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Stability and convergence of a modified Picard-Ishikawa hybrid iterative process

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Abstract

In this work, a variant of the Picard–Ishikawa hybrid scheme is introduced which is designed for approximation of fixed points for contractive mappings in Banach spaces. The corresponding iterative sequence is shown to be stable and to approach the fixed point more rapidly than a known Picard–S process. Furthermore, data dependence results associated with the process are obtained. These findings demonstrate that suitable modifications of existing hybrid iterations can significantly enhance convergence performance.

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1. Introduction

Fixed point methods are crucial for the development of numerical schemes for nonlinear equations. Motivated by the classical Banach contraction principle (BCP), many authors have proposed different one-step and multi-step iterative constructions for locating fixed points of nonlinear mappings. In recent years, hybrid schemes like the Picard–Mann [1] and Picard–Ishikawa [2] processes have gained considerable attention.

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These methods have been shown to exhibit superior convergence behavior compared to many classical schemes, particularly when analyzed in the framework introduced by Berinde [3]. Gursoy and Karakaya [4] demonstrated that the Picard–S iteration converges faster than several well-known iterative processes for contraction mappings. Motivated by these developments, we propose a modified Picard–Ishikawa hybrid iteration and investigate its convergence properties. The primary objectives of this paper are threefold. First, we establish the strong convergence result of the proposed scheme. Second, we prove stability and data dependency. Third, we prove its superiority, in term of convergence rate in Berinde's sense, over the Picard–S iteration. In addition, data dependence results are also obtained for the scheme.

2. Preliminaries

Throughout this paper, we let $(X, \|\cdot\|)$ be a Banach space and $C \neq \emptyset$ a convex and closed subset of X.

A point $p \in C$ is a fixed-point of $T: C \to C$ if $Tp = p$.

Let $Fix(T) = \{p \in C |\ Tp = p\}$.

$T: C \to C$ is a contraction if $\exists\, \theta \in (0, 1)$ such that

$$\|Tx - Ty\| \leqslant \theta \|x - y\| \text{ for all } x, y \in C \tag{1}$$

We now recall some well-known iterative processes:

Picard iteration [5]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$p_{n+1} = Tp_n \tag{2}$$

Mann iteration [6]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$p_{n+1} = (1 - \alpha_n) p_n + \alpha_n T p_n \tag{3}$$

where $\{\alpha_n\} \subset (0, 1)$.

Ishikawa iteration [7]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} p_{n+1} &= (1-\alpha_n) p_n + \alpha_n T q_n \\ q_n &= (1-\beta_n) p_n + \beta_n T p_n \end{aligned} \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.

Noor iteration [8]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} p_{n+1} &= (1-\alpha_n) p_n + \alpha_n T q_n \\ q_n &= (1-\beta_n) p_n + \beta_n T r_n \\ r_n &= (1-\gamma_n) p_n + \gamma_n T p_n \end{aligned} \tag{5}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$.

For further reading see [9]-[14].

Definition 1: [3] Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two real sequences with $x_n \to x$ and $y_n \to y$. If $\lim\limits_{n\to\infty} \frac{|x_n - x|}{|y_n - y|} = 0$, then $\{x_n\}_{n=1}^{\infty}$ converges faster to x than $\{y_n\}_{n=1}^{\infty}$ does to y.

Definition 2: *[3] Let $\{x_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be two iterative sequences such that $x_n \to p$, $t_n \to p$. Suppose $\|x_n - p\| \leqslant a_n$ & $\|t_n - p\| \leqslant b_n$* for all $n \in \mathbb{N}$, *where $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ $(a_n > 0,\ b_n > 0)$ are such that $a_n \to 0$, $b_n \to 0$. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then $\{x_n\}_{n=1}^{\infty}$ converges faster than $\{t_n\}_{n=1}^{\infty}$ to p.*

Picard-Mann hybrid iteration [1]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} p_{n+1} &= T q_n \\ q_n &= (1-\alpha_n) p_n + \alpha_n T p_n \end{aligned} \tag{6}$$

where $\{\alpha_n\} \subset (0,1)$.

Picard-Ishikawa hybrid iteration [2]

For a given initial point $p_1 \in C$, this iteration $\{p_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} p_{n+1} &= Tq_n \\ q_n &= (1-\alpha_n)\, p_n + \alpha_n T r_n \\ r_n &= (1-\beta_n)\, p_n + \beta_n T p_n \end{aligned} \tag{7}$$

where $\{\alpha_n\},\{\beta_n\} \subset (0,1)$.

Picard-S hybrid iteration [4]

For a given initial point $t_1 \in C$, this iteration $\{t_n\}_{n=1}^{\infty}$ is defined by:

$$\begin{aligned} t_{n+1} &= Tu_n \\ u_n &= (1-\alpha_n)Tt_n + \alpha_n T v_n \\ v_n &= (1-\beta_n)t_n + \beta_n T t_n \end{aligned} \tag{8}$$

where $\{\alpha_n\},\{\beta_n\} \subset (0,1)$.

We introduce a Modified Picard-Ishikawa scheme here: For a given initial point $x_1 \in C$, the Modified Picard-Ishikawa iteration $\{x_n\}_{n=1}^{\infty}$ is defined for $i \in \mathbb{N}$ by:

$$\begin{aligned} x_{n+1} &= T^i y_n \\ y_n &= (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n &= (1-\beta_n)x_n + \beta_n T x_n \end{aligned} \tag{9}$$

where $\{\alpha_n\},\{\beta_n\} \subset (0,1)$, and T^i is the i^{th} iterate of T.

Lemma 1: *Let the real sequences* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ $(a_n \geqslant 0,\ b_n \geqslant 0)$ *satisfy* $a_{n+1} \leqslant (1-\lambda_n)a_n + b_n$, *where* $\lambda_n \in (0,1)\ \forall\ n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ *and* $\frac{b_n}{\lambda_n} \to 0$ *as* $n \to \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2: *Let* $\{\xi_n\}_{n=1}^{\infty}$ $(\xi_n \geqslant 0)$, *be a sequence or reals. Suppose* $\exists\, n_1 \in \mathbb{N}$, *such that* $\forall n \geqslant n_1$, *the inequality* $\xi_{n+1} \leqslant (1-\zeta_n)\xi_n + \zeta_n \lambda_n$ *is satisfied, where* $\zeta_n \in (0,1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \zeta_n = \infty$ *and* $\lambda_n \geqslant 0,\ \forall\ n \in \mathbb{N}$. *Then*

$$0 \leqslant \limsup_{n\to\infty} \xi_n \leqslant \limsup_{n\to\infty} \lambda_n \tag{10}$$

3. Convergence and Stability

Theorem 1: *Let* $T: C \to C$ *satisfy (1). Suppose (9) generates the iterative sequence* $\{x_n\}_{n=1}^{\infty}$, *where* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *with* $\sum_{k=1}^{\infty} \alpha_k \beta_k = \infty$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges to a unique* $p \in Fix(T)$.

Proof: By BCP, T has a unique $p \in Fix(T)$.

It remains to show that $x_n \to p$ as $n \to \infty$.

From the definition of z_n, y_n and x_{n+1} in (9), we obtain

$$\begin{aligned}
\|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n T x_n - p\| \\
&\leqslant (1-\beta_n)\|x_n - p\| + \beta_n \|Tx_n - Tp\| \\
&\leqslant (1-\beta_n)\|x_n - p\| + \beta_n \theta \|x_n - p\| \\
&= (1-\beta_n(1-\theta))\|x_n - p\|
\end{aligned} \tag{11}$$

$$\begin{aligned}
\|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n T z_n - p\| \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n \|Tz_n - Tp\| \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n \theta \|z_n - p\| \\
&= (1-\alpha_n)\|x_n - p\| + \alpha_n \theta [(1-\beta_n(1-\theta))\|x_n - p\|] \\
&\leqslant (1-\alpha_n)\|x_n - p\| + \alpha_n (1-\beta_n(1-\theta))\|x_n - p\| \\
&= [1-\alpha_n \beta_n (1-\theta)]\|x_n - p\|
\end{aligned} \tag{12}$$

$$\begin{aligned}
\|x_{n+1} - p\| &= \|T^i y_n - Tp\| \\
&= \|T(T^{i-1} y_n) - Tp\| \\
&\leqslant \theta \|T^{i-1} y_n - p\| \\
&= \theta \|T^{i-1} y_n - Tp\| \\
&\leqslant \theta^2 \|T^{i-2} y_n - p\| \\
&\cdots \\
&\leqslant \theta^i \|y_n - p\| \\
&\leqslant \theta^i \{[1-\alpha_n \beta_n (1-\theta)]\|x_n - p\|\} \\
&= \theta^i [1-\alpha_n \beta_n (1-\theta)]\|x_n - p\|
\end{aligned} \tag{13}$$

Repeated application of the above process yields

$$\left.\begin{aligned}&\|x_{n+1}-p\|\leqslant\theta^{i}[1-\alpha_n\beta_n(1-\theta)]\,\|x_n-p\|,\\&\|x_n-p\|\leqslant\theta^{i}[1-\alpha_{n-1}\beta_{n-1}(1-\theta)]\,\|x_{n-1}-p\|,\\&\|x_{n-1}-p\|\leqslant\theta^{i}[1-\alpha_{n-2}\beta_{n-2}(1-\theta)]\,\|x_{n-2}-p\|,\\&\ldots\\&\|x_2-p\|\leqslant\theta^{i}[1-\alpha_1\beta_1(1-\theta)]\,\|x_1-p\|\end{aligned}\right\}\tag{14}$$

From (14) we get

$$\|x_{n+1}-p\|\leqslant\|x_1-p\|\theta^{in}\prod_{k=1}^{n}[1-\alpha_k\beta_k(1-\theta)]\tag{15}$$

Since $\theta,\ \alpha_n,\beta_n\in(0,1)$, we have

$$1-\alpha_n\beta_n(1-\theta)<1\tag{16}$$

We know that $\forall\ x\in(0,1),\ \ 1-x<e^{-x}$. Using these facts and (16) we get

$$\|x_{n+1}-p\|\leqslant\|x_1-p\|\theta^{in}e^{-(1-\theta)\sum_{k=1}^{n}\alpha_k\beta_k}\tag{17}$$

Taking limit as $n\to\infty$ on both sides of (17), we get $\lim_{n\to\infty}\|x_n-p\|=0$.

Therefore, $\{x_n\}_{n=1}^{\infty}$ converges uniquely to $p\in Fix(T)$.

Theorem 2: *Let $T:C\to C$ satisfy (1). If (9) generates the iterative sequence $\{x_n\}_{n=1}^{\infty}$, with $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfying $\sum_{k=1}^{\infty}\alpha_k\beta_k=\infty$, then (9) is T-stable.*

Proof. Consider a sequence $\{t_n\}_{n=1}^{\infty}\subset C$. Let (9) generate the sequence $x_{n+1}=F_{T,x_n}$ which converge to a unique $x^*\in Fix(T)$ (using Theorem 1) and $\varepsilon_n=\|t_{n+1}-F_{T,x_n}\|$. We will show that

$$\lim_{n\to\infty}\varepsilon_n=0\iff\lim_{n\to\infty}t_n=x^*.$$

Assume that $\lim_{n\to\infty}\varepsilon_n=0$. We have

$$\begin{aligned}\|t_{n+1}-x^*\| &\leqslant \|t_{n+1}-F_{T,x_n}\|+\|F_{T,x_n}-x^*\| \\ &\leqslant \varepsilon_n+\theta^i(1-\alpha_n\beta_n(1-\theta))\|t_n-x^*\| \\ &\leqslant \varepsilon_n+(1-\alpha_n\beta_n(1-\theta))\|t_n-x^*\|\end{aligned}$$

Now, $\theta\in(0,1)$, $\alpha_n,\ \beta_n\in(0,1)\ \forall\ n\in\mathbb{N}$ and $\lim_{n\to\infty}\varepsilon_n=0$.

Using Lemma 1, we get $\lim_{n\to\infty}t_n=x^*$.

Conversely, assume that $\lim_{n\to\infty}t_n=x^*$.

$$\begin{aligned}\varepsilon_n &=\|t_{n+1}-F_{T,x_n}\| \\ &\leqslant \|t_{n+1}-x^*\|+\|F_{T,x_n}-x^*\| \\ &\leqslant \|t_{n+1}-x^*\|+\theta^i(1-\alpha_n\beta_n(1-\theta))\|t_n-x^*\| \\ &\leqslant \|t_{n+1}-x^*\|+(1-\alpha_n\beta_n(1-\theta))\|t_n-x^*\|\end{aligned}$$

This implies that $\lim_{n\to\infty}\varepsilon_n=0$.

Hence the proof.

Theorem 3: *Let $T:C\to C$ satisfy (1) with unique $p\in Fix(T)$. For $t_1=x_1\in C$, let (8) and (9) generate the sequences $\{t_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ respectively, with $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ satisfying*

(i) $\alpha\leqslant\alpha_n<1$ and $\beta\leqslant\beta_n<1$, for some $\alpha,\ \beta>0$ and $\forall\ n\in\mathbb{N}$.

Then for $i>2$, $\{x_n\}_{n=1}^{\infty}$ converges to p faster than $\{t_n\}_{n=1}^{\infty}$.

Proof: (15) gives

$$\|x_{n+1}-p\|\leqslant\|x_1-p\|\theta^{in}\prod_{k=1}^{n}[1-\alpha_k\beta_k(1-\theta)] \tag{18}$$

Using assumption (i) to (18), we get

$$\|x_{n+1}-p\|\leqslant\theta^{in}[1-\alpha\beta(1-\theta)]^n\|x_1-p\| \tag{19}$$

$$\text{Let } a_n=\theta^{in}[1-\alpha\beta(1-\theta)]^n\|x_1-p\| \tag{20}$$

Using (8), we have

$$\begin{aligned}\|v_n - p\| &= \|(1-\beta_n)t_n + \beta_n T t_n - (1-\beta_n+\beta_n)p\| \\ &\leqslant (1-\beta_n)\|t_n - p\| + \beta_n \|Tt_n - Tp\| \\ &\leqslant (1-\beta_n)\|t_n - p\| + \beta_n \theta \|t_n - p\| \\ &\leqslant [1-\beta_n(1-\theta)]\|t_n - p\|\end{aligned} \tag{21}$$

$$\begin{aligned}\|u_n - p\| &= \|(1-\alpha_n)Tt_n + \alpha_n T v_n - p\| \\ &\leqslant (1-\alpha_n)\|Tt_n - Tp\| + \alpha_n \|Tv_n - Tp\| \\ &\leqslant (1-\alpha_n)\theta\|t_n - p\| + \alpha_n \theta \|v_n - p\| \\ &\leqslant (1-\alpha_n)\theta\|t_n - p\| + \alpha_n \theta [1-\beta_n(1-\theta)]\ \|t_n - p\| \\ &= \theta[1-\alpha_n\beta_n(1-\theta)]\|t_n - p\|\end{aligned} \tag{22}$$

$$\begin{aligned}\|t_{n+1} - p\| &\leqslant \|Tu_n - Tp\| \\ &\leqslant \theta\|u_n - p\| \\ &\leqslant \theta^2[1-\alpha_n\beta_n(1-\theta)]\|t_n - p\|\end{aligned} \tag{23}$$

Repeated application of the above process yields

$$\left.\begin{aligned}&\|t_{n+1} - p\| \leqslant \theta^2[1-\alpha_n\beta_n(1-\theta)]\|t_n - p\|, \\ &\|t_n - p\| \leqslant \theta^2[1-\alpha_{n-1}\beta_{n-1}(1-\theta)]\|t_{n-1} - p\|, \\ &\|t_{n-1} - p\| \leqslant \theta^2[1-\alpha_{n-2}\beta_{n-2}(1-\theta)]\|t_{n-2} - p\|, \\ &\ \ldots \\ &\|t_2 - p\| \leqslant \theta^2[1-\alpha_1\beta_1(1-\theta)]\|t_1 - p\|\end{aligned}\right\} \tag{24}$$

From (24), we get

$$\|t_{n+1} - p\| \leqslant \|t_1 - p\|\theta^{2n}\prod_{k=1}^{n}[1-\alpha_k\beta_k(1-\theta)] \tag{25}$$

Using assumption (i) to (25) we obtain

$$\|t_{n+1} - p\| \leqslant \theta^{2n}[1-\alpha\beta(1-\theta)]^n\|t_1 - p\| \tag{26}$$

Let
$$b_n = \theta^{2n}[1-\alpha\beta(1-\theta)]^n\|t_1 - p\| \tag{27}$$

$$\frac{a_n}{b_n} = \frac{\theta^{in}[1-\alpha\beta(1-\theta)]^n \ \|x_1 - p\|}{\theta^{2n}[1-\alpha\beta(1-\theta)]^n \|t_1 - p\|}$$
$$= \frac{\theta^{(i-2)n}\|x_1 - p\|}{\|t_1 - p\|}$$
$$\to 0 \ \text{ as } \ n\to\infty \quad (\text{for } \ i > 2)$$

Hence, (9) is faster than (8).

Example 1: Let $C=[1,7]\subseteq X=\mathbb{R}$ and $T:C\to C$ be defined in C by

$Tx=\sqrt[3]{x+6}$ $\forall x\in C$.

Choose $\alpha_n=\beta_n=\gamma_n=\frac{1}{2}$ $\forall n\in\mathbb{N}$ with $x_1=5$.

Evidently, T satisfies (1) and $Fix(T)=\{2\}$.

We compare the iterative process (9) for $i=3,4$ with the (6), (7) and (8). Numerical values obtained at successive iterations are presented in Table 1.

Table 1

Comparison of convergence of the Modified Picard-Ishikawa iteration against the Picard-S, Picard-Mann, and Picard-Ishikawa iterations.

Step	**Modified Picard-Ishikawa (for $i=4$)**	**Modified Picard-Ishikawa (for $i=3$)**	**Picard-S**	**Picard-Mann**	**Picard-Ishikawa**
1	5.00000000000	5.00000000000	5.00000000000	5.00000000000	5.00000000000
2	2.00007055197	2.00084665354	2.01448573161	2.12620158444	2.12259063800
3	2.00000000178	2.00000025603	2.00007749876	2.00567817894	2.00532360284
4	**2.00000000000**	2.00000000008	2.00000041485	2.00025626919	2.00023180111
5	2.00000000000	**2.00000000000**	2.00000000222	2.00001156763	2.00001009429
6	2.00000000000	2.00000000000	2.00000000001	2.00000052215	2.00000043958
7	2.00000000000	2.00000000000	**2.00000000000**	2.00000002357	2.00000001914
8	2.00000000000	2.00000000000	2.00000000000	2.00000000106	2.00000000083
9	2.00000000000	2.00000000000	2.00000000000	2.00000000004	2.00000000004
10	2.00000000000	2.00000000000	2.00000000000	**2.00000000000**	**2.00000000000**
11	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000
12	2.00000000000	2.00000000000	2.00000000000	2.00000000000	2.00000000000

From Table 1, it is evident that the modified Picard–Ishikawa iteration reaches point $p = 2$ in less number of iterations as compared to the Picard–S, Picard–Mann and Picard–Ishikawa processes. This clearly supports the theoretical conclusion that the iterative process defined in (9) exhibits a faster convergence (6), (7), and (8).

4. Data Dependency

Theorem 4: *Let $T: C \to C$ satisfy (1) with $\tilde{T}$ being its approximate operator. Let (9) generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$ for T. We define $\{\tilde{x}_n\}_{n=1}^{\infty}$ as follows*

$$\begin{aligned} \tilde{x}_{n+1} &= \tilde{T}^i \tilde{y}_n \\ \tilde{y}_n &= (1-\alpha_n)\tilde{x}_n + \alpha_n \tilde{T}\tilde{z}_n \\ \tilde{z}_n &= (1-\beta_n)\tilde{x}_n + \beta_n \tilde{T}\tilde{w}_n \end{aligned} \tag{28}$$

where $\alpha_n,\ \beta_n \in (0,1)$ satisfying

(i) $\dfrac{1}{2} \leqslant \alpha_n (1 - \theta(1 - \beta_n(1-\theta)))\ \ \forall n \in \mathbb{N}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \alpha_n (1 - \theta(1 - \beta_n(1-\theta))) = \infty$

If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ be such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$, then $\|p - \tilde{p}\| \leqslant \dfrac{5\varepsilon}{1-\theta}$, where $\varepsilon > 0$ is a fixed number.

Proof:

$$\begin{aligned} \|x_{n+1} - \tilde{x}_{n+1}\| &= \left\|T^i y_n - \tilde{T}^i \tilde{y}_n\right\| \\ &= \left\|T^i y_n - T^i \tilde{y}_n + T^i \tilde{y}_n - \tilde{T}^i \tilde{y}_n\right\| \\ &\leqslant \left\|T^i y_n - T^i \tilde{y}_n\right\| + \left\|T^i \tilde{y}_n - \tilde{T}^i \tilde{y}_n\right\| \\ &\leqslant \theta\left\|T^{i-1} y_n - T^{i-1}\tilde{y}_n\right\| + \frac{\varepsilon}{i} \\ &\leqslant \theta^2\left\|T^{i-2} y_n - T^{i-2}\tilde{y}_n\right\| + \frac{2\varepsilon}{i} \\ &\cdots \\ &\leqslant \theta^i \left\|y_n - \tilde{y}_n\right\| + \varepsilon \end{aligned} \tag{29}$$

$$\begin{aligned}\|z_n-\tilde{z}_n\| &= \left\|(1-\beta_n)x_n+\beta_n Tx_n-(1-\beta_n)\tilde{x}_n-\beta_n\tilde{T}\tilde{x}_n\right\| \\ &\leqslant (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\left\|Tx_n-\tilde{T}\tilde{x}_n\right\| \\ &\leqslant (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\left\{\|Tx_n-T\tilde{x}_n\|+\left\|T\tilde{x}_n-\tilde{T}\tilde{x}_n\right\|\right\} \\ &\leqslant (1-\beta_n)\|x_n-\tilde{x}_n\|+\beta_n\theta\|x_n-\tilde{x}_n\|+\beta_n\varepsilon \\ &= (1-\beta_n(1-\theta))\|x_n-\tilde{x}_n\|+\beta_n\varepsilon\end{aligned} \tag{30}$$

$$\begin{aligned}\|y_n-\tilde{y}_n\| &= \left\|(1-\alpha_n)x_n+\alpha_n Tz_n-(1-\alpha_n)\tilde{x}_n-\beta_n\tilde{T}\tilde{z}_n\right\| \\ &\leqslant (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\left\|Tz_n-\tilde{T}\tilde{z}_n\right\| \\ &\leqslant (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\left\{\|Tz_n-T\tilde{z}_n\|+\left\|T\tilde{z}_n-\tilde{T}\tilde{z}_n\right\|\right\} \\ &\leqslant (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\theta\|z_n-\tilde{z}_n\|+\alpha_n\varepsilon \\ &\leqslant (1-\alpha_n)\|x_n-\tilde{x}_n\|+\alpha_n\theta\{(1-\beta_n(1-\theta))\|x_n-\tilde{x}_n\|+\beta_n\varepsilon\} \\ &\quad +\alpha_n\varepsilon \\ &= [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\|+\beta_n\varepsilon(1+\beta_n\theta)\end{aligned} \tag{31}$$

From (29) and (31), we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}_{n+1}\| &\leqslant \theta^i\left\{\begin{matrix}[1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\| \\ +\alpha_n\varepsilon(1+\beta_n\theta)\end{matrix}\right\}+\varepsilon \\ &\leqslant [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\| \\ &\quad +\alpha_n\theta^i\varepsilon(1+\beta_n\theta)+\varepsilon \\ &= [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\| \\ &\quad +\alpha_n\theta^i\varepsilon+\alpha_n\beta_n\theta^{i+1}\varepsilon+\varepsilon \\ &\leqslant [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\|+\alpha_n\beta_n\varepsilon+2\varepsilon\end{aligned} \tag{32}$$

From assumption (i) we obtain

$$\begin{aligned}\|x_{n+1}-\tilde{x}_{n+1}\| &\leqslant [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\| \\ &\quad +\alpha_n\beta_n\varepsilon+2(1-\alpha_n\beta_n+\alpha_n\beta_n)\varepsilon \\ &\leqslant [1-\alpha_n(1-\theta(1-\beta_n(1-\theta)))]\|x_n-\tilde{x}_n\|\end{aligned}$$

$$+\alpha_n\beta_n(1-\theta)\frac{5\varepsilon}{1-\theta} \tag{33}$$

Let

$$\xi_n := ||x_n - \tilde{x}_n||,\ \zeta_n := \alpha_n(1-\theta(1-\beta_n(1-\theta))) \in (0,1),\ \lambda_n := \frac{5\varepsilon}{1-\theta}$$

Using Lemma 2, we get

$$0 \leqslant \limsup_{n\to\infty} \|x_n - \tilde{x}_n\| \leqslant \limsup_{n\to\infty} \frac{5\varepsilon}{1-\theta} \tag{34}$$

Now, $\lim_{n\to\infty} x_n = p$, and by assumption, $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$. Therefore, we have $\|p-\tilde{p}\| \leqslant \frac{5\varepsilon}{1-\theta}$.

5. Conclusion

In this paper, a modified Picard–Ishikawa hybrid iterative process has been proposed and analyzed in detail, with convergence, stability and data dependence results being rigorously established for contraction mappings in Banach spaces. The theoretical analysis, supported by a numerical illustration, demonstrates that the proposed iteration is superior to Picard–S iteration in terms of convergence rate. The study highlights that appropriate modifications of existing iterative algorithms—particularly through the use of suitable iterates of contraction mappings—can lead to significant improvements in convergence performance. These findings suggest considerable scope for further investigation into accelerated fixed point iterative methods and their extensions to broader classes of nonlinear mappings.

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