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Multivalued $(\theta; \Gamma)$-generalized contraction in strong b-metric spaces with order structure

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Abstract

This paper establishes fixed-point theorems for multivalued mappings in the setting of complete partially ordered strong b-metric spaces. By employing appropriate rational contractive conditions together with suitable auxiliary functions, the obtained results extend and unify some well-known fixed-point theorems of literature. An illustrative example is presented to demonstrate and validated the applicability of the proposed result in theoretical framework.

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1. Introduction And Foundational Concept

The investigation of fixed points for multivalued functions covers a broad spectrum of functional analysis. In past three decades number of fixed point

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results has been obtained for multivalued maps within the framework of partially ordered metric spaces under certain inequalities defined by a control functions. The idea behind weak contraction principle was firstly placed by Alber and Guerre-Delabriere [1] in Banach Space, later Rhoades [2] utilized this notion to admit fixed points in complete metric spaces under some additional assumptions. The notion of weak contractions and weak contractive conditions have been extensively studied in last 20 years and so, some of these are [3, 4, 5, 6].

In 2011, Choudhury and Metiya [7] proved certian fixed point theorems in multivalued mappings in partially ordered metric spaces. In 2019, Mani et al. [8] derived common fixed point theorem for two pairs of multivalued mapping with δ-distance. Bakhtin [9] and Czerwik [10] introduced b-metric spaces as a generalization of metric spaces. Differing from the framework of b-metric spaces, strong b-metric spaces(SbMS) was introduced in [11], Doan established several extensions of fixed point theorems for both single-valued and multivalued mappings defined on complete strong b-metric spaces. Several fixed point results for Kannan-type contractions were obtained by Kumari, Rathee, and Mor [12] in 2023, specifically in the setting of strong partial b-metric spaces. Some authors [13, 14] further explored some fixed-point results in generalized metric type spaces and highlighted their theoretical and practical significance.

Motivated by above mentioned discussions, this article determine the criteria under which multivalued functions and the space (strong b–metric space) guarantee that map T admits a unique fixed point.

Following are the definitions which are required to prove our main result and we denote the set $[0,\infty) = \mathbb{E}$ throughout this paper.

Consider a metric space $(\mathbb{Y},\rho)$. Throughout this work, for each $A \in \mathcal{B}(\mathbb{Y})$ satisfies $A \neq \emptyset$ and is contained in some closed ball of finite radius in $\mathbb{Y}$. For $\mathbb{Y}_1, \mathbb{Y}_2 \in \mathcal{B}(\mathbb{Y})$, functions $d_H(\mathbb{Y}_1,\mathbb{Y}_2)$ and $d_M(\mathbb{Y}_1,\mathbb{Y}_2)$ are defined as follows:

$$d_M(\mathbb{Y}_1,\mathbb{Y}_2) = \sup\{\rho(\varsigma,\tau): \varsigma \in \mathbb{Y}_1, \tau \in \mathbb{Y}_2\}, d_H(\mathbb{Y}_1,\mathbb{Y}_2) = \inf\{\rho(\varsigma,\tau): \varsigma \in \mathbb{Y}_1, \tau \in \mathbb{Y}_2\}.$$

In general, $d_H(\mathbb{Y}_1,\mathbb{Y}_2) \leq d_M(\mathbb{Y}_1,\mathbb{Y}_2)$. For all $\mathbb{Y}_1,\mathbb{Y}_2,\mathbb{Y}_3 \in \mathcal{B}(\mathbb{Y})$, the definition of $d_M(\mathbb{Y}_1,\mathbb{Y}_2)$ yields the following:

$$\begin{aligned} d_M(\mathbb{Y}_1,\mathbb{Y}_2) &= d_M(\mathbb{Y}_2,\mathbb{Y}_1), \\ d_M(\mathbb{Y}_1,\mathbb{Y}_2) &\leq d_M(\mathbb{Y}_1,\mathbb{Y}_3) + d_M(\mathbb{Y}_3,\mathbb{Y}_2), \\ d_M(\mathbb{Y}_1,\mathbb{Y}_2) &= 0 \text{ iff} \mathbb{Y}_1 = \mathbb{Y}_2 = \{\varsigma\}. \end{aligned}$$

Definition 1.1: [15] For nonempty subsets $\mathbb{Y}_1, \mathbb{Y}_2 \subseteq \mathbb{Y}$ of a POSet $(\mathbb{Y},\preceq)$, we write $\mathbb{Y}_1 \prec_1 \mathbb{Y}_2$ if for each $\varsigma \in \mathbb{Y}_1$ $\exists$ $\tau \in \mathbb{Y}_2$ with $\varsigma \preceq \tau$.

Definition 1.2: [11] Let $\mathbb{Y} \neq \phi$ and $\rho: \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}$ be a map. Additionally, assume that $\forall$ $\varsigma,\tau,z \in \mathbb{Y}$ with a constant $s \geq 1$, the following assertions satisfies:

1. $\rho(\varsigma,\tau) \geq 0$,
2. $\rho(\varsigma,\tau) = 0 \Leftrightarrow \varsigma = \tau$,
3. $\rho(\varsigma,\tau) = \rho(\tau,\varsigma)$,
4. $\rho(\varsigma,\tau) \leq \rho(\varsigma,z) + s\,\rho(z,\tau)$.

Then the map ρ is a *strong b-metric* on $\mathbb{Y}$, and the 3-tuples $(\mathbb{Y}, \rho, s)$ are referred to as a strong b-metric space (SbMS).

Definition 1.3: For a sequence $\{\varsigma_\alpha\} \subseteq \mathbb{Y}$ with $\varsigma \in \mathbb{Y}$, in a strong b-metric space $(\mathbb{Y}, \rho)$, we say

1. $\{\varsigma_\alpha\} \to \varsigma$ if $\lim_{\alpha\to\infty}\rho(\varsigma_\alpha, \varsigma) = 0$.
2. $\{\varsigma_\alpha\}$ is Cauchy, if $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that $\rho(\varsigma_\alpha, \varsigma_\beta) < \varepsilon$ for $\beta, \alpha > N$.

2. $(\boldsymbol{\theta}, \boldsymbol{\Gamma})$-Generalized Multivalued Contractions in Ordered Strong $\boldsymbol{b}$-Metric Spaces

Here,we first establish a lemma for our result.

Lemma 2.1: *Consider $\mathbb{Y} \neq \phi$ together with a map $\rho: \mathbb{Y} \times \mathbb{Y} \to \mathbb{E}$ such that $(\mathbb{Y}, \rho)$ is a SBMS. Additionally, assume that the sequence $\{\varsigma_\alpha\} \in \mathbb{Y}$ satisfies*

$$\lim_{\alpha\to\infty}\rho(\varsigma_\alpha, \varsigma_{\alpha+1}) = 0. \tag{1}$$

If a sequence $\{\varsigma_\alpha\}$ fails to be Cauchy in $\mathbb{Y}$, then one can find a $\varepsilon > 0$ together with two sequences $\{\beta_p\}$ and $\{\alpha_p\}$ with $\beta_p > \alpha_p > p \forall p \in \mathbb{N}$, satisfying

$$\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) \geq \varepsilon \text{ and } \rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p-1}}) < \varepsilon.$$

Then following holds:

$$\lim_{p\to\infty}\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) = \varepsilon, \lim_{p\to\infty}\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p+1}}) = \varepsilon, \lim_{p\to\infty}\rho(\varsigma_{\alpha_p}, \varsigma_{\beta_{p+1}}) = \varepsilon..$$

Proof: It's given that the sequence $\{\varsigma_\alpha\} \in \mathbb{Y}$ satisfies

$$\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) \geq \varepsilon, \text{and } \rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p-1)}}) < \varepsilon, \text{for each } \alpha(p) > \beta(p) > p.$$

Now,

$$\varepsilon \leq \rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) \leq \rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p-1}}) + s\rho(\varsigma_{\alpha_{p-1}}, \varsigma_{\alpha_p}),$$

on letting $p \to \infty$ in the preceding inequality and applying Eq. (1), we obtain

$$\lim_{p\to\infty}\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) = \varepsilon. \tag{2}$$

Again, on using triangle inequality, we have

$$\rho(\varsigma_{\alpha_p}, \varsigma_{\beta_{p+1}}) \leq \rho(\varsigma_{\alpha_p}, \varsigma_{\beta_p}) + s\,\rho(\varsigma_{\beta_p}, \varsigma_{\beta_{p+1}}).$$

Letting $p \to \infty$ in the preceding inequality and applying Eq.(1) and Eq.(2), we have

$$\lim_{p\to\infty}\rho(\varsigma_{\alpha_p}, \varsigma_{\beta_{p+1}}) = \varepsilon. \tag{3}$$

Using the triangle inequality again, we obtain

$$\rho(\varsigma_{\beta_{p+1}}, \varsigma_{\alpha_{p+1}}) \leq \rho(\varsigma_{\beta_{p+1}}, \varsigma_{\alpha_p}) + s\,\rho(\varsigma_{\alpha_p}, \varsigma_{\alpha_{p+1}}).$$

Further, letting $p \to \infty$, and applying Eq.(1) and Eq.(3), we obtain $\lim_{p\to\infty}\rho(\varsigma_{\beta_{p+1}}, \varsigma_{\alpha_{p+1}}) = \varepsilon$.

We start this section with the definition of (θ, Γ)- generalized multivalued contractions followed by our first result.

Definition 2.2: Suppose $\mathbb{Y}$ be a non-void set along with a metric ρ such that $(\mathbb{Y}, \rho)$ is a complete SBMS with $s \geq 1$. Then a multivalued map $T: \mathbb{Y} \to \mathcal{B}(\mathbb{Y})$ is said to be (θ, Γ)-generalized multivalued contractions, if for all comparable $\varsigma, \tau \in \mathbb{Y}$

$$\theta\big(d_M(\mathrm{T}\varsigma, \mathrm{T}\tau)\big) \leq \Gamma\left(\max\left\{\rho(\varsigma, \tau), d_H(\varsigma, \mathrm{T}\varsigma), d_H(\tau, \mathrm{T}\tau), \frac{d_H(\varsigma, \mathrm{T}\tau) + d_H(\tau, \mathrm{T}\varsigma)}{2}\right\}\right), \quad (4)$$

where $\Gamma: \mathbb{E} \to \mathbb{E}$ and θ are continuous and altering distance function satisfying condition $0 < \Gamma(t) < \theta(t)$, for all $t > 0$.

Theorem 2.3: *Suppose $(\mathbb{Y}, \preceq)$ is POSet equipped with a metric ρ for which $(\mathbb{Y}, \rho)$ is a complete SBMS with $s \geq 1$, and let $T: \mathbb{Y} \to \mathcal{B}(\mathbb{Y})$ be a multivalued mapping satisfying the assumptions:*

(i) $\exists\ \varsigma_0 \in \mathbb{Y}$ such that $\{\varsigma_0\} \preceq_1 \mathrm{T}\varsigma_0$,

(ii) for $\varsigma, \tau \in \mathbb{Y}$, $\varsigma \preceq \tau$ implies $\mathrm{T}\varsigma \preceq_1 \mathrm{T}\tau$,

(iii) if $\varsigma_\alpha \to \varsigma$ for a non-decreasing sequence in $\mathbb{Y}$, then $\varsigma_\alpha \preceq \varsigma, \forall \alpha \geq 0$,

(iv) *T is a (θ, Γ)- generalized multivalued contractions.*

Then T possess a fixed point.

Proof: From condition (i), we find that there exists $\varsigma_1 \in \mathrm{T}\varsigma_0$ satisfying the relation $\varsigma_0 \preceq \varsigma_1$. By condition (ii), we also have $\mathrm{T}\varsigma_0 \preceq_1 \mathrm{T}\varsigma_1$. Consequently, there exists $\varsigma_2 \in \mathrm{T}\varsigma_1$ such that $\varsigma_1 \preceq \varsigma_2$. Continuing with this approach, we construct a sequence $\{\varsigma_\alpha\} \subset \mathbb{Y}$ (monotone increasing) such that $\varsigma_{\alpha+1} \in \mathrm{T}\varsigma_\alpha, \forall \alpha \geq 0$. Thus, we get

$$\varsigma_0 \preceq \varsigma_1 \preceq \varsigma_2 \preceq \varsigma_3 \preceq \cdots \preceq \varsigma_\alpha \preceq \varsigma_{\alpha+1} \preceq \cdots$$

Suppose that $\varsigma_\alpha \neq \varsigma_{\alpha+1}$, $\forall \alpha \geq 0$. By the monotonicity of θ, condition (iv), and Eq.(4), it follows that $\forall \alpha \geq 0$,

$$\theta\big(\rho(\varsigma_{\alpha+1}, \varsigma_{\alpha+2})\big) \leq \theta\big(d_M(\mathrm{T}\varsigma_\alpha, \mathrm{T}\varsigma_{\alpha+1})\big) \leq \Gamma(\max\{\rho(\varsigma_\alpha, \varsigma_{\alpha+1}), \rho(\varsigma_{\alpha+1}, \varsigma_{\alpha+2})\}). \quad (5)$$

On simplification of these two cases, we obtain sequence of distances $\{\rho(\varsigma_\alpha, \varsigma_{\alpha+1})\}$ decreases monotonically. Hence $\exists$ an $\eta \geq 0$ such that

$$\rho(\varsigma_\alpha, \varsigma_{\alpha+1}) \to \eta \text{ as } \alpha \to \infty. \quad (6)$$

Suppose $\eta > 0$. On letting $\alpha \to \infty$ in (5) and by the use of property on θ, we obtain $\theta(\eta) \leq \Gamma(\eta)$ which leads to a contradiction due to the property of the condition Γ and θ, and so $\eta = 0$. Thus Eq. (6) reduces to

$$\lim_{\alpha \to \infty} \rho(\varsigma_\alpha, \varsigma_{\alpha+1}) = 0. \quad (7)$$

Assume that the defined sequence $\{\varsigma_\alpha\}$ is not Cauchy. For an $\varepsilon > 0$ and sequences $\{\beta_p\}, \{\alpha_p\}$ with $\alpha_p > \beta_p > p$ for all $p \in \mathbb{N}$ such that

$$\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}) \geq \varepsilon \text{ and } \rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p-1}}) < \varepsilon.$$

Since ς_{β_p} and ς_{α_p} are comparable, therefore from Eq. (4), by monotonicity of θ and using condition (iv), we get

$$\theta(\rho(\varsigma_{\beta_{p+1}}, \varsigma_{\alpha_{p+1}})) \leq \theta(d_M(\mathrm{T}\varsigma_{\beta_p}, \mathrm{T}\varsigma_{\alpha_p})) \leq \Gamma\left(\max\left\{\begin{matrix}\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_p}), \rho(\varsigma_{\beta_p}, \varsigma_{\beta_{p+1}}), \\ , \rho(\varsigma_{\alpha_p}, \varsigma_{\alpha_{p+1}}), \\ \frac{\rho(\varsigma_{\beta_p}, \varsigma_{\alpha_{p+1}}) + \rho(\varsigma_{\alpha_p}, \varsigma_{\beta_{p+1}})}{2}\end{matrix}\right\}\right).$$

Letting $p \to \infty$ in the preceding inequality and using Lemma 2.1 and Eq. (6), we get $\theta(\varepsilon) \leq \Gamma(\varepsilon)$, contradicting $0 < \Gamma(t) < \theta(t)$. Hence $\{\varsigma_\alpha\}$ is a Cauchy, so $\exists l \in \mathbb{Y}$ (using completeness of $\mathbb{Y}$) such that

$$\varsigma_\alpha \to l \text{ as } \alpha \to \infty. \tag{8}$$

By assumption (iii), $\varsigma_\alpha \preceq l$, $\forall\ \alpha \geq 0$. Again, using monotonicity of θ and from condition (iv), we get

$$\theta(d_M(\varsigma_{\alpha+1}, \mathrm{T}l)) \leq \theta(d_M(\mathrm{T}\varsigma_\alpha, \mathrm{T}l)) \leq \Gamma\left(\max\left\{\rho(\varsigma_\alpha, l), d_H(\varsigma_\alpha, \mathrm{T}\varsigma_\alpha), d_H(l, \mathrm{T}l), \frac{d_H(\varsigma_\alpha, \mathrm{T}l) + d_H(l, \mathrm{T}\varsigma_\alpha)}{2}\right\}\right)$$

Letting $\alpha \to \infty$ in the preceding inequality, using Eq. (6) and Eq. (7), we get

$$\theta(d_M(l, \mathrm{T}l)) \leq \Gamma(d_H(l, \mathrm{T}l)) \leq \Gamma(d_M(l, \mathrm{T}l)),$$

which leads to a contradiction unless $d_M(l, \mathrm{T}l) = 0$, i.e. $\{l\} = \mathrm{T}l$. Hence, l is a fixed point of T. Hence proved.

Theorem 2.4: *Suppose* $(\mathbb{Y}, \rho)$ *be a complete SBMS, and* $F, G: \mathbb{Y} \to \mathcal{B}(\mathbb{Y})$ *are mappings. If* $\exists \theta \in \Theta$ *such that* $\forall \varsigma, \tau \in \mathbb{Y}$

$$\theta(d_M(\mathrm{F}\varsigma, \mathrm{G}\tau)) \leq \Gamma(N(\varsigma, \tau)) \tag{9}$$

where

$$N(\varsigma, \tau) = \max\left\{\rho(\varsigma, \tau), d_H(\varsigma, \mathrm{F}\varsigma), d_H(\tau, \mathrm{G}\tau), \frac{1}{2}(d_H(\varsigma, \mathrm{G}\tau) + d_H(\tau, \mathrm{F}\varsigma))\right\}. \tag{10}$$

Moreover, if θ *is continuous, and* $\forall \varsigma \in \mathbb{Y}$, $G\varsigma$ *is closed, then maps* F *and* G *admits a common fixed point within* $\mathbb{Y}$.

Proof: Let ς_0 be an arbitrary element in $\mathbb{Y}$. We define the subsequent terms iteratively as $\varsigma_1 = \mathrm{F}\varsigma_0$ and $\varsigma_2 = \mathrm{G}\varsigma_1$. Continuing this iterative process, we construct two sequences, $\{\varsigma_\alpha\}$ and $\{\tau_\alpha\}$, within $\mathbb{Y}$ such that

$$\varsigma_{2\alpha+1} = \mathrm{F}\varsigma_{2\alpha} = \tau_{2\alpha}, \varsigma_{2\alpha+2} = \mathrm{G}\varsigma_{2\alpha+1} = \tau_{2\alpha+1}, \forall \alpha \in \mathbb{N}. \tag{11}$$

If $\exists\ \alpha_0 \in \mathbb{N} \cup \{0\}$ for which $\tau_{\alpha_0} = \tau_{\alpha_{0+1}} = \varsigma$ (say), then clearly, F and G has a common fixed point ς. Assume, therefore, that $\tau_{2\alpha} \neq \tau_{2\alpha+1}\ \forall\ \alpha \in \mathbb{N} \cup \{0\}$. Clearly, $\rho(\tau_{2\alpha}, \tau_{2\alpha+1}) > 0$ and $d_M(\tau_{2\alpha}, \tau_{2\alpha+1}) > 0\ \forall\ \alpha \in \mathbb{N} \cup \{0\}$. Then from Eq. (9) we get

$$\theta\big(\rho(\tau_{2\alpha},\tau_{2\alpha+1})\big) \le \theta\big(d_M(\tau_{2\alpha},\tau_{2\alpha+1})\big) = \theta\big(d_M(\mathrm{F}\varsigma_{2\alpha},\mathrm{G}\varsigma_{2\alpha+1})\big) \le \Gamma\big(N(\varsigma_{2\alpha},\varsigma_{2\alpha+1})\big) \tag{12}$$

where,

$$N(\varsigma_{2\alpha},\varsigma_{2\alpha+1}) \le \max\left\{\rho(\tau_{2\alpha-1},\tau_{2\alpha}),\rho(\tau_{2\alpha-1},\tau_{2\alpha}),\rho(\tau_{2\alpha},\tau_{2\alpha+1}),\tfrac{1}{2}\rho(\tau_{2\alpha-1},\tau_{2\alpha+1})\right\} \tag{13}$$

If $\rho(\tau_{2\alpha-1},\tau_{2\alpha}) \le \rho(\tau_{2\alpha},\tau_{2\alpha+1})$ and using the fact that

$$\tfrac{1}{2}\rho(\tau_{2\alpha-1},\tau_{2\alpha+1}) \le \max\{\rho(\tau_{2\alpha-1},\tau_{2\alpha}),\rho(\tau_{2\alpha},\tau_{2\alpha+1})\},$$

then from Eq.(12) and Eq.(13), we have $\theta(\rho(\tau_{2\alpha},\tau_{2\alpha+1})) \le \Gamma(\rho(\tau_{2\alpha},\tau_{2\alpha+1}))$, which leads to a contradiction to our supposition that $\theta(t) \le \Gamma(t)$. Therefore, $\rho(\tau_{2\alpha},\tau_{2\alpha+1}) < \rho(\tau_{2\alpha-1},\tau_{2\alpha}) \forall \alpha \ge 0$ and so from Eq.(12) and Eq.(13) we get

$$\theta(\rho(\tau_{2\alpha},\tau_{2\alpha+1})) \quad \le \Gamma(\rho(\tau_{2\alpha-1},\tau_{2\alpha})). \tag{14}$$

As $\rho(\tau_{2\alpha},\tau_{2\alpha+1}) \le \rho(\tau_{2\alpha-1},\tau_{2\alpha})$, it follows that the sequence $\{\rho(\tau_{2\alpha}, \tau_{2\alpha+1})\}$ decreases monotonically. Hence $\exists$ an $\eta \ge 0$ such that

$$\rho(\tau_{2\alpha},\tau_{2\alpha+1}) \to \eta \text{ as } \alpha \to \infty. \tag{15}$$

Suppose $\eta > 0$. On letting $\alpha \to \infty$ in (14) and by the use of property on θ, we obtain $\theta(\eta) \le \Gamma(\eta)$, which leads to a contradiction, and hence $\eta = 0$. Thus Eq.(15) reduces to

$$\lim_{\alpha\to\infty} \rho(\tau_{2\alpha},\tau_{2\alpha+1}) = 0. \tag{16}$$

Assume that the defined sequence $\{\tau_\alpha\}$ is not Cauchy. Then for an $\varepsilon > 0$ and sequences $\{\beta_p\},\{\alpha_p\}$ with $\alpha_p > \beta_p > p$ for all $p \in \mathbb{N}$ such that

$$\rho(\tau_{\beta_p},\tau_{\alpha_p}) \ge \varepsilon \text{ and } \rho(\tau_{\beta_p},\tau_{\alpha_{p-1}}) < \varepsilon.$$

Since τ_{β_p} and τ_{α_p} are comparable, therefore by using monotonicity of θ and the Eq.(9) and Eq.(10), we get

$$\theta\left(\rho(\tau_{\beta_{p+1}},\tau_{\alpha_{p+1}}))\right) \le \theta\left(d_M(\mathrm{F}\tau_{\beta_p},\mathrm{G}\tau_{\alpha_p}))\right) \le$$

$$\Gamma\left(\max\begin{Bmatrix}\rho\left(\tau_{\beta_p},\tau_{\alpha_p}\right), d_H\left(\tau_{\beta_p},\mathrm{F}\tau_{\beta_p}\right), \\ d_H\left(\tau_{\alpha_p},\mathrm{G}\tau_{\alpha_p}\right), \\ \frac{d_H\left(\tau_{\beta_p},\mathrm{G}\tau_{\alpha_p}\right)+d_H\left(\tau_{\alpha_p},\mathrm{F}\tau_{\beta_p}\right)}{2}\end{Bmatrix}\right).$$

Letting $p \to \infty$ in the preceding inequality and using Lemma 2.1, we get $\theta(\varepsilon) \le \Gamma(\varepsilon)$, contradicting $0 < \Gamma(t) < \theta(t)$. Hence $\{\tau_\alpha\}$ is Cauchy, so $\exists\, l \in \mathbb{Y}$ such that

$$\lim_{\alpha\to\infty} \tau_{2\alpha} = l. \tag{17}$$

From Eq. (11), we have

$$\lim_{\alpha\to\infty} \mathrm{F}\varsigma_{2\alpha} = l, \quad \lim_{\alpha\to\infty} \mathrm{G}\varsigma_{2\alpha+1} = l, \forall \alpha \in \mathbb{N}. \tag{18}$$

Suppose $\rho(\mathrm{F}l,l)\neq 0$. Then $\rho(\mathrm{F}l,l)>0$ and $d_M(\mathrm{F}l,l)>0 \ \forall \ l\in\mathbb{Y}$. Then by Eq. (9) and Eq. (10), we get

$$\theta(\rho(\mathrm{F}l,l)) \ \leq \theta(d_M(\mathrm{F}l,l)) = \lim_{\alpha\to\infty}\theta(d_M(\mathrm{F}l,\mathrm{G}\varsigma_{2\alpha+1})) \leq \lim_{\alpha\to\infty}\Gamma(N(l,\varsigma_{2\alpha+1})), \tag{19}$$

where,

$$\begin{aligned}\lim_{\alpha\to\infty}N(l,\varsigma_{2\alpha+1}) \ &= \lim_{\alpha\to\infty}\max\{\rho(l,\varsigma_{2\alpha+1}),d_H(l,\mathrm{F}l),d_H(\varsigma_{2\alpha+1},\mathrm{G}\varsigma_{2\alpha+1}),\\ &\qquad \tfrac{1}{2}(d_H(l,\mathrm{G}\varsigma_{2\alpha+1})+d_H(\varsigma_{2\tau+1},\mathrm{F}l))\}\\ &\leq \rho(l,\mathrm{F}l)\end{aligned} \tag{20}$$

and so from Eq. (18) and by continuity of θ, we get $\theta(\rho(\mathrm{F}l,l))\leq\Gamma(\rho(\mathrm{F}l,l))$, which leads to a contradiction. Thus $\rho(\mathrm{F}l,l)=0$. Next we claim that $l\in\mathrm{G}l$. We proceed by contradiction, let us assume that $l\notin\mathrm{G}l$. So $\exists$ a sub-sequence $\{\varsigma_{2\alpha_p}\}$ of $\{\varsigma_{2\alpha}\}$ such that $d_H(\varsigma_{2\alpha_{p+1}},\mathrm{G}l)>0\forall\alpha_p>\alpha_0\in\mathbb{N}$. Then from Eq.(9), we get

$$\theta(d_H(\varsigma_{2\alpha_{p+1}},\mathrm{G}l))\leq\theta(d_M(\mathrm{F}\varsigma_{2\alpha_p},\mathrm{G}l))\leq\Gamma(N(\varsigma_{2\alpha_p},l)), \tag{21}$$

where,

$$N(\varsigma_{2\alpha_p},l)\leq\max\left\{\rho(\varsigma_{2\alpha_p},l),\rho(\varsigma_{2\alpha_p},\mathrm{F}\varsigma_{2\alpha_p}),d_H(l,\mathrm{G}l),\tfrac{1}{2}(\rho(\varsigma_{2\alpha_p},\mathrm{G}l)+\rho(l,\varsigma_{2\alpha_p}))\right\}.$$

Letting $p\to\infty$ in Eq. (18), Eq. (19) and by the continuity of θ, we get $\theta(d_H(l,\mathrm{G}l))\leq\Gamma(d_H(l,\mathrm{G}l))$, which lead to a contradiction, and so $l\in\mathrm{G}l$. Hence the result is proved.

3. Numerical Results

Example 3.1: Let $\mathbb{Y}=\left\{\left(0,-\frac{1}{4}\right),(0,0),\left(-\frac{1}{7},0\right)\right\}\subset\mathbb{R}^2$ with the order $^\circ$ defined as: for $(\varsigma_1,\tau_1),(\varsigma_2,\tau_2)\in\mathbb{Y}$, $(\varsigma_1,\tau_1)^\circ(\varsigma_2,\tau_2)$ iff $\varsigma_1\leq\varsigma_2,\ \tau_1\leq\tau_2$.

Let $\rho:\mathbb{Y}\times\mathbb{Y}\to\mathbb{R}$ be given as

$$\rho(\varsigma,\tau)=\max\{|\varsigma_1-\varsigma_2|,|\tau_1-\tau_2|\},\ \text{for}\ \varsigma=(\varsigma_1,\tau_1),\tau=(\varsigma_2,\tau_2)\in\mathbb{Y}.$$

Then $(\mathbb{Y},\rho)$ is a complete SBMS and satisfies all conditions of Theorem (2.3) Define $\mathrm{T}:\mathbb{Y}\to\mathcal{B}(\mathbb{Y})$ by

$$\mathrm{T}\left(0,-\tfrac{1}{4}\right)=\left\{(0,0),\left(-\tfrac{1}{7},0\right)\right\},\mathrm{T}\left(-\tfrac{1}{7},0\right)=\{(0,0)\},\mathrm{T}(0,0)=\{(0,0)\}.$$

Then T has the properties mentioned in Theorem

Finally, let $\theta,\Gamma:\mathbb{E}\to\mathbb{E}$ be defined as $\theta(t)=t^3,\Gamma(t)=t^2$ respectively. Then θ and Γ satisfies the conditions of Theorem (2.3). Suppose that $\varsigma\preceq\tau$ and we examine the following cases:

Cases (i)–(v): (i) $\varsigma=\left(0,-\frac{1}{4}\right),\tau=(0,0)$; (ii) $\varsigma=\left(-\frac{1}{7},0\right),\tau=(0,0)$; (iii) $\varsigma=\left(0,-\frac{1}{4}\right),\tau=\left(0,-\frac{1}{4}\right)$; (iv) $\varsigma=\left(-\frac{1}{7},0\right),\tau=\left(-\frac{1}{7},0\right)$; (v) $\varsigma=(0,0),\tau=(0,0)$;.

From the computations in each case, we obtain

$$d_M(\mathrm{T}\varsigma, \mathrm{T}\tau) \in \left\{0, \frac{1}{7}\right\},$$

$$\max\left\{\rho(\varsigma,\tau), d_H(\varsigma, \mathrm{T}\varsigma), d_H(\tau, \mathrm{T}\tau), \frac{d_H(\varsigma,\mathrm{T}\tau)+d_H(\tau,\mathrm{T}\varsigma)}{2}\right\} \in \left\{0, \frac{1}{4}, \frac{1}{7}\right\},$$

Thus, in all five cases it follows that inequality (4) holds.

Example 3.2: Let $\mathbb{Y} = [0,1]$. Consider a function $\rho: \mathbb{Y} \times \mathbb{Y} \to \mathbb{E}$ by $\rho(\varsigma,\tau) = |\varsigma - \tau|^2$. Clearly, $(\mathbb{Y}, \rho)$ forms a complete SBMS. Define $\theta, \Gamma: \mathbb{E} \to \mathbb{E}$ by $\theta(t) = t$ and $\Gamma(t) = \frac{1}{2}t \forall\ t > 0$. It follows that $\theta \in \Theta$ and Γ are continuous functions. Define the mappings $G, F: \mathbb{Y} \to \mathcal{B}(\mathbb{Y})$ by

$$\mathrm{G}(\varsigma) = \left[0, \frac{\varsigma}{3}\right] \text{ and } \mathrm{F}(\varsigma) = \left[0, \frac{\varsigma}{5}\right].$$

Suppose, without any loss of generality, that all $\varsigma, \tau \neq 0$ and $\varsigma < \tau$ and $d_M(\mathrm{F}\varsigma, \mathrm{G}\tau) > 0$. Then

$$\theta(d_M(\mathrm{F}\varsigma, \mathrm{G}\tau)) = \theta\left(d_M\left(\left[0, \frac{\varsigma}{5}\right], \left[0, \frac{\tau}{3}\right]\right)\right) = \theta\left(\left(\left|\frac{\varsigma}{5} - \frac{\tau}{3}\right|\right)^2\right) = \left(\left|\frac{\varsigma}{5} - \frac{\tau}{3}\right|\right)^2$$

and

$$\Gamma(N(\varsigma,\tau)) = \Gamma(\max\{\rho(\varsigma,\tau), d_H(\varsigma, \mathrm{F}\varsigma), d_H(\tau, \mathrm{G}\tau), \frac{1}{2}(d_H(\varsigma, \mathrm{G}\tau) + d_H(\tau, \mathrm{F}\varsigma))\}) = \frac{1}{2}(|\varsigma - \tau|)^2$$

Clearly, $\theta(d_M(\mathrm{F}\varsigma, \mathrm{G}\tau)) \leq \Gamma(N(\varsigma,\tau))$. Thus, the hypothesis of Theorem (2.4) holds. Moreover $\mathrm{F}(0) = \mathrm{G}(0) = 0$ and is distinct.

4. Conclusion

In this study, we first present and explain the notion of fixed points in the context of partially ordered metric spaces for multivalued mappings. Moreover, two key theorems (Theorem 2.3 and Theorem 2.4) have been proposed to confirm that a fixed point exists uniquely under the specified conditions. To substantiate the theoretical results, illustrative examples are provided. This study both extends and generalizes various results previously established in the literature.

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