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# Machine learning approaches for solving nonlinear differential equations in sustainability science

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## Abstract

Sustainability scientists often work with complex, nonlinear systems that are governed by differential equations. Traditional numerical methods are error-free, but when used in dynamic or uncertain environments, they can be rigid and computationally intensive. For approximating solutions to nonlinear differential equations (NLDEs), machine learning (ML), particularly deep learning, offers a versatile and data-driven framework. With an emphasis on sustainability applications, such as climate modeling, renewable energy systems, resource optimization, and environmental dynamics, this paper examines cutting-edge machine learning techniques applied to NLDEs. We show how operator learning techniques, Gaussian processes, and physics-informed neural networks (PINNs) can enhance computational efficiency and solution accuracy, underscoring their potential to facilitate sustainable decision-making in real time.

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## Introduction

Sustainability research often involves modeling complex, nonlinear systems that are dynamic and multi-scale. These systems are very sensitive to initial or boundary conditions. They include climate modeling, renewable energy, environmental resource management, and ecological forecasting. They are usually modeled using nonlinear differential equations (NLDEs) normally. Finite difference, finite element, or spectral approaches are traditional numerical solvers that guarantee the solutions, but they become expensive, rigid, or inappropriate when applied in uncertain, data-scarce, or real-time cases [1].

To overcome these limitations, machine learning (ML), and more so deep learning, has been a new promising alternative or complementary strategy for resolving NLDEs. Methods based on ML are flexible, scalable, and can learn from data. Thus, they are especially useful in sustainability science, where models tend to have to adapt to dynamic environmental conditions. Techniques like physics-informed neural networks (PINNs) [2], Gaussian processes, and operator learning paradigms (e.g., DeepONets, Fourier Neural Operators) can most closely emulate complex solution manifolds, reduce computational expense, and improve robustness in the presence of uncertainty [3].

In green applications, ML techniques have been successfully applied for climate system simulation [4], renewable energy forecasting [5], groundwater and hydrology estimation [6], and resource planning [7, 8].These developments have the capability to revolutionize real-time decision-making by providing fast, precise, and adaptive solutions to problems that were otherwise limited by the complexity of their governing equations.

This paper discusses the application of machine learning in the approximation of NLDEs' solutions in the context of sustainability. We refer to recent advances in data-driven solvers and physics-informed hybrids, showing that these can enhance computational efficiency as well as solution accuracy—ingredients required for credible and timely results in sustainability analysis.

## Methodology

Let a nonlinear differential equation for a sustainability related system be given by

$$\mathcal{N}[u(x,t);\theta] = 0, \quad x \in \Omega, \quad t \in [0,T]$$

Where

$u(x,t)$ is unknown solution (e.g. temperature, concentration, flow rate),
$\mathcal{N}[.]$ is nonlinear differential operator,
$\theta$ reprents system parameters (e.g. material properties, source terms),
$\Omega \subset \mathbb{R}^n$ is the spatial domain
And initial/boundary conditions are specified accordingly.

Machine learning aims to approximate the solution$u(x,t) \approx \hat{u}(x,t;\phi)$, where:
$\hat{u}(x,t;\phi)$ is neural network or other ML model,

$\phi$ are the trainable parameters of the ML model.

In Physics-Informed Neural Networks (PINNs), the loss function is a combination of data fidelity and physical constraints:

$$\mathcal{L}(\phi) = \sum_{i=1}^{N} \parallel \hat{u}(x_i, t_i; \phi) - u_i \parallel^2 + \lambda \sum_{j=1}^{M} \parallel \mathcal{N}\left[\hat{u}\left(x_j, t_j; \phi\right); \theta\right] \parallel^2$$

Where,

$\sum_{i=1}^{N} \parallel \hat{u}(x_i, t_i; \phi) - u_i \parallel^2$ : Data loss and $\sum_{j=1}^{M} \parallel \mathcal{N}\left[\hat{u}\left(x_j, t_j; \phi\right); \theta\right] \parallel^2$ : Physics loss

The minimization of this loss allows the ML model to learn a solution that adheres to both the observed data and the governing nonlinear equation.

In order to find analytically a nonlinear differential equationwe require a particular equation. The general form:

$$\mathcal{N}[u(x,t); \theta] = 0$$

Is too general to be solved analytical unless we have the explicit form of $\mathcal{N}$. But the paper shows how one can achieve an analytical solution for a typical NLDE that is used in sustainability applications.

Fluid Dynamics is a very rich area of nonlinear differential equations, particularly nonlinear partial differential equations that rule the motion of fluids. The incompressible Navier-Stokes equations are a classic and well-studied example.

Take the **Burger's equation,** a simplified model for the study shock waves, turbulence and viscous flow. It is nonlinear and in contrast to full Navier-Stokes, there exist exact solutions in certain cases

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Where:

$u(x,t)$ is the velocity field, $\nu$ is the kinematic viscosity

Apply the Cole-Hopf transformation:

With the initial condition:

$$u(x,0) = f(x) = \frac{-2\nu k}{1+e^{kx}}$$

ApplyingCole-Hopf transformation

$$u(x,t) = -2\nu\frac{\partial_x \phi(x,t)}{\phi(x,t)} == -2\nu\frac{\phi_x}{\phi}$$

Substituting in Burger's equations, finds that $\phi(x,t)$ satisfies the heat equation:

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial x^2}$$

Find the Initial condition for $\phi(x,0)$

Given

$$u(x,0) = f(x) = -2\nu \frac{\phi_x(x,0)}{\phi(x,0)} \Longrightarrow \frac{\phi_x(x,0)}{\phi(x,0)} = \frac{k}{1+e^{kx}}$$

Integrating

$$\phi(x,0) = \frac{e^{kx}}{1+e^{kx}}$$

Now solve

$$\frac{\partial\phi}{\partial t} = \nu\frac{\partial^2\phi}{\partial x^2}, \quad \phi(x,0) = \frac{e^{kx}}{1+e^{kx}}$$

The basic solution of the heat equation:

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\nu t}} exp\left(-\frac{(x-y)^2}{4\nu t}\right).\phi(y,0)dy$$

or

$$\phi(x,t) = \int_{-\infty}^{\infty} G(x-y,t).\phi(y,0)dy,$$

Where, $G(x,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-x^2/4\nu t}$(Green's Function)

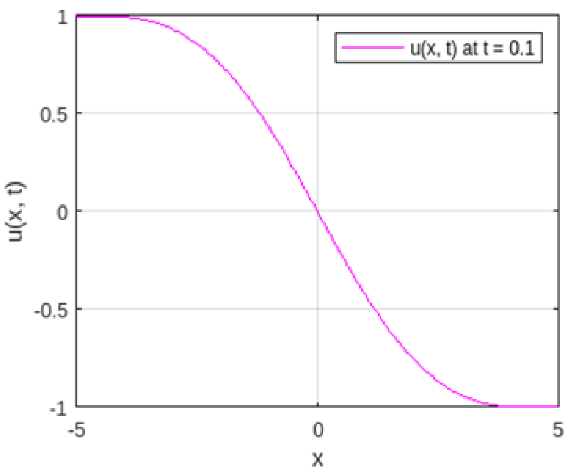


**Figure 1**

**Viscous Burgers' equation at t=0.1, computed using the Cole–Hopf transformation**.

Secondly tackle an **analytical solution** of a simplified case of the **Navier–Stokeswith forcing** (e.g., wind or pressure gradient)

*2D Incompressible Navier-Stokes Equations with Forcing:*

Let's look at the Navier–Stokes equations for incompressible flow in two dimensions:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \vec{F}(x,y,t)$$

$$\nabla.\vec{u} = 0$$

Where $\vec{F}(x,y,t)$ is the external force (wind, pressure gradient, buoyancy etc.)

Let's suppose a constant wind stress in the x-direction (e.g. simulating wind blowing over the ocean):

$$\vec{F} = (F_0, 0), \quad \text{with } F_0 = \text{constant}$$

So the momentum equation becomes:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + (F_0, 0)$$

To make the Navier-Stokes equations simpler for steady, one dimensional flow, make some assumptions. Assume first a steady state condition, which means that the time derivative of velocity is zero $\frac{\partial \vec{u}}{\partial t} = 0$. Secondly, flow to the x-direction alone, so the velocity vector is of the form $\vec{u} = (u(y), 0)$, meaning there is no y-direction velocity component. Moreover, suppose that the velocity is a function only of the vertical coordinate y, and thereby reduce the analysis to a single spatial dimension. Lastly, any pressure gradient in the system is assumed constant or included directly within a generic forcing term. These assumptions greatly decrease the complexity of the governing equations and are standard within the applications of boundary layer theory and driven channel flow problems.

Then the Navier-Stokes equation reduces to

$$0 = \nu\frac{d^2u}{dy^2} + F_0$$

$$\frac{d^2u}{dy^2} = -\frac{F_0}{\nu}$$

Integrate once:

$$\frac{du}{dy} = -\frac{F_0}{\nu}y + C_1$$

Integrate again:

$$u(y) = -\frac{F_0}{2\nu}y^2 + C_1 y + C_2$$

Boundary Conditions:

Fluid layer between two plates (a channel), such as the atmosphere over the earth or ocean water between sea bottom and surface. Assume:

No slip at both boundaries:

$$u(0) = 0, u(H) = 0$$

After applying these conditions,

$$\text{At y=0,} \quad u(0) = C_2 \Rightarrow C_2 = 0$$

$$y = H, \qquad u(H) = -\frac{F_0}{2\nu}H^2 + C_1 H = 0 \quad \Rightarrow C_1 = \frac{F_0}{2\nu}H$$

So the solution is:

$$u(y) = \frac{F_0}{2\nu}(Hy - y^2)$$

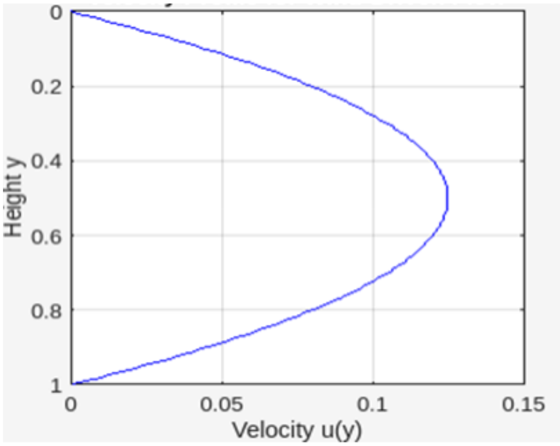


**Figure 2**
**Velocity profile for the wind driven flow.**

**Result**

Figure 1 depict that the analytical solution of the time-dependent Navier–Stokes equation under the above assumption brings forth important physical behaviors typical of viscous flows. Most significantly, the solution shows smoothed wave propagation, wherein the velocity profile changes gradually over time. This smoothing is due to viscous diffusion, which suppresses any steep gradients that would otherwise develop as a result of nonlinear convection terms.Compared to the inviscid situation e.g., in Burgers' equation without viscosity where the solutions can develop shocks (discontinuities in velocity), the viscosity in the Navier-Stokes regime makes sure the gradient of the velocity is smooth and continuous. This is critical in sustainability-oriented modeling applications (e.g., pollution dispersion or air flow over a terrain), where realistic transitions of flows are important to be captured and non-physical phenomena such as infinite gradients or discontinuities need to be avoided.

In general, this interpretation emphasizes the manner in which viscous forces balance the flow, allowing physically reasonable and mathematically well-defined solutions critical to environmental and engineering simulations.

Figure 2 depict that the resulting velocity profile is parabolic, typical of bounded-domain flows like ocean currents driven by wind or atmospheric boundary layers. The flow is driven here by a uniform external forcing, e.g., wind stress, as opposed to a pressure gradient. The velocity is maximum at the middle of the vertical domain, i.e., at y=H/2, where viscous drag from the boundaries is least. Although this solution is similar to the traditional

Poiseuille flow, which is a pressure gradient-driven flow, the distinguishing feature here is that the movement is caused by a uniform bodyforce simulating wind or some other environmental factor. This makes the model of most use in sustainability scenarios such as simulating near-surface oceanic currents or air flow over land surfaces driven by meteorological factors.

**Conclusion**

This reduced fluid dynamics model has significant uses in a number of sustainability-related applications. One such area is wind-driven ocean current modeling, where it forms the basis of understanding effects such as Ekman transport that affect large-scale ocean circulation and climate systems. It is also crucial in pollution dispersion modeling, serving to simulate how pollutants disperse through air or water bodies due to the effects of wind or flow. In atmospheric science, the model helps in surface layer modeling that is, to understand airflow over urban areas or forested areas, which helps in assessing heat exchange, carbon flux, and air quality. In addition, in renewable energy, this technique is used in modeling how wind farms affect airflow, which helps in determining the best position for turbines and efficiency while minimizing harm to the environment.

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