

Capturing nonlinear dynamical behavior of the (3+1)-dimensional Zakharov-Kuznetsov equation arising in shallow water waves by using Lie group theory

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Abstract

In this research work, the Zakharov–Kuznetsov equation is solved analytically utilizing the similarity transformations approach via classical theory of Lie-group. At each stage, the similarity transformation technique (STM) lowers the number of independent variables by one. The STM is applied to Zakharov–Kuznetsov (ZK) equation and after successive application it reduces an ordinary differential equation. This can be solved further to get invariant solutions of the Zakharov–Kuznetsov equation. This research has the potential to solve Zakharov–Kuznetsov equation in a more general approach. Fluid mechanics, oceanography, optical fiber, fluid dynamics are all applications of the system.

Subject Classification: 35Q53, 35C05, 35A30, 37K10.

Keywords: Lie symmetry analysis, Exact solution, Zakharov–Kuznetsov (ZK) equation, Plasma physics, Nonlinear optics, Similarity transformation method.

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1. Introduction

Nonlinear partial differential equations (NPDEs) are useful tools for simulating various complex processes in a wide range of domains, such as biology, fluid dynamics, oceanography, magneto-fluid dynamics, plasma physics, optics, neural networks, and many more, as discussed by Singhal, Joshi, and Arora, [1]. Several variables and functions illustrate the evolutionary aspects of practical value in the solutions to this problem. These can assist us in comprehending and forecasting the status and evolution of real-world issues. Solitons, compactons, and stable solutions are common phenomena arising from NPDE solutions. Analytical solutions and numerical simulations can be used to acquire these qualities. The study of NPDEs is essential to explain the complex nature of the universe. Within this framework of plasma physics, Zakharov and Kuznetsov [5] introduced the Zakharov-Kuznetsov (ZK) equation to describe the propagation of ion-acoustic waves in magnetized plasma, as discussed by Rehman et al. [2]. The ZK equation models weakly nonlinear, long-wavelength dispersive waves in a magnetized plasma and can be viewed as a multidimensional extension of the Korteweg-de Vries (KdV) equation. It incorporates the effects of nonlinearity, dispersion, and magnetic-field-induced anisotropy, as reported by Rehman et al. [2]. Owing to its applications in fluid dynamics, plasma physics, and nonlinear optics, the ZK equation has attracted considerable attention. Moreover, it supports stable multidimensional soliton structures and serves as an important model for studying nonlinear dispersive wave phenomena.

According to Anco and Gandarias [3], previous studies have primarily focused on solitary-wave solutions of the (2+1)-dimensional ZK equation, whereas the dynamics of the corresponding (3+1)-dimensional model remain relatively unexplored. Therefore, the present work aims to investigate invariant solutions of the (3+1)-dimensional ZK equation to enhance the understanding of wave propagation in anisotropic media under the influence of magnetic fields. The obtained results may contribute to the analysis of space plasmas and controlled fusion systems.

The Sophus Lie [16]-[20] by transformation group method was developed in the 19th century and is a potent analytical technique for determining invariant solutions to differential equations. This approach allows large systems of partial differential equations (PDEs) to be systematically reduced into simpler ordinary differential equations (ODEs), which may be applied to both dependent and independent variables. It also makes it easier to reduce the order of ODEs, which makes it much easier to discover precise or approximate solutions.

The (3+1)-dimensional Zakharov-Kuznetsov equation, in its general version, describes the evolution of weakly nonlinear ion-acoustic waves in a plasma that contains both warm thermal particles and cold ions under the influence of a homogenous magnetic field.

$$u_t + auu_x + u_{xx} + u_{yy} + u_{zz} = 0, \quad (1)$$

Where $u=u(x, y, z, t)$ represents the wave amplitude, and the constant a is determined by the physical parameters of the plasma medium.

You and Ning [6] studied the generalized initial value problem, Kuo [7] developed the simplest equation method, Wazwaz [8] employed the sine-cosine method, Telezhko et al. [11] proposed the (G)-expansion method, Rehman et al. [13] utilized the Riccati method and the Backlund transformations are some methodological approaches that have been discovered to find invariant and exact solutions of nonlinear PDEs, as discussed by Zhao et al. [9].

Furthermore, the ZK equation for simplifying Lie symmetry analysis has become a potent technique, which makes it possible to derive conservation laws and investigate fractional-order generalizations.

In addition to its theoretical appeal, the (3+1)-dimensional ZK equation is important for wave stability, plasma confinement and energy transmission in magnetic settings. The objective of this study is to examine the equation's analytical structure, investigate its solution space, and draw attention to its applications in modern mathematical physics.

The studies inspire the authors to use the similarity transformations technique (STM) to answer Eq. (1) analytically. This method's basic idea is to reduce a system of PDEs' number of independent variables by one. The symmetry and invariance property are used to decrease the variables in the system of partial differential equations (PDEs) under Lie group theory. The theory and its applications are understandable to researchers and textbooks by Bluman and Cole [4]. In this article, the ZK equation system is reduced to a new class of PDEs with one dependent and four independent variables using the similarity transformation technique (STM). The PDEs are transformed into ordinary differential equations (ODEs) with one dependent and three independent variables by repeating the similarity reduction procedure. In the end, explicit forms that bifurcate in various instances were used to represent the solutions of the resultant ODEs. With the help of their graphical representations, the answers produced for various circumstances have been physically examined.

1.1 Literature survey

The Zakharov–Kuznetsov equation's initial boundary value problem is posed by on a half strip, and a thorough literature review pertaining to the many kinds of ZK equation (1) solutions is presented by Singhal [1]. The generalized (2 + 1)-dimensional nonlinear wave equation has been studied by Xin Zhao et al. [9] in the domains of nonlinear optics, plasma physics, and fluid mechanics. Hua et al. [10] suggested an extended (2+1)-dimensional Hirota bilinear equation as a unified model to investigate nonlinear dynamical phenomena in plasma physics, shallow sea waves, and nonlinear optics. Gao and Zhao [12] used the (G'/G)-expansion approach to get accurate solutions for higher-order nonlinear equations. The mKdV–ZK equation was subjected to the Improved Generalized Riccati Equation Mapping (IGREM) approach by Rehman et al. [13]. The Hirota bilinear approach and Lie symmetry analysis have been used to obtain soliton and other exact solutions by Tiwari, Sharma, and Arora, [14]. Wang et al. [24] derived a variety of traveling wave solutions for the extended QZK equation. A variety of researchers have studied the (2+1)-dimensional KD system using different

analytical methods to find exact and soliton solutions, demonstrating that similarity transformations effectively simplify nonlinear partial differential equations, as discussed by Kumar, Kumar, and Kumar [15]. Several analytical techniques were employed in earlier research on the (2+1)-dimensional DJKM problem to derive soliton and interaction solutions. The motivation behind this study was to use Lie group theory to obtain symmetry reductions and solitary wave solutions, as Lie symmetry analysis had not yet been utilized, as discussed by Chauhan, Sharma, and Arora, [17]. Inderjeet and Bhardwaj [22] proposed an integrated numerical method for solving nonlinear equations with improved accuracy and convergence. Abed [21] applied the differential transform method combined with Grey Wolf and Cuckoo optimization algorithms to obtain approximate solutions of the Sawada–Kotera equation. Both studies demonstrated the effectiveness of advanced computational techniques for nonlinear differential equations. Kumar and Parihar [23] investigated the (3+1)-dimensional Zakharov–Kuznetsov equation and derived several new exact solutions using analytical techniques. The obtained solutions were expressed in different wave forms and illustrated graphically to examine their physical characteristics. Their results demonstrated the effectiveness of exact solution methods for analyzing nonlinear evolution equations.

2. Infinitesimals via Lie-Symmetry Analysis

In this section, the one-parameter Lie group similarity transformations technique (STM) is used to generate infinitesimals of the Zakharov–Kuznetsov equation.

$$\begin{aligned}u^* &= u + \varepsilon \eta^{(u)}(\Xi) + o(\varepsilon^2), \\x^* &= x + \varepsilon \xi^{(x)}(\Xi) + o(\varepsilon^2), \\y^* &= y + \varepsilon \xi^{(y)}(\Xi) + o(\varepsilon^2), \\z^* &= z + \varepsilon \xi^{(z)}(\Xi) + o(\varepsilon^2), \\t^* &= t + \varepsilon \tau(\Xi) + o(\varepsilon^2),\end{aligned}\tag{2}$$

Where $\eta^{(u)}$, $\xi^{(x)}$, $\xi^{(y)}$, $\xi^{(z)}$, τ are the infinitesimals for u , x , y , z and t respectively and $\Xi \equiv (u, x, y, z, t)$ and ε is a small parameter.

Let $u = \theta^{(u)}(x, y, z, t)$ be the solutions of Eq. (1). Then, the invariance conditions is:

$$[\eta_t^{(u)}] + a \theta_x^{(u)} \theta^{(u)} + a u [\eta_x^{(u)}] + [\eta_{xx}^{(u)}] + [\eta_{yy}^{(u)}] + [\eta_{zz}^{(u)}] = 0, [1]\tag{3}$$

One can follow the books [19], [20] for getting the values of the extensions $[\eta_x^{(1)}]$, $[\eta_{xx}^{(1)}]$, $[\eta_{yy}^{(2)}]$ and $[\eta_{zz}^{(3)}]$ etc.

Making use of Eq. (1) into Eq. (3) which gives

$$\begin{aligned}\tau_x &= \tau_y = \tau_z = \tau_u = 0, \tau_{tt} = 0, \\ \xi_{1u} &= \xi_{2u} = \xi_{3u} = 0, \xi_{1t} = \xi_{2t} = \xi_{3t} = 0, \\ \xi_{1x} &= \xi_{2y} = \xi_{3z}, \\ \xi_{1y} &= \xi_{1z} = \xi_{2x} = \xi_{2z} = \xi_{3x} = \xi_{3y} = 0,\end{aligned}\tag{4}$$

$$\begin{aligned} 2\xi_{1x} &= \tau_t, \xi_{1xx} = \xi_{2yy} = \xi_{3zz} = 0, \\ 2\eta_u &= -u\tau_t. \end{aligned}$$

The resulting infinitesimal generators Eq. (1) for can be derived by solving the above determining Eq. (4)

$$\begin{aligned} \xi_x &= b_5 + b_2 t + \frac{b_1 x}{2}, \\ \xi_y &= b_6 + b_3 z + \frac{b_1 y}{2}, \\ \xi_z &= b_4 - b_3 y + \frac{b_1 z}{2}, \\ \tau &= b_7 + b_1 t, \\ \eta &= \frac{b_2}{a} - \frac{b_1 u}{2}, \end{aligned} \quad (5)$$

where all b_i 's are arbitrary constants.

Now, vector field of above Lie symmetry is given by

$$B = \tau B_1 + \xi_x B_2 + \xi_y B_3 + \xi_z B_4 + \eta^{(u)} B_5 \quad (6)$$

Where B_1, B_2, B_3, B_4 and B_5 are the infinitesimal generators. The infinitesimal generators obtained by solving the determining Eq. (4) are summarized in Table 1.

$$\begin{aligned} B_1 &= t \partial_t + \frac{x}{2} \partial_x + \frac{y}{2} \partial_y + \frac{z}{2} \partial_z - \frac{u}{2} \partial_u, \\ B_2 &= t \partial_x + \frac{1}{a} \partial_u, \\ B_3 &= z \partial_y - y \partial_z, \\ B_4 &= \partial_z, \\ B_5 &= \partial_x, \\ B_6 &= \partial_y, \\ B_7 &= \partial_t. \end{aligned} \quad (7)$$

The commutative relations are determined as:

$$\begin{aligned} [B_i, B_j] &= B_i B_j - B_j B_i, i, j = 1, 2, 3, 4, \dots \\ [B_1, B_2] &= \frac{1}{2} B_2, & [B_1, B_4] &= -\frac{1}{2} B_4, & [B_1, B_5] &= -\frac{1}{2} B_5, \\ [B_1, B_6] &= -\frac{1}{2} B_6, & [B_1, B_7] &= -B_7, & [B_2, B_7] &= B_5, \\ [B_3, B_4] &= -B_6, & [B_3, B_6] &= B_4, & [B_2, B_1] &= -\frac{1}{2} B_2, \\ [B_4, B_1] &= \frac{1}{2} B_4, & [B_4, B_3] &= B_6, & [B_5, B_1] &= \frac{1}{2} B_5, \\ [B_6, B_1] &= \frac{1}{2} B_6, & [B_6, B_3] &= -B_4, & [B_7, B_1] &= B_7, \\ [B_7, B_2] &= -B_5. \end{aligned} \quad (8)$$

All other commutators are zero. The commutator table of the Lie algebra for the ZK equation is presented in Table 1.

Table 1
Commutator table of Lie algebra for the ZK equation.

$[B_i, B_j]$	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_1	0	$\frac{1}{2}B_2$	0	$-\frac{1}{2}B_4$	$-\frac{1}{2}B_5$	$-\frac{1}{2}B_6$	$-B_7$
B_2	$-\frac{1}{2}B_2$	0	0	0	0	0	B_5
B_3	0	0	0	$-B_6$	0	B_4	0
B_4	$\frac{1}{2}B_4$	0	B_6	0	0	0	0
B_5	$\frac{1}{2}B_5$	0	0	0	0	0	0
B_6	$\frac{1}{2}B_6$	0	$-B_4$	0	0	0	0
B_7	B_7	$-B_5$	0	0	0	0	0

3. Similarity reductions and invariant solutions

In order to derive analytical solutions for ZK equation (1), Lagrange's characteristic equations are

$$\frac{dx}{b_5+b_2t+\frac{b_1x}{2}} = \frac{dy}{b_6+b_3z+\frac{b_1y}{2}} = \frac{dz}{b_4-b_3y+\frac{b_1z}{2}} = \frac{dt}{b_7+b_1t} = \frac{du}{\frac{b_2}{a}-\frac{b_1u}{2}} \quad (9)$$

Now, in order to proceed further, we considered the following cases:

Case (I): For $b_3 = 0$, the Eq.(9) provides,

$$\frac{dx}{b_5+b_2t+\frac{b_1x}{2}} = \frac{dy}{b_6+\frac{b_1y}{2}} = \frac{dz}{b_4+\frac{b_1z}{2}} = \frac{dt}{b_7+b_1t} = \frac{du}{\frac{b_2}{a}-\frac{b_1u}{2}} \quad (10)$$

On solving , one can have

$$\begin{aligned} X &= x(b_7 + b_1t)^{-\frac{1}{2}} - \frac{2b_2(b_7+b_1t)^{\frac{1}{2}}}{b_1^2} - \frac{2(b_2b_7-b_5b_1)(b_7+b_1t)^{-\frac{1}{2}}}{b_1^2} \\ X &= y(b_7 + b_1t)^{-\frac{1}{2}} - \frac{2b_2(b_7+b_1t)^{-\frac{1}{2}}}{b_1} \\ Z &= z(b_7 + b_1t)^{-\frac{1}{2}} - \frac{2b_4(b_7+b_1t)^{-\frac{1}{2}}}{b_1} \quad \text{and} \\ u &= \frac{2b_2}{ab_1} + U(b_7 + b_1t)^{-\frac{1}{2}} \end{aligned} \quad (11)$$

Where X,Y,Z similarity variable and U is similarity functions. The similarity reduction of the ZK Eq. (1) yields the following

$$b_1XU_X + b_1YU_Y + b_1ZU_Z + b_1U - 2aUU_X - 2U_{XX} - 2U_{YY} - 2U_{ZZ} = 0 \quad (12)$$

Where a and b_1 are arbitrary constants.

The infinitesimals ξ_X, ξ_Y, ξ_Z and η_1 for X,Y,Z and U can be found in order to solve this problem.

$$\xi_x = 0, \xi_Y = Z, \xi_z = -Y, \eta_1 = 0 \quad (13)$$

Further steps can be taken by dividing into the following scenarios to obtain the precise solutions:

Case (Ia): Then Lagrange's system for (12) is

$$\frac{dX}{0} = \frac{dY}{kZ} = \frac{dZ}{-kY} = \frac{dU}{0} \quad (14)$$

This provides the following similarity forms

$$U = F(X_1, Z_1) \quad (15)$$

Where $U = F(X_1, Z_1)$ is a similarity function of the variables $X_1 = X$, $Z_1 = -\left(\frac{Y^2 + Z^2}{2}\right)$.

Substituting U into Eqs.(12), one can have

$$b_1 X_1 F X_1 + 2b_1 Z_1 F Z_1 + b_1 F - 2a F F X_1 - 2F X_1 X_1 + 4Z_1 F Z_1 Z_1 = 0 \quad (16)$$

Where a and b_1 are arbitrary constants.

To obtain the exact solutions, one can determine the following set of infinitesimals ξ_{X_1} , ξ_{Z_1} , and η_1 for the variables X_1 , Z_1 , and F .

$$\xi_{X_1} = 1, \xi_{Z_1} = 0, \eta_1 = \frac{b_1}{2a} \quad (17)$$

By dividing the procedure into the following situations, the precise answers may be obtained:

Case (Ib): Then the characteristic equation for the system (16) is

$$\frac{dX_1}{i} = \frac{dZ_1}{0} = \frac{dF}{\frac{b_1 F}{2a}} \quad (18)$$

Where a , b_1 and i are arbitrary constant.

After solving it, we obtain the similarity reduction shown below.

$$F = G(X_2) \quad (19)$$

Where similarity variable, $X_2 = X_1$.

Therefore, the reduction of PDEs (16) provides

$$b_1 X_2 G_{X_2} + b_1 G - 2a G G_{X_2} - 2G_{X_2 X_2} = 0 \quad (20)$$

This ODE system is fulfilled by

$$G = \frac{b_1}{2a} X_2 + C \quad (21)$$

Where C is an arbitrary constant of integration.

So, the solution of ZK equation (1) is

$$u = \frac{2b_2}{ab_1} + \frac{1}{(b_7 + b_1 t)^{1/2}} \left[\frac{b_1 x (b_7 + b_1 t)^{-1/2}}{2a} - \frac{b_2 (b_7 + b_1 t)^{1/2}}{ab_1} - \frac{(b_2 b_7 - b_5 b_1) (b_7 + b_1 t)^{-1/2}}{ab_1} + C \right] \quad (22)$$

Where a , b_1 , b_2 , b_5 , b_7 and C are arbitrary constant.

This system of ODEs is satisfy by

$$\frac{1}{\frac{a}{b_1 X_2} + D \exp\left(\frac{b_1 X_2}{2}\right)} \quad (23)$$

And solution of ZK equation (1) is

$$u = \frac{2b_2}{ab_1} + \frac{1}{(b_7 + b_1 t)^{1/2}} \left[\frac{1}{\frac{a}{b_1 \phi}} + D \exp\left(\frac{b_1 \phi}{2}\right) \right], \quad (24)$$

$$\phi = \frac{b_1 x (b_7 + b_1 t)^{-1/2}}{2a} - \frac{b_2 (b_7 + b_1 t)^{\frac{1}{2}}}{b_1 a} - \frac{(b_2 b_7 - b_5 b_1) (b_7 + b_1 t)^{-1/2}}{a b_1}$$

Where D is an arbitrary constant.

4. Analysis and Discussions

In this section, physical behaviour of the analytical solutions of the ZK Equation (1) is described. Lump type, stationary, asymptotic and parabolic nature of the solutions are explored. The main aim of this exercise is to look into what the answers are like based on how they are shown graphically. The animated behaviour of the solutions is discovered using symbolic computations. With respect to variations in time and space, the wave amplitude function varies. Thus, as seen in Figures (1-2), the authors were successful in capturing the dominating behaviour of an animation frame. The authors have given the constants suitable values, which reflect the dynamics of profiles since arbitrary constants are an vital part of analytical solutions.

Figure 1: The wave amplitude u shows parabolic type profile and helpful to suggest an offshore structure, which describes the fluid flow during $-10 \leq t \leq 10$. The arbitrary constants are taken as $a = 2$, $b_1 = 3$, $b_2 = 3.25$, $b_5 = 2$, $b_7 = 3.15$ and $C = 1$. The 3-D type profiles are extensively helpful in describing localized waves in the ocean, localized chemical waves and localized optical pulses that do not dissipate as they are non-dispersive, localized in space and preserve the shape.

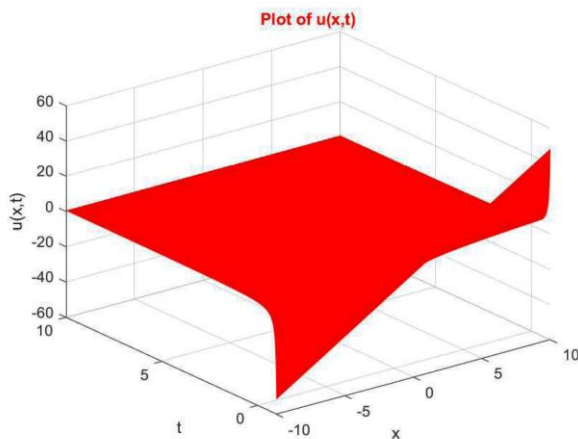


Figure 1
3-D type profile at $t = 10$

Figure 2: The wave nature shows asymptotic behaviour for u at time $-10 \leq t \leq 4000$ treating $a = 2$, $b_1 = 3$, $b_2 = -3.25$, $b_5 = 2$, $b_7 = 3.15$ and $D = 1$. As the space range for x expands, the wave amplitude function becomes 3-D type profile in nature.

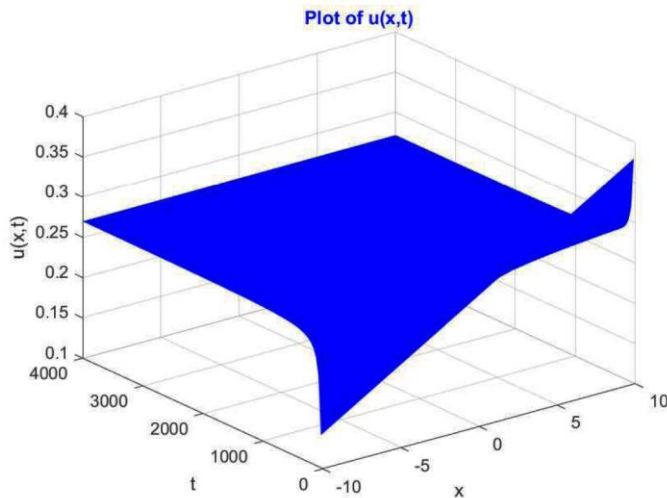


Figure 2
3-D type profile at $t = 4000$.

5. Conclusions

We have applied STM to attain invariant solutions of the $(3 + 1)$ -ZK equation by generating similarity forms via us Lie-group symmetry theory. On solving these similarity forms further we obtained explicit form of analytical solutions expressed via (22), (24). We have added to the provided answers by using MATLAB to trace the animation profiles of the chosen values of arbitrary constants. Asymptotic, multisolitons, periodic, stationary, and parabolic wave profiles are noted. To the best of the authors' knowledge, the results of this study are novel for the ZK equation. By capturing phenomena like dispersive shock waves and undular bores, this effort eventually results in a scientific contribution to the solution of a physically difficult system like shallow water waves. The stationary behavior of the wave and multi-solitons is thus shown by the analytical results, which may be utilized to assess, contrast, and examine numerical techniques in this area.

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Received January, 2026