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Some results in multiplicative-modular b-metric spaces and their applications

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Abstract

This article introduces a novel Multiplicative-Modular b-Metric Space (MMbMS), which generalizes two Metric Spaces by relaxing the Triangle Inequality. We establish its properties and prove the Banach and Kannan Contraction Theorems with adjusted constants, ensuring their validity in this broader framework. The Banach Contraction Principle is extended to this generalized space, proving its validity under appropriate Contraction Conditions, and its potential applications in solving Integral and Differential Equations are explored.

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1. Introduction

Czerwik [2] developed the idea of b-metric Spaces in 1993 as a way to generalize Metric Spaces. It relaxes the Triangle Inequality by adding a coefficient $b \geq 1$. This extension has proven useful in Fixed Point (FP) Theory, allowing theorems to hold in broader classes of spaces where standard metrics may not apply.

Bashirov, Kurpınar, and Özyapıcı [5] examined Multiplicative Calculus and formulated foundational theorems, illustrating its applicability through various examples. Multiplicative Calculus provides a straightforward approach to the derivatives of products and quotients. They defined Multiplicative

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Distances between nonnegative reals and positive matrices, laying the groundwork for Multiplicative Metric Spaces.

Building on papers [1, 3-11], this work introduces Multiplicative-Modular b-metric Spaces (MMbMS), which combine Multiplicative Metrics, Modular Structures, and the b-relaxation parameter. These spaces generalize Metric Spaces where distances are multiplicative (in $[1, \infty)$) rather than additive, with a relaxed Triangle Inequality involving a coefficient $b \geq 1$. The MMbMS framework is novel and holds potential for significant study.

2. Main Results

Definition 2.1: For $\Delta \neq \phi$, $b \geq 1$, and $P: (0, \infty) \times \Delta \times \Delta \to [1, \infty]$ a function with the following properties, for $\hbar, \varpi, \nu \in \Delta$ and $\eta, \mu > 0$:

(i) $P_\eta(\hbar, \varpi) = 1$ if and only if $\hbar = \varpi$,

(ii) $P_\eta(\hbar, \varpi) = P\eta(\varpi, \hbar)$,

(iii) $P_{\eta+\mu}(\hbar, \varpi) \leq [P_\eta(\hbar, \nu) \cdot P_\mu(\nu, \varpi)]b$.

Then (Δ, P) is defined as a Multiplicative-Modular b-metric Space (MMbMS). We write $P(\eta, \hbar, \varpi) = P_\eta(\hbar, \varpi)$.

Definition 2.2: Let (Δ, P) be a MMbMS.

(i) A sequence $\{\hbar_n\}$ is P-convergent to $\hbar \in \Delta$ if $P\eta(\hbar_n, \hbar) \to 1$ as $n \to \infty$ for all $\eta > 0$.

(ii) A sequence $\{\hbar_n\}$ is P-Cauchy if $P\eta(\hbar_m, \hbar_n) \to 1$ as $m, n \to \infty$ for all $\eta > 0$.

(iii) The MMbMS (Δ, P) is P-complete if every P-Cauchy sequence converges to some $\hbar \in \Delta$.

Example 2.3: Let $\Delta = [0, \infty)$, $b = 2$, and define $P : (0, \infty) \times \Delta \times \Delta \to [1, \infty)$ by $P_\eta(\hbar, \varpi) = a(\hbar-\varpi)^2/\eta$, where $a > 1$ is a fixed real number, for all $\hbar, \varpi \in \Delta$ and $\eta > 0$. Then (Δ, P) is a MMbMS.

Hence, $P_{\eta+\mu}(\hbar, \varpi) \leq [P_\eta(\hbar, \nu) \cdot P(\nu, \varpi)]^2$, satisfying the Triangle Inequality for $b = 2$.

Example 2.4: Consider a b-metric Space (Δ, σ) with $b \geq 1$, so that $\sigma(\hbar, \varpi) \leq b[\sigma(\hbar, \nu) + \sigma(\nu, \varpi)]$ for all $\hbar, \varpi, \nu \in \Delta$. Define the function $P : (0, \infty) \times \Delta \times \Delta \to [1, \infty)$ by $P_\eta(\hbar, \varpi) = e\sigma(\hbar,\varpi)/\eta$ for all $\hbar, \varpi \in \Delta$ and $\eta > 0$. Then (Δ, P) is a Multiplicative-Modular b-metric Space (MMbMS).

Definition 2.5: Let (Δ, P) be a MMbMS with $b \geq 1$. If for any $\hbar, \varpi \in \Delta$ and $\eta \in (0, \infty)$, there exists a real constant $k \in [0, 1/b)$ such that $P_\eta(R\hbar, R\varpi) \leq [P_\eta(\hbar, \varpi)]^k$, then a mapping $R : \Delta \to \Delta$ is called a Multiplicative-b-Modular Contraction.

Theorem 2.6: (Banach Contraction Principle for MMbMS).

Let $R: \Delta \to \Delta$ be a Multiplicative-b-Modular Contraction of the Complete MMbMS (Δ, P). Then, there exists a unique Fixed Point in R.

Proof: Let $\hbar_0 \in \Delta$. Define $\{\hbar_n\}$ in Δ such that $\hbar n = R\hbar_{n-1}$, n = 1, 2, 3, ….

By the Multiplicative-b-Modular Contraction property of R,

$$P_\eta(\hbar_n, \hbar_{n+1}) = P_\eta(R\hbar_{n-1}, R\hbar_n) \leq \ldots \leq [P_\eta(\hbar_0, \hbar_1)]k^n.$$

For m, n $\in \mathbb{N}$ with m > n, let M = m − n. Using the MMbMS Triangle Inequality,

$$P_\eta(\hbar_n, \hbar_m) \leq [P\eta(\hbar_0, \hbar_1)]b\cdot k^n/(1-k).$$

As n, m → ∞, since k < 1/b, the exponent b · kn/(1 − k) → 0. Thus, lim $P_\eta(\hbar_n, \hbar_m) = 1$. By completeness, there exists $\hbar \in \Delta$ such that $\hbar_n \to \hbar$ as n → ∞. Considering

$$P_\eta(R\hbar, \hbar) \leq [P_\eta/2(\hbar, \hbar_{n-1})]k \cdot P_\eta/2(\hbar_n, \hbar)]b,$$

as n → ∞ both factors tend to 1, so $P_\eta(R\hbar, \hbar) \leq 1$. Since $P_\eta(R\hbar, \hbar) \geq 1$, we obtain $P_\eta(R\hbar, \hbar) = 1$, implying $R\hbar = \hbar$. Uniqueness follows: if ϖ is also a Fixed Point, then $P_\eta(\hbar, \varpi) = P_\eta(R\hbar, R\varpi) \leq [P_\eta(\hbar, \varpi)]k$. Since k < 1, this forces $P\eta(\hbar, \varpi) = 1$, so $\hbar = \varpi$.

Hence, R has a unique Fixed Point.

Theorem 2.7: (Kannan-Type Contraction for MMbMS). *Consider a Complete MMbMS (Δ, P) with b ≥ 1, and assume that the mapping R : Δ → Δ satisfies the Contraction Condition*

$$P_\eta(R\hbar, R\varpi) \leq [P_\eta(R\hbar, \hbar) \cdot P_\eta(R\varpi, \varpi)]^k,$$

where k ∈ [0, 1/(1+b)) is a constant. Then R has a unique Fixed Point.

Proof: Choose $\{\hbar_n\} \in \Delta$ with $\hbar n = R\hbar_{n-1} = R_n\hbar_0$, n = 1, 2, 3, …. By the Contraction Condition, setting an = $P_\eta(\hbar_n, \hbar_{n+1})$, we derive $a_n \leq ah_{n-1}$ where h = k/(1−k). Since k < 1/(1+b), we have h < 1. For m > n, with M = m − n,

$$P_\eta(\hbar n, \hbar m) \leq [P_\eta/M(\hbar_0, \hbar_1)]b\cdot h^n/(1-h) \to 1.$$

Thus $\{\hbar n\}$ is a P-Cauchy sequence. By completeness, there exists $\hbar \in \Delta$ such that $\hbar n \to \hbar$. For the Fixed Point verification, $P\eta/2(\hbar n, R\hbar) \leq [P\eta/2(\hbar n-1, \hbar)\cdot P\eta/2(\hbar, R\hbar)]k$. Letting n → ∞ yields $P\eta(\hbar, R\hbar) \leq [P\eta/2(\hbar, R\hbar)]bk$. If $P\eta/2(\hbar, R\hbar) > 1$ and bk < 1, this is a contradiction, so $P\eta/2(\hbar, R\hbar) = 1$, giving $R\hbar = \hbar$. If ϖ is another Fixed Point, $P_\eta(\hbar, \varpi) = P_\eta(R\hbar, R\varpi) \leq [P\eta(\hbar, \hbar) \cdot P\eta(\varpi, \varpi)]k = 1$, so $\hbar = \varpi$. Hence R has a unique Fixed Point.

Corollary 1: *Let (Δ, P) be a P-complete MMbMS with coefficient b ≥ 1. Let $\hbar_0 \in \Delta$, η > 0, and $B\lambda(\hbar_0, r) = \{y \in \Delta : P_\eta(\hbar_0, y) \leq r\}$ be a w-closed ball with radius r > 1. Suppose a mapping f : Δ → Δ satisfies*

$$P_\eta(f\hbar, fy) \leq [P_\eta(\hbar, y)]a \cdot [P_\eta(\hbar, f\hbar)]b \cdot [P\eta(y, fy)]c,$$

for all $\hbar$, y ∈ $B_\lambda(\hbar_0, r)$, with 0 < b(a + b + c) < 1 and $P_\eta(\hbar_0, f\hbar_0) \leq$ r1/(1−b(a+b+c)). Then f has a unique Fixed Point in $B\lambda(\hbar_0, r)$.

Proof: Define $\{\hbar_n\}$ by $\hbar_{n+1} = f\hbar n$. Since $P_\eta(\hbar_0, \hbar_1) \leq r1/(1-b(a+b+c)) \leq r$, we have $\hbar_1 \in B_\lambda(\hbar_0, r)$. By induction, all iterates remain in the ball. Setting h =

$(a+b)/(1-c) < 1$, we obtain $a_n \le a h^{n}_0$, and Cauchy convergence follows exactly as in Theorem 2.6. The limit $\hbar^* \in B_\lambda(\hbar_0, r)$ satisfies $f\hbar^* = \hbar^*$, and uniqueness gives $\hbar^* = \vartheta$.

3. Applications

We apply the Banach-type Contraction Theorem in MMbMS to establish the solutions to Fredholm Integral Equations.

Theorem 3.1: *For the following Non-linear Fredholm Integral Equation:*

$$u(\hbar) = I(\hbar) + \int_0^1 J(\hbar, R, u(t))\, dt, \quad \hbar \in [0, 1],$$

where $I : [0, 1] \to \mathbb{R}$ and $J : [0, 1]2 \times \mathbb{R} \to \mathbb{R}$ are continuous with the following conditions:

1. $|J(\hbar, R, u) - J(\hbar, R, v)| \le M|u - v|$ for all $\hbar, R \in [0, 1]$ and $u, v \in \mathbb{R}$, with $M2 < 1/2$.
2. $\sup_{\hbar\in[0,1]} \int_0^1 |J(\hbar, R, u(t))|\, dt < \infty$.

Then the Integral Equation has a unique solution in the space C[0, 1].

Proof: Define $P_\eta(u, v) = \exp(d(u, v)/\eta)$ for $\eta > 0$. Then (Δ, P) is an MMbMS. The integral operator R satisfies

$$P_\eta(Ru, Rv) = \exp(d(Ru, Rv)/\eta) \le \exp(M2d(u, v)/\eta) = [P_\eta(u, v)]^{M^2}.$$

Since $M2 < 1/2$, setting $k = M2$, Theorem 2.6 guarantees that R has a unique Fixed Point $u^* \in \Delta$, which is the unique solution of the Integral Equation.

4. Conclusions

The introduction of MMbMS advances Fixed Point results by unifying Multiplicative, Modular, and b-metric structures. This paper establishes the foundational properties of MMbMS and proves generalized Banach and Kannan Contraction Theorems, adapting Contraction Constants for this broader framework. The application to Fredholm Integral Equations demonstrates the practical utility of these results in obtaining unique solutions.

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