

Fractional diffusion model for removing additive noise with forward-backward diffusivity

Monika Rani [†]

Santosh Kumar ^{*}

Department of Mathematics & Data Science

Sharda School of Engineering & Science

Sharda University

Greater Noida 201310

Uttar Pradesh

India

Abstract

The PDE-based diffusion model is very effective in reducing noise and preserving edges, which are the major problems in image processing. This paper aims to propose a time-fractional diffusion model to remove additive noise from noisy images while keeping important edges clear. The proposed model uses both spatial derivatives and a time-fractional derivative. This fractional order helps control the diffusion process more effectively than the classical model. The model is discretized using a finite difference method for numerical implementation. The outcomes of the fractional model are evaluated using peak signal-to-noise ratio (PSNR), and the results show that the model improves noise removal while preserving image edges.

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1. Introduction

A nonlinear partial differential equation that contains various applications in image processing. The first method in this regard involves image denoising and preserving edges using a diffusion model, which was introduced by Perona and Malik [1]. Catte et al. [2] introduced an improved PM model for removing noise by smoothing the initial image using a Gaussian filter. The advantage of this model is that it preserves edges more effectively while smoothly changing

* ORCID-ID: 0000-0001-9500-7229

[†] E-mail: reachmonika11@gmail.com

* E-mail: skykumar87@gmail.com (Corresponding Author)

regions in the input image are converted into piecewise constant regions in the output image. The diffusion tensor-based model for image restoration was proposed by Weickert [3, 4]. Welk et al. [5] introduced a diffusion model based on additive noise to address image deblurring and denoising. The Gaussian kernel-based diffusion model proposed by Chauhan, Kumar, and Alam [6]. They have used the convolution of $G_\sigma * u$ in the diffusion model and proved the viscosity solution. Kumar, Sarfaraz, and Ahmad [7] developed a wavelet transform-based model for decreasing noise. They used a wavelet-based decomposition technique to break down the noisy images and then reconstructed them using a wavelet filter. Various diffusion models and image processing techniques were proposed by the researchers, see references [8-12]. Rudin, Osher, and Fatemi [13] introduced a time-dependent model and they introduced a numerical scheme that is based on a constrained optimization, and the time-dependent model was derived from the gradient projection method.

In recent years, researchers have shown interest in using fractional derivative-based non-linear models for image denoising, see reference [14, 15]. The partial differential equations based on fractional derivatives contain many applications, see reference [16-18]. These definitions are valid for only real-valued functions. The conformable fractional derivative (CFD) is proposed by Khalil et al. [19]. The CFD, which contains many applications in science, engineering, etc. Zhao and Luo [20] defined physical interpretation and application of the general conformable fractional derivative. Many other approach for image processing, see reference [21-26].

In this paper, we extend a classical diffusion model into a fractional diffusion model to remove additive noise. The enhanced model includes the time-fractional derivative with smoothness diffusivity parameters and uses the CFD definition. Section 2 presents the definition of conformable fractional derivatives. In Section 3, a given denoising algorithm for the fractional diffusion model. Section 4 introduces the discretization of both classical and fractional diffusion models with Forward-Backward Diffusivities. In Section 5, we present results in Table 1 and Figures 1-6. Finally, the conclusion.

2. Conformable Fractional Derivative

The definition of CFD, see references [19, 27] of the order α , is a mapping $v: (0, \infty) \rightarrow R$ defined as

$$U_\alpha(v)(t) = U_\alpha v(t) = \lim_{\epsilon \rightarrow 0} \frac{v(t+\epsilon t^{1-\alpha}) - v(t)}{\epsilon},$$

provided the limit exists for all values $t > 0$ and $\alpha \in (0, 1)$, and it can be considered the same as $U_\alpha v(t) = t^{1-\alpha} v'(t)$.

3. Denoising-Based Time-Fractional Diffusion Algorithm

The 2-dimensional additive noise-based diffusion equation studied by Welk et al. [5]. The prior smoothness-based improved Catte et al. [2] model for removing additive noise can be defined as:

$$\frac{\partial u}{\partial t} = \operatorname{div}(\zeta(|\nabla G_\sigma * u|)\nabla u) - \lambda(u - u_0). \quad (3.1)$$

Here $u = u(x, y, t)$ represents the original image, Ω is a bounded domain of R^2 , $\vec{n}$ the unit outer normal to Ω , $T > 0, \lambda > 0, 0 < t < T, \frac{\partial u}{\partial \vec{n}}$ on the boundary of $\partial\Omega$ is homogeneous Neumann boundary conditions and $u(x, y, 0) = u_0(x, y), \frac{\partial u}{\partial t}(x, y, 0) = 0, x, y \in \Omega, (u_0(x, y)$ is an initial condition) and the diffusivity $\zeta(s)$ is linked to $\psi'(s^2)$. The variance of Gaussian noise is denoted by σ^2 and $G_\sigma(x, y)$ is the low-pass filter or Gaussian kernel with spatial parameter $a = \sigma^{\frac{1}{2}}$ and

$$G_\sigma(x, y) = \frac{1}{2\pi a^2} \exp\left(-\frac{(x^2 + y^2)}{2a^2}\right).$$

The diffusivity ζ , the diffusion threshold parameter K and $\zeta(s)$ is a nonnegative function. The threshold parameter K is used to control the even enhancement of edges preserved. The function $\zeta(s) = \frac{1}{\sqrt{1 + \left(\frac{|s|^2}{K^2}\right)}}$, is similar to the

convex regularizer $\psi(s^2) = \sqrt{K^4 + K^2 s^2} - K^2$, see reference [3, 28], and it is called Charbonnier diffusivity, which is used in our numerical experiments.

The conformable fractional derivative is good because its formula involves a controller that can be applied to a complex problem, such as image restoration, edge detection, etc., and also satisfies the numerical solution in terms of the time-fractional heat equation, see [29, 30]. Using the definition of conformable as given in section 2, the diffusion model (3.1) is converted into a time fractional diffusion model:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \operatorname{div}(\zeta(|\nabla G_\sigma * u|)\nabla u) - \lambda(u - u_0). \quad (3.2)$$

Here, α is the fractional order.

4. Discrete scheme

In this section, we discuss an explicit scheme to discretize for both anisotropic diffusion and the fractional anisotropic diffusion model. Let $u_n^{i,j}$ represent the approximation for the value $u(x_i, y_j, t_n)$. We denote some notation, i.e., $\Delta x = \Delta y, \Delta t$ and $t_n = n\Delta t, n \geq 1$ are spatial, time step size and time variable, respectively. Specifically, we set $x_i = i\Delta x, y_j = j\Delta x, i, j = 1, 2, 3, \dots, N, N\Delta x = 1$. We express the operation $G_\sigma * u$ as u .

The discrete explicit scheme of the anisotropic diffusion model (3.1) is:

$$u_{i,j}^t = \frac{1}{2\Delta x} [(\zeta_{i+1,j}^n + \zeta_{i,j}^n)(u_{i+1,j}^n - u_{i,j}^n) - (\zeta_{i,j}^n + \zeta_{i-1,j}^n)(u_{i,j}^n - u_{i-1,j}^n)] + \frac{1}{2\Delta x} [(\zeta_{i,j+1}^n + \zeta_{i,j}^n)(u_{i,j+1}^n - u_{i,j}^n) - (\zeta_{i,j}^n + \zeta_{i,j-1}^n)(u_{i,j}^n - u_{i,j-1}^n)] - \lambda(u_{i,j}^n - u_{i,j}^0).$$

The discrete explicit scheme of the fractional anisotropic diffusion model (3.2) is:

$$u_{i,j}^t = t^{\alpha-1} \frac{1}{2\Delta x} [(\zeta_{i+1,j}^n + \zeta_{i,j}^n)(u_{i+1,j}^n - u_{i,j}^n) - (\zeta_{i,j}^n + \zeta_{i-1,j}^n)(u_{i,j}^n - u_{i-1,j}^n)] + \\ t^{\alpha-1} \frac{1}{2\Delta x} [(\zeta_{i,j+1}^n + \zeta_{i,j}^n)(u_{i,j+1}^n - u_{i,j}^n) - (\zeta_{i,j}^n + \zeta_{i,j-1}^n)(u_{i,j}^n - u_{i,j-1}^n)] - \\ t^{\alpha-1} \lambda(u_{i,j}^n - u_{i,j}^0).$$

Here $\zeta(|\nabla u|^2)$ can be written in the discrete form,

$$\zeta_{i,j}^n = \psi' \left(\left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{\Delta x} \right)^2 + \left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{\Delta x} \right)^2 \right).$$

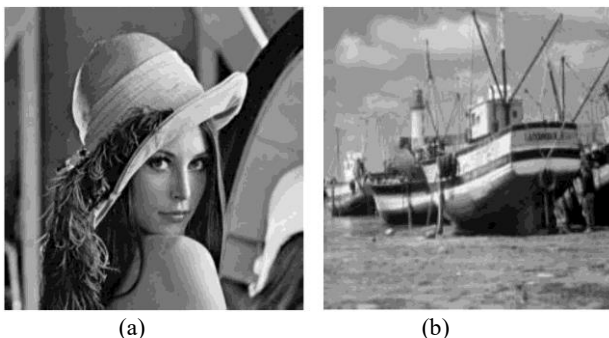
5. Experimental Result

The experimental findings are shown in this section, by the anisotropic nonlinear fractional diffusion model and the classical model on real original images. The original images contain the pixel value $[0, 255]$. Firstly, we reduce the pixel value of the original images to lie between $[0, 1]$. The original images ‘Lena’ and ‘Boat’ were corrupted by the additive Gaussian white noise. For noisy images, we use normal imnoise function and $\text{Imnoise}(u, \text{Gaussian}, m, \sigma^2)$ in Matlab, where m represents the mean and σ^2 signifies the variance. In experiments, we have added different levels of Gaussian white noise for the different values of variance with time parameter $t = 0.2, \Delta t/\Delta x^2 \leq 0.4$, diffusivity parameter $K = 5, \lambda = 0.85$, see reference [21, 31-34].

The experimental results for different fractional orders significantly reduce the iteration step and better PSNR values. The fractional-order α is very important in the experiment. A very small fractional-order α will get a clearer denoising image at a same number of iterations. We check the clarity of the denoising image by the PSNR value. A PSNR value that is larger indicates a good result. The fractional model provides fast processing of images for image denoising. The metric PSNR for smoothness images:

$$PSNR = 10 \log_{10} \left(\frac{S^2}{\frac{1}{MN} \sum_{i,j} (u_1(i,j) - u(i,j))^2} \right).$$

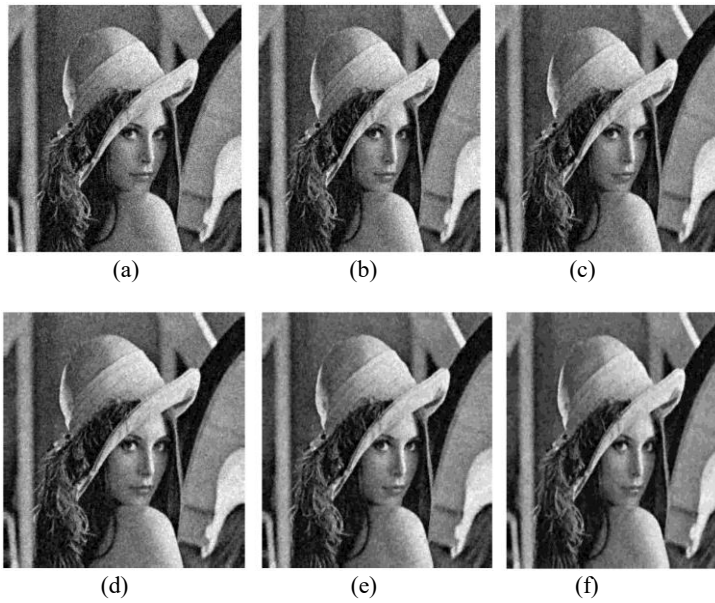
The restored image is represented by $u_1(i, j)$. Here, S corresponds to the largest pixel value of u , and MN indicates the dimensions of the matrix.



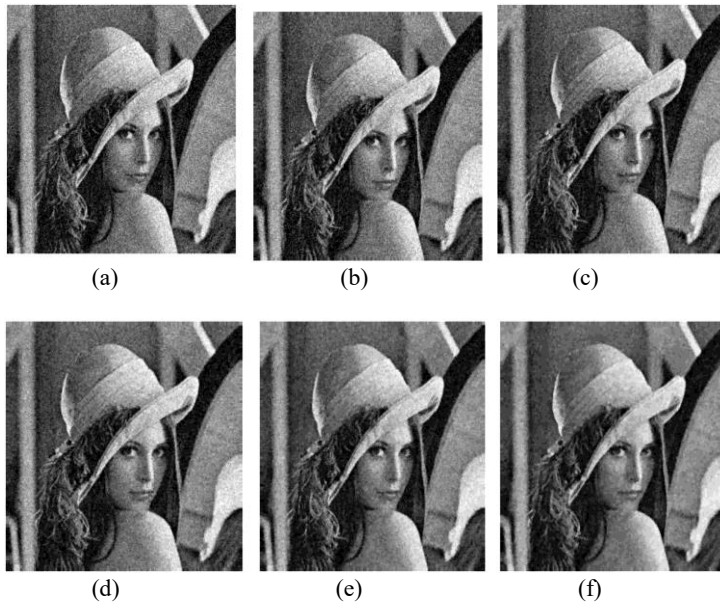
(a)

(b)

Figure 1**(a) Lena image and (b) Boat image.**

**Figure 2**

(a) represents a noisy corrupted image with ($\sigma^2 = 0.006$); (b) denoised image by (3.1); (c-f) denoised images by (3.2) at $\alpha=0.7, 0.5, 0.3$ and 0.1 , respectively.

**Figure 3**

(a) represents a noisy corrupted image with ($\sigma^2 = 0.008$); (b) denoised image by (3.1); (c-f) denoised images by (3.2) at $\alpha = 0.7, 0.5, 0.3$ and 0.1 , respectively.

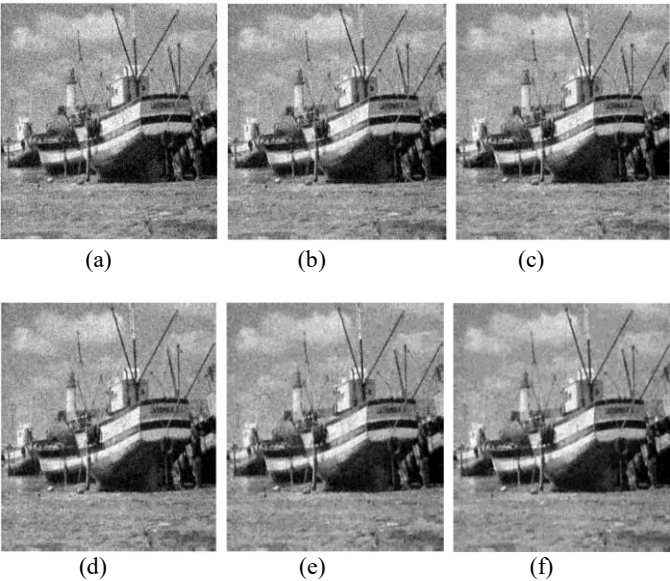


Figure 4

(a) represents a noisy corrupted image with ($\sigma^2 = 0.006$); (b) denoised image by (3.1); (c-f) denoised images by (3.2) at $\alpha = 0.7, 0.5, 0.3$ and 0.1 , respectively.

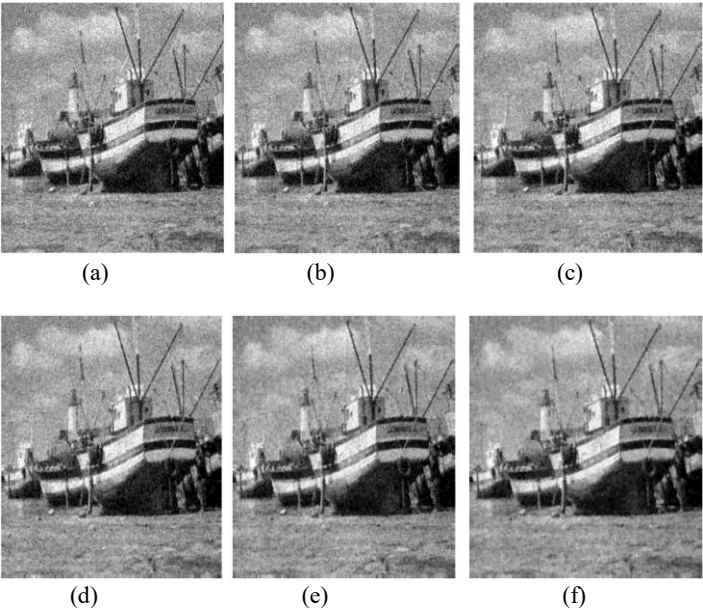


Figure 5

(a) represents a noisy corrupted image with ($\sigma^2 = 0.008$); (b) denoised image by (3.1); (c-f) denoised images by (3.2) at $\alpha = 0.7, 0.5, 0.3$ and 0.1 , respectively.

Table 1

The results are given in terms of PSNR values of the diffusion model (3.1) and the fractional diffusion model (3.2) at different levels of noisy images.

Images	PSNR values of the noisy images	PSNR values by the old model (3.1)	PSNR values by our model (3.2)			
		$\alpha = 1$	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.1$
Lena, $\sigma^2 = 0.006$	22.35	23.54	24.61	25.67	26.88	28.11
Lena, $\sigma^2 = 0.008$	21.16	22.27	23.27	24.32	25.55	26.84
Boat, $\sigma^2 = 0.006$	22.30	23.33	24.37	25.35	26.49	27.56
Boat, $\sigma^2 = 0.008$	21.07	22.09	23.06	24.03	25.22	26.43
Number of iteration	-	100	100	100	100	100

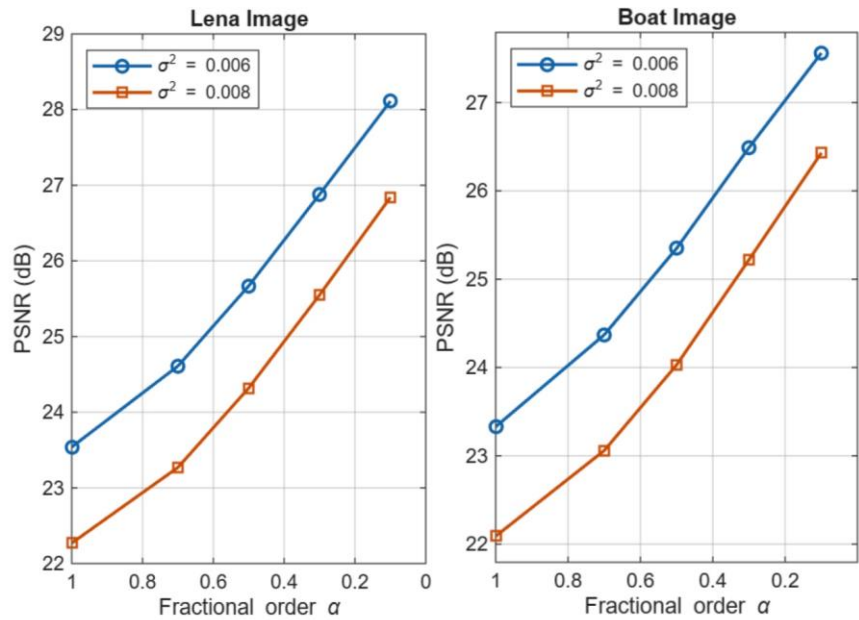


Figure 6
The graphs represent the outcomes for Lena and Boat images.

Conclusion

We have introduced a smoothness diffusivity-based fractional nonlinear anisotropic diffusion model to remove Gaussian white noise. The fractional diffusion model extended the traditional diffusion model. In the nonlinear PDE,

fractional-order derivatives are present in both space and time. The finite-difference explicit scheme computes the experimental results in terms of PSNR values of the fractional and classical diffusion models. The proposed model (3.2) has larger PSNR values corresponding to (3.1) at the same iteration numbers. It is observed that the proposed model contains good results in comparison to the old model. The proposed model can be extended to speckle noise, salt and pepper noise, etc.

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