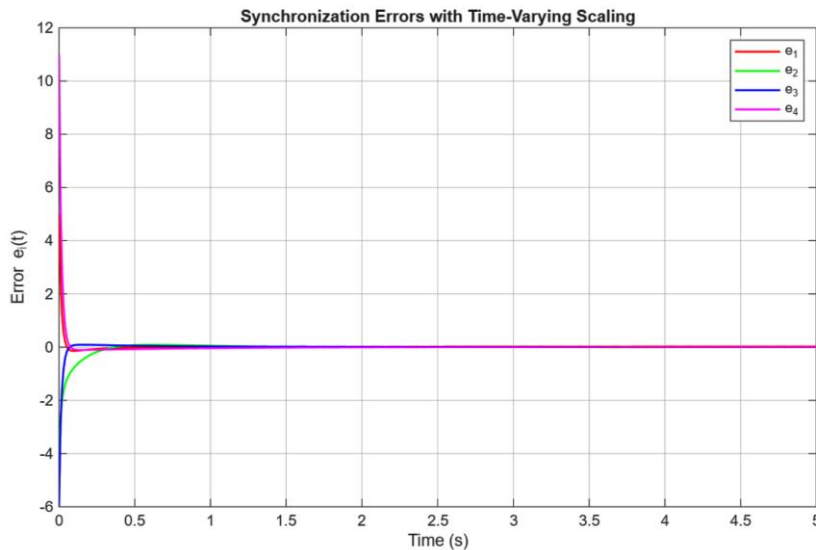
The estimated errors $\tilde{d}_i(t) = d_i(t) - \hat{d}_i(t), i = 1, 2, 3, 4$ are plotted in Figure 4 (a) and (b). In Figure 4 (a), all the curves in the four plots overlap almost perfectly across the entire time interval, that is, initial time $t = 0$ to the end time $t = 10$. The observer successfully tracks low-frequency disturbances like $d_1(t)$ and $d_4(t)$ as well as high-frequency disturbances like $d_2(t)$ and $d_3(t)$. This demonstrates that the observer is capable of accurately estimating external disturbances in real time. Now, in Figure 4 (b), it is observed that the divergence estimation errors converge rapidly after a short time period, that is, $t \approx 0.1$ confirming the effectiveness of the proposed observer (6) and (7).

Hence, these numerical results show that this paper is successful in achieving the bounded hybrid projective synchronization using the designed adaptive sliding mode controller (24) and the adaptive update law (25) between the master (1) and slave (2) systems.

**Figure 4**

(a) Results of distance observer synchronization (b) Time waveform graph for synchronization of disturbances error.

5.1 Comparison of error synchronization in fractional and integer model

**Figure 5**

Error synchronization graphs of the Tuberculosis master (1) and slave (2) systems for integer-order model using adaptive control method

The two graphs in Figure 3 (b) and Figure 5 shows the convergence of synchronization errors in the Tuberculosis system (1). Figure 3 (b) uses FODO and our proposed method whereas Figure 5 uses integer-order model via Adaptive control method based on [12]. Both the graphs shows that the system achieves complete synchronization but even though the parameter values $a = 1, b = 2, c = 1.5, d = 0.8$ and initial conditions $x_0(t) = [0.5, 0.3, 0.2, 0.1]$, $y_0(t) = [0.8, 0.5, 0.4, 0.3]$ are same in both the graphs but, as compared to Figure 5 which

converge at $t \approx 2.5$, Figure 3 (b) converges much quickly at $t \approx 0.2$. This indicates that the control strategy and synchronization mechanism in Figure 3 (b) is much more effective as compared to Figure 5, thus, indicating the effectiveness and robustness of the methodology shown in this paper.

6. Conclusion

This research validates a FODO-based adaptive sliding-mode synchronization strategy for a chaotic Tuberculosis (TB) system. Utilizing Caputo fractional derivatives ($0 < q < 1$), the model captures the memory and hereditary properties inherent in disease transmission. Chaotic oscillations were confirmed via numerical simulations in MATLAB. The study employs a master-slave synchronization framework where the slave system is subjected to unknown disturbances and control inputs. To manage these uncertainties, a Fractional-Order Disturbance Observer was designed to estimate perturbations in real time, with stability guaranteed through fractional Lyapunov methods. The control laws we made integrates four components: non-linear cancellation, proportional error feedback, adaptive switching, and observer-based compensation. Lyapunov analysis confirmed that synchronization errors and adaptive parameters converge to zero, ensuring robust stability. Numerical simulations in MATLAB demonstrated that this fractional-order approach outperforms integer-order models in controlling chaos and handling parameter variations.

Conflicts of Interest

The authors declare no conflicts of interest.

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