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Redefining Kannan contractions through harmonic mean perspectives

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Abstract

This paper introduces a novel approach to Kannan-type contractions by integrating the concept of harmonic mean, offering a fresh perspective on fixed-point (FP) theory in metric spaces (MS). We introduce a broad framework that leverages the harmonic mean to refine the contractive conditions of Kannan-type mappings in metric spaces. Our findings extend numerous pioneering research from the existing literature, providing a richer understanding of the topic. Additionally, an applied example is presented to validate our theoretical findings. More over the existence of solutions for specific type of functional equation that emerge in dynamic programming is explored.

Subject Classification: *47H10, 54H25.*

Keywords: *Banach contraction, Kannan contraction, Metric completeness, Suzuki contraction.*

1. Introduction

In 1922, a pivotal breakthrough in FP theory was achieved by Banach [1]. He introduced the renowned Banach Contraction Principle (BCP), paving a significant path for research in contraction mappings:

Theorem 1.1: *Suppose* $(\mathcal{E}, \mathfrak{d})$ *is a complete metric space (CMS) with* $\mathcal{T}: \mathcal{E} \to \mathcal{E}$ *such that*

$$\mathfrak{d}(\mathcal{T}\mathfrak{s}, \mathcal{T}\mathfrak{t}) \leq r\, \mathfrak{d}(\mathfrak{s}, \mathfrak{t})\ \forall\ \mathfrak{s}, \mathfrak{t} \in \mathcal{E};\ r \in [0, 1).$$

Then $\mathcal{T}$ has a unique FP.

In a contraction mapping, the distance between the images of any two points is reduced as compared to the distance between the points themselves.

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The BCP stands as one of the most celebrated principles in FP theory, having been expanded, generalized, and unified by numerous researchers in various distinct ways (see, for example, [2], [3], [4] and the references therein).

In 1968, by adopting an entirely different condition, Kannan [3] proved a FP theorem for operators, which need not be contractions. Actually, the FP theorem for a discontinuous map was first put forth by Kannan. The FP theorem that follows, also known as the Kannan contraction theorem (KCT), was in fact demonstrated by Kannan [3].

Theorem 1.2: *Suppose $Ę$ is a CMS and $Ŧ: Ę \to Ę$. Assume there exists $a \in [0, 1/2)$ such that*

$$đ(Ŧş, Ŧt) \leq a\, đ(ş, Ŧş) + a\, đ(t, Ŧt)\ \forall\ ş, t \in Ę.$$

Then $Ŧ$ has a unique FP in $Ę$.

Kannan's theorem motivated numerous extensions and generalizations of the BCP and KCT.

In 2018, the celebrated Kannan theorem was re-examined by taking the interpolation theory into consideration by Karapinar [5] as outlined below:

Theorem 1.3: *Suppose $Ę$ is a CMS and $Ŧ: Ę \to Ę$. Assume $\exists\ ƛ \in [0, 1)$ and $\kappa \in (0, 1)$ such that*

$đ(Ŧş, Ŧ t) \leq ƛ\ [đ(ş, Ŧş)]^{\kappa}[đ(t, Ŧt)]^{1-\kappa}$ for each $ş, t \in Ę$ with $ş \neq Tş$.

Then $Ŧ$ has a FP.

The aforementioned theorems, have all contributed to the enormous expansion of FP theorems and their various applications. Nevertheless, the contractive or contractive conditions must hold for every point $ş, t$ in the domain in order for any of these FP theorems to apply. Thus, one should naturally anticipate the day when this condition is significantly loosened without compromising the theorem's conclusions. This laid the groundwork for FP theorems of the Suzuki type.

In 2007, a notable prominent statement in the field of metric FP is Suzuki contraction theorem, was obtained by Suzuki [6]. It characterizes the metric completeness and was a new kind of generalization of BCP (see also [7] and [8]).

The Kannan version of the Suzuki contraction theorem, a novel and robust extension, was demonstrated in the following way by Kikkawa and Suzuki [9] in 2008.

Theorem 1.4: *Take a function $ς$ that is non-increasing from [0, 1) to $\left(\frac{1}{2}, 1\right]$ and is defined as*

$$ς(\alpha) = \begin{cases} 1 & \text{if } 0 \leq \alpha \leq \frac{1}{\sqrt{2}} \\ \frac{1}{1+\alpha} & \text{if } \frac{1}{\sqrt{2}} \leq \alpha < 1. \end{cases}$$

Suppose E be a CMS and $T: E \to E$. Let $a \in \left[0, \frac{1}{2}\right)$ and assume $\exists\ \alpha = \frac{a}{(1-a)} \in [0, 1)$ in a way that,

$\varsigma(\alpha)\ d(s, Ts) \leq d(s, t)$ implies $d(Ts, Tt) \leq a\ d(s, Ts) + a\ d(t, Tt)\ \forall\ s, t \in E$.

Then T has a unique FP.

This paper introduces a fixed-point theorem that generalizes the BCP and KCT. While numerous extensions of the BCP and KCT exist, our approach introduces a novel direction, stemming from the concept of the harmonic mean for a particular value of α, that is $\alpha = \frac{1}{2}$.

2. Main Result

First, let's start by redefining Suzuki type Kannan contraction through harmonic mean perspective.

Definition 2.1: Suppose (E,d) is a MS and $T: E \to E$ be a selfmap. We say T is interpolative Suzuki-type Kannan contraction in rational form if $\exists\ \alpha, \lambda \in [0,1)$ such that

$$\frac{1}{2}d(s,Ts) \leq d(s, t) \text{ implies } d(Ts, Tt) \leq \lambda \left[\frac{d(s,Ts)\ d(t, Tt)}{\alpha d(s,Ts)+(1-\alpha)d(t, Tt)}\right] \forall s,t \in E.$$

Theorem 2.2: *Suppose (E, d) is a CMS and T is interpolative Suzuki-type Kannan contraction in rational form. Then T has a FP in E.*

***Proof*:** Let $s_0 \in E$ and consider $\{s_n\}$, given as $s_n = T^n(s_0)\ \forall\ n \in N$. If $\exists\ n_0 \in N$ such that $s_{n_0} = s_{n_0+1}$, then s_{n_0} is a FP of T which completes the proof. So assume that $s_n \neq s_{n+1}\ \forall\ n \geq 0$.

Now $\frac{1}{2}d(s_{n-1},Ts_{n-1}) = \frac{1}{2}d(s_{n-1},s_n) < d(s_{n-1},s_n)$ and this implies

$$d(Ts_{n-1},Ts_n) = d(s_n, s_{n+1})) \leq \lambda \left[\frac{d(s_{n-1}, Ts_{n-1})\ d(s_n, Ts_n)}{\alpha d(s_{n-1}, Ts_{n-1})+(1-\alpha)d(s_n, Ts_n)}\right].$$

That is,

$$d(s_n, s_{n+1}) \leq \lambda \left[\frac{d(s_{n-1}, s_n)\ d(s_n, s_{n+1})}{\alpha d(s_{n-1}, s_n)+(1-\alpha)d(s_n,s_{n+1})}\right].$$

That is, $\alpha d(s_{n-1},s_n) + (1-\alpha)d(s_n,s_{n+1}) \leq \lambda\ d(s_{n-1},s_n)$.

That is, $d(s_n,s_{n+1}) \leq \frac{\lambda-\alpha}{1-\alpha}\ d(s_{n-1},s_n)$.

Let $\beta = \frac{\lambda-\alpha}{1-\alpha}$.

Since $0 \leq \lambda < 1$, $0 \leq \alpha < 1$ implies $\beta = \frac{\lambda-\alpha}{1-\alpha} < 1$.

Now, $d(s_n,s_{n+1}) \leq \beta d(s_{n-1},s_n);\ \beta < 1\ \forall n \in N$.

This implies, $d(s_n,s_{n+1}) \leq \beta^n d(s_0,s_1)$.

And, hence $\sum_{n=1}^{\infty} d(s_n,\ s_{n+1}) \leq \sum_{n=1}^{\infty} \beta^n d(s_0,\ s_1) = (\sum_{n=1}^{\infty} \beta^n) d(s_0,s_1) < \infty$.

This implies, the sequence $\{\xi_n\}$ is Cauchy in E. As E is CMS, $\exists$ some $z \in E$ with $\xi_n \to z$.

We shall now establish that z is a FP of T.

Suppose not, that is, $z \neq Tz$ and so $d(z,Tz) > 0$.

We assert that $\frac{1}{2}d(\xi_n,T\xi_n) \leq d(\xi_n,z)$ or $\frac{1}{2}d(T\xi_n, T^2\xi_n) \leq d(T\xi_n, z)$.

Suppose not $\frac{1}{2}d(\xi_n,T\xi_n) > d(\xi_n,z)$ and $\frac{1}{2}d(T\xi_n, T^2\xi_n) > d(T\xi_n, z)$.

Now, $d(\xi_n,\xi_{n+1}) \leq d(\xi_n, z) + d(z,\xi_{n+1}) < \frac{1}{2}d(\xi_n,T\xi_n) + \frac{1}{2}d(T\xi_n, T^2\xi_n)$

$$= \frac{1}{2}d(\xi_n,\xi_{n+1}) + \frac{1}{2}d(\xi_{n+1},\xi_{n+2})$$
$$= \frac{1}{2}[d(\xi_n,\xi_{n+1}) + \beta d(\xi_n,\xi_{n+1})]$$
$$= \frac{1}{2}(1+\beta)d(\xi_n,\xi_{n+1}).$$

That is, $d(\xi_n,\xi_{n+1}) < d(\xi_n,\xi_{n+1})$, a contradiction.

So, for every $n \in N$, either

$\frac{1}{2}d(\xi_n,\xi_{n+1}) \leq d(\xi_n,z)$ or $\frac{1}{2}d(\xi_{n+1},\xi_{n+2}) \leq d(\xi_{n+1},z)$.

If, $\frac{1}{2}d(\xi_n,\xi_{n+1}) \leq d(\xi_n,z)$, then $d(T\xi_n,Tz) \leq \lambda\left[\frac{d(\xi_n,T\xi_n)\,d(z,Tz)}{\alpha d(\xi_n,T\xi_n)+(1-\alpha)d(z,Tz)}\right]$.

That is,

$$d(\xi_{n+1},Tz) \leq \lambda\left[\frac{d(\xi_n,\xi_{n+1})\,d(z,Tz)}{\alpha d(\xi_n,\xi_{n+1})+(1-\alpha)d(z,Tz)}\right]. \tag{2.1}$$

If, $\frac{1}{2}d(\xi_{n+1},\xi_{n+2}) \leq d(\xi_{n+1},z)$, then

$$d(T\xi_{n+1},Tz) \leq \lambda\left[\frac{d(\xi_{n+1},T\xi_{n+1})\,d(z,Tz)}{\alpha d(\xi_{n+1},T\xi_{n+1})+(1-\alpha)d(z,Tz)}\right].$$

That is,

$$d(\xi_{n+2},Tz) \leq \lambda\left[\frac{d(\xi_{n+1},\xi_{n+2})\,d(z,Tz)}{\alpha d(\xi_{n+1},\xi_{n+2})+(1-\alpha)d(z,Tz)}\right]. \tag{2.2}$$

Letting $n \to \infty$ in (2.1), (2.2), we obtain $d(z,Tz) = 0$.

In other words, $z = Tz$. Thus, z is FP of T.

The following example illustrate that our result is generalizations of Theorem 1.1 and Theorem 1.2.

Example 2.3: Assume $E = \{\xi, t, z, v\}$ is endowed with a metric d satisfying $d(\xi, \xi) = d(t, t) = d(z, z) = d(v, v) = 0$, $d(\xi, t) = d(t, \xi) = d(\xi, z) = d(z, \xi) = 1$, $d(v, t) = d(t, v) = d(z, v) = d(v, z) = 1$, $d(t, z) = d(z, t) = 1.5$, $d(v, \xi) = d(\xi, v) = 1.2$

We define a self-map T on E by $T(\xi) = \xi$, $T(t) = z$, $T(z) = \xi$, $T(v) = v$.

It is clear that T is neither Banach contraction nor Kannan contraction. Indeed, there is no $\lambda \in [0, 1)$ such that $d(T\xi,Tt) \leq \lambda d(\xi,t)$ for all $\xi, t \in E$. And there is no $\lambda \in [0, 1/2)$ such that $d(T\xi,Tt) \leq \lambda(d(\xi,T\xi) + d(t,Tt))$ for all $\xi, t \in E$. On the other hand, for $\lambda = 0.99$ and $\alpha = 0.1$, the requirements of Theorem 2.2 are met and T has ξ, v as FP.

3. Application to functional equations arising in Dynamic Programming problem

We exemplify our results with the study of existence for the solution of functional equation

$$u(\text{ş}) = \sup_{\text{ŧ} \in D}\{\text{g}(\text{ş}, \text{ŧ}) + \text{J}(\text{ş}, \text{ŧ}, u(\text{T}(\text{ş}, \text{ŧ}))\}. \tag{3.1}$$

which appear in dynamic programming where $\text{ş} \in G$; G is state space, $Ł$ is decision space, $T: G \times Ł \to G$, $g: G \times Ł \to R$ and $J: G \times Ł \times R \to R$ are given mappings.

We consider a non empty set G and work in the space $B(G)$ which denotes the family of real functions which are bounded and defined on G. Define the norm $\|.\|$ given by $\|u\|_\infty = \sup_{\text{ş} \in G} |u(\text{ş})| \; \forall \; u \in B(G)$. We have that $(B(G), \|.\|_\infty)$ serves as a Banach space and

$$đ(u, v) = \sup_{\text{ş} \in G} |u(\text{ş}) - v(\text{ş})| \; \forall \; u, v \in B(G).$$

Theorem 3.1: *Suppose* $g: G \times Ł \to R$ *and* $J: G \times Ł \times R \to R$ *are bounded. Assume* $\exists \, 0 < \lambda, \alpha < 1$ *so that for all* $(\text{ş}, \text{ŧ}) \in G \times Ł$, $g, h \in B(G)$ *and* $\epsilon \in G$,

$$\frac{1}{2} \, |h(\epsilon) - Kh(\epsilon)| \; \leq \; |h(\epsilon) - g(\epsilon)| \tag{3.2}$$

implies

$$|J(\text{ş}, \text{ŧ}, h(\text{T}(\text{ş}, \text{ŧ})) - J(\text{ş}, \text{ŧ}, g(\text{T}(\text{ş}, \text{ŧ}))| \leq \lambda \left[\frac{đ(h(\epsilon), Kh(\epsilon)) \; đ(g(\epsilon), Kg(\epsilon))}{\alpha \, đ(h(\epsilon), Kh(\epsilon)) + (1-\alpha) đ(g(\epsilon), Kg(\epsilon))}\right] \tag{3.3}$$

where

$$Kh(\text{ş}) = \sup_{y \in Ł}\{g(\text{ş}, \text{ŧ}) + J(\text{ş}, \text{ŧ}, h(\text{T}(\text{ş}, \text{ŧ})))\}. \tag{3.4}$$

Proof : Since K is a self map of $B(G)$. Pick $\text{ş} \in G$ and $h_1, h_2 \in B(G)$, let $đ(h_1, h_2) = \sup\{|h_1(\text{ş}) - h_2(\text{ş})| : \text{ş} \in G\}$. Then $(B(G), đ)$ is a CMS.

Let $\mu \in R^+$, we can choose $\text{ŧ}_1, \text{ŧ}_2 \in Ł$ such that

$$Kh_i < g(\text{ş}, \text{ŧ}_i) + J(\text{ş}, \text{ŧ}_i, h_i(\text{ş}_i)) + \mu, \tag{3.5}$$

where $\text{ş}_i = \text{T}(\text{ş}, \text{ŧ}_i)$, $i = 1, 2$.

Further, we have

$$Kh_1(\text{ş}) \geq g(\text{ş}, \text{ŧ}_2) + J(\text{ş}, \text{ŧ}_2, h_1(\text{ş}_2)), \tag{3.6}$$

$$Kh_2(\text{ş}) \geq g(\text{ş}, \text{ŧ}_1) + J(\text{ş}, \text{ŧ}_1, h_2(\text{ş}_1)), \tag{3.7}$$

Therefore, (3.2) becomes

$$\frac{1}{2} \, |h_1(\text{ş}) - Kh_1(\text{ş})| \; \leq \; |h_1(\text{ş}) - h_2(\text{ş})| \tag{3.8}$$

Let $M(\lambda, \alpha) = \lambda \left[\frac{đ(h_1(\epsilon), Kh_1(\epsilon)) \; đ(h_2(\epsilon), Kh_2(\epsilon))}{\alpha đ(h_1(\epsilon), Kh_1(\epsilon)) + (1-\alpha) đ(h_2(\epsilon), Kh_2(\epsilon))}\right]$ (3.9)

From (3.5), (3.7) and (3.8), we have

$$Kh_1(\text{ş}) - Kh_2(\text{ş}) < J(\text{ş}, \text{ŧ}_1, h_1(\text{ş}_1)) - J(\text{ş}, \text{ŧ}_1, h_2(\text{ş}_1)) + \mu$$

$$\leq \left|\mathcal{F}(\mathfrak{s}, \mathfrak{t}_1, \mathfrak{h}_1(\mathfrak{s}_1)) - \mathcal{F}(\mathfrak{s}, \mathfrak{t}_1, \mathfrak{h}_2(\mathfrak{s}_1))\right| + \mu$$

$$\leq \lambda \left[\frac{\mathfrak{d}(\mathfrak{h}_1(\mathfrak{e}), \mathcal{K}\mathfrak{h}_1(\mathfrak{e}))\, \mathfrak{d}(\mathfrak{h}_2(\mathfrak{e}), \mathcal{K}\mathfrak{h}_2(\mathfrak{e}))}{\alpha \mathfrak{d}(\mathfrak{h}_1(\mathfrak{e}), \mathcal{K}\mathfrak{h}_1(\mathfrak{e})) + (1-\alpha)\mathfrak{d}(\mathfrak{h}_2(\mathfrak{e}), \mathcal{K}\mathfrak{h}_2(\mathfrak{e}))}\right] + \mu = \mathcal{M}(\lambda, \alpha) + \mu. \tag{3.10}$$

Similarly, from (3.5), (3.6) and (3.8), we have

$$\mathcal{K}\mathfrak{h}_2(\mathfrak{s}) - \mathcal{K}\mathfrak{h}_1(\mathfrak{s}) \leq \mathcal{M}(\lambda, \alpha) + \mu. \tag{3.11}$$

So, from (3.10) and (3.11), we obtain

$$|\mathcal{K}\mathfrak{h}_1(\mathfrak{s}) - \mathcal{K}\mathfrak{h}_2(\mathfrak{s})| \leq \mathcal{M}(\lambda, \alpha) + \mu. \tag{3.12}$$

Since this inequality (3.12) is true for any $\mathfrak{s} \in G$, and $\mu > 0$ is inconsistent, on taking supremum, from (3.8), we determine

$$\frac{1}{2}\, \mathfrak{d}(\mathfrak{h}_1, \mathcal{K}\mathfrak{h}_1) \leq \mathfrak{d}(\mathfrak{h}_1, \mathfrak{h}_2) \text{ implies } \mathfrak{d}(\mathcal{K}\mathfrak{h}_1, \mathcal{K}\mathfrak{h}_2) \leq \mathcal{M}(\lambda, \alpha).$$

So $\mathcal{K}$ has a FP say $\mathfrak{h}^*$, that is, $\mathfrak{h}^*(\mathfrak{s})$ is a solution of (3.1).

Conclusion

This paper outlines a fresh perspective to Kannan-type contractions by incorporating the harmonic mean, thereby establishing a generalized framework that refines the contractive conditions for mappings in metric spaces. This work broadens and unifies existing results in FP theory, offering a deeper and more comprehensive understanding of the subject. This work not only enriches the existing literature but also opens new avenues for further exploration in the analysis of FP results and their applications in MS.

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