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Fuzzy points relation with nonfuzzy or crisp set

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Abstract

This Manuscript explores fixed point theory in both fuzzy and non-$\mathcal{FMS}$s, focusing on two distinct contractive mappings and their associated fixed point theorems. We redefine $\mathcal{FMS}$s by employing fuzzy constant with real numbers, providing a approach that differs from previous definitions.

It is shown that any corresponding $\mathcal{FMS}$ that preserves completeness. The findings highlight significance of mappings in establishing fixed point results and shows how the nature of these mappings influences the proof and application of fixed point theorems.

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1. Introduction

The concept of $\mathcal{FMS}$ serves as a crucial element in the study of fuzzy sets and their mathematical structures. Over time, multiple definitions have emerged, each adopting different approaches to characterize $\mathcal{FMS}$. Additionally, $\mathcal{FIP}$ results in $\mathcal{FMS}$s have been widely explored by Felbin [2], demonstrating their applicability in probabilistic $\mathcal{MS}$s and multivalued mappings. The development of $\mathcal{FMS}$s has been shaped by early research on fuzzy sets and $\mathcal{MS}$s, including statistical $\mathcal{MS}$s and $\mathcal{FP}$ metrics by Kaleva and Seikkala [8]. These studies laid the foundation for further exploration. [1, 5, 6] Two main approaches exist: using fuzzy numbers to define metrics, as studied by Diamond and Kloeden [9].

Definition 1.1: In 1982 Zike [3] $\mathcal{FP}$s form the fuzzy sets with respect to $\Xi_{\tilde{\mu}}$ and $\mathcal{T}$ with $0 \leq \alpha \leq 1$ can be defined as,

$$\omega_\alpha(\eta_1) = \begin{cases} \alpha & \eta_1 = \omega \\ 0 & \eta_1 \neq \omega \end{cases} \text{ for every } \eta_1 \in \mathcal{T}$$

The relationship between these spaces and classical $\mathcal{MS}$s is explored, with $\mathcal{FIP}$ theorems formulated based on the distance function defined between $\mathcal{FP}$s. This research contributes to the broader understanding of $\mathcal{FMS}$s and their potential applications [10, 11, 12].

2. Preliminaries and Basic Concepts

The fundamental concepts in $\mathcal{FMS}$s utilized and $\tilde{\mathcal{Z}}^{\tilde{\jmath}}$ refers to the collection of all fuzzy subsets of $\tilde{\jmath}$, notation $(\tilde{\jmath}, \tilde{\delta}_{\tilde{\jmath}})$ is used to describe a nonfuzzy (or crisp) $\mathcal{MS}$, while $(\tilde{\mathcal{Z}}^{\tilde{\jmath}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}})$ represents the associated $\mathcal{FMS}$. Additionally, $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ signifies a $\mathcal{FP}$ where the support set consists of a single element $\tilde{\mu} \in \tilde{\jmath}$ and the membership function is defined as $\tilde{\eta}_1$ which from $[0,1]$. The following remark can be established and verified straightforwardly based on the fuzzy distance function defined in (2.1).

Remark 2.1: Let $\mathcal{FP}$s $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >$, where $\tilde{\mu}_n, \tilde{\eta}_{1_n} \in [0,1]$ converge $\mathcal{FP}$ $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ if and only when ever monotonic real numbers $< \tilde{\eta}_{1_n} >$, $n \in \mathbb{N}$, converges to $\tilde{\eta}_1 \in \mathbb{R}$, along with a sequence of support points $\langle \tilde{\mu}_n \rangle$, where $\tilde{\mu}_n \in \tilde{\jmath}$ and $n \in \mathbb{N}$, that converges to $\tilde{\mu} \in \tilde{\jmath}$ as $n \to \infty$.

In this following section, we examine a non-fuzzy function positioned between two distinct $\mathcal{FP}$s. Additionally, we introduce and define the fuzzy distance function in this context.

Definition 2.1: [10] Suppose the $(\tilde{\jmath}, \tilde{\delta}_{\tilde{\jmath}})$ $\mathcal{MS}$ which having the function $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}} : \tilde{\mathcal{Z}}^{\tilde{\jmath}} \times \tilde{\mathcal{Z}}^{\tilde{\jmath}} \to \overline{\mathbb{R}}^+$ serves as a distance function between $\mathcal{FP}$s if, for all $\mathcal{FP}$s $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}, \tilde{\Xi}_{z}^{\tilde{\eta}_3} \in \tilde{\mathcal{Z}}^{\tilde{\jmath}}$, the mapping $\tilde{\mathfrak{p}}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}$ satisfies the following four points: where $\tilde{\mu}, \tilde{\nu}, z \in \tilde{\jmath}$ and $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3 \in [0,1]$,

1. $\mathcal{FM}_1$: $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) = \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{u}}}\left(\tilde{\Xi}_{\tilde{\nu}}^{1-\tilde{\eta}_2}, \tilde{\Xi}_{\tilde{\mu}}^{1-\tilde{\eta}_1}\right)$.
2. $\mathcal{FM}_2$: Whenever $\tilde{\mu} = \tilde{\nu}$ and $\tilde{\eta}_1 = \tilde{\eta}_2$ then it gives $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) = 0$.
3. $\mathcal{FM}_3$: $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) \leq \tilde{\zeta}$, with $\tilde{\zeta} \geq 0$, which has $\widehat{\tilde{\eta}_1} > \tilde{\eta}_1$ gives $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\widehat{\tilde{\eta}_1}}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) < \tilde{\zeta}$.
4. $\mathcal{FM}_4$: $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) \leq \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{z}^{\tilde{\eta}_3}\right) + \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{u}}}\left(\tilde{\Xi}_{z}^{\tilde{\eta}_3}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right)$. Then we say $\left(\tilde{\mathcal{Z}}^{\tilde{\jmath}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\right)$ be a $\mathcal{FMS}$

Above definition we apply on, which will be applied in the subsequent discussion, is also outlined in the following remark. Its verification follows directly from Definition (2.1).

Remark 2.2: Consider a non$\mathcal{FMS}$ represented as $\left(\tilde{\mathrm{u}}, \tilde{\delta}\tilde{\mathfrak{Z}}\right)$ and its corresponding $\mathcal{FMS}$ $\left(\tilde{\mathcal{Z}}^{\tilde{\jmath}}, \tilde{\delta}\tilde{\mathcal{Z}}^{\tilde{\jmath}}\right)$. The measurement of distance between two $\mathcal{FP}$s can be established based on the structure of the underlying nonfuzzy metric. This definition plays a crucial role in understanding the behavior of $\mathcal{FMS}$s in applications. For every $\tilde{\mu}, \tilde{\nu} \in \tilde{\mathfrak{Z}}$ and $\tilde{\eta}_1, \tilde{\eta}_2 \in [0,1]$ we write,

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) = |\tilde{\eta}_1 - \tilde{\eta}_2| + \tilde{\delta}_{\tilde{\jmath}}(\tilde{\mu}, \tilde{\nu}) \tag{2.1}$$

It is important to highlight that various fuzzy distance functions can be derived based on how the nonfuzzy distance function $\tilde{\delta}_{\tilde{\jmath}}(\tilde{\mu}, \tilde{\nu})$ is defined in equation (2.1).

3. Complete $\mathcal{FMS}$

This section establishes the fundamental conditions necessary for formulating and demonstrating the fuzzy $\mathcal{FIP}$ theorem, The focus is on the completeness of $\mathcal{FMS}$ and their connection to the completeness of crisp $\mathcal{MS}$s. In particular, we analyze the convergence of sequences of $\mathcal{FP}$s for sequence $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >, n \in \mathbb{N}$ as below,

$$\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} \to \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$$

Consequently, the main properties are explored, serving as a basis for further studies (for additional insights, refer to [1, 7, 10, 12]). convergence of $\mathcal{FP}$s is introduced initially, by a Cauchy sequence in $\mathcal{FMS}$s.

Definition 3.1: [4] Suppose $\tilde{\mathcal{Z}}^{\tilde{\jmath}}$, a $\mathcal{FP}$ s sequence $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >, n \in \mathbb{N}$ be converges to $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1} \in \tilde{\mathcal{Z}}^{\tilde{\jmath}}$ that is the mapping $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} \to \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ for all $\varepsilon > 0$, which has $N \in \mathbb{N}$, we write

$$\tilde{\delta}_{\tilde{\mathcal{Z}}}^{\tilde{\jmath}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) < \varepsilon, \text{for every } n \geq N$$

whereas $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1} \in \tilde{\mathcal{Z}}^{\tilde{\jmath}}, \tilde{\mu}_n, \tilde{\mu} \in \tilde{\mathfrak{Z}}$ aloong with $\tilde{\eta}_{1_n}, \tilde{\eta}_1 \in [0,1]$.

Moreover, the subsequent theorem establishes the connection between convergent sequences of $\mathcal{FP}$s and Cauchy sequences of $\mathcal{FP}$s.

Theorem 3.1: *Consider* $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >, \tilde{\mu}_n \in \tilde{\mathfrak{Z}}$, *and* $\tilde{\eta}_{1_n} \in [0,1], n \in \mathbb{N}$ *be a sequence of* $\mathcal{FP}$*s that converges. Then, the sequence* $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} \right\rangle$ *must also satisfy the conditions of a Cauchy sequence in the* $\mathcal{FMS}$.

Proof: Let's Assume that $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} \right\rangle$, according to Definition (3.1), for every $\varepsilon > 0$, which has $N \in \mathbb{N}$ such that the fuzzy distance function satisfies $\tilde{\eth}_{\tilde{Z}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) < \varepsilon$ for all $n \geq N$. then $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >$ forms a Cauchy sequence in the $\mathcal{FMS}$, it is required to show that $\tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1m}}\right) < \varepsilon$ holds.

By applying the triangle inequality given in Definition (2.1(3)) this result directly follows.

$$\tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1m}}\right) < \tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) + \tilde{\mathfrak{p}}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1m}}\right)$$

$< \frac{1}{2}(\tilde{\varepsilon} + \tilde{\varepsilon}) = \tilde{\varepsilon}$, Finally which gives $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} >$ be a Cauchy.

The concept of completeness in a $\mathcal{FMS}$ is defined similarly to that in a non $\mathcal{FMS}$, with the distinction that it involves $\mathcal{FP}$s, as outlined below.

Definition 3.2: Lets suppose $\mathcal{FMS}$ $\left(\tilde{Z}^{\tilde{\mathfrak{Z}}}, \tilde{\eth}_{\tilde{Z}^{\tilde{\mu}}}\right)$ holds completeness $\mathcal{FMS}$, whenever all Cauchy sequence of given $\mathcal{FP}$s inside $\tilde{Z}^{\tilde{\mathfrak{Z}}}$ convergent to the $\mathcal{FP}$ inside $\tilde{Z}^{\tilde{\mathfrak{Z}}}$.

One of the key findings of this paper is the established connection between the completeness of the non-$\mathcal{FMS}$ $(\tilde{\mathfrak{Z}}, \tilde{\eth})$ and the completeness of the $\mathcal{FMS}$ $\left(\tilde{Z}^{\tilde{\mathfrak{Z}}}, \tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\right)$.

Theorem 3.2: *Lets consider* $\left(\tilde{Z}^{\tilde{\mathfrak{Z}}}, \tilde{\partial}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\right)$ *is the* $\mathcal{FMS}$ *induced in non-fuzzy or crisp-* $\mathcal{MS}$ $\left(\tilde{\mathfrak{Z}}, \tilde{\partial}_{\tilde{\mathfrak{Z}}}\right)$, *then* $\left(\tilde{Z}^{\tilde{\mathfrak{Z}}}, \tilde{\partial}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\right)$ *be a complete whenever* $\left(\tilde{\mathfrak{Z}}, \tilde{\eth}_{\tilde{\mathfrak{Z}}}\right)$ *be a complete.*

Proof: Assume that $\left(\tilde{\mathfrak{Z}}, \tilde{\eth}_{\tilde{\mathfrak{Z}}}\right)$ represents a complete non-$\mathcal{FMS}$, and we aim to demonstrate the completeness of $\left(\tilde{Z}^{\tilde{\mathfrak{Z}}}, \tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\right)$. Consider a Cauchy sequence of $\mathcal{FP}$s $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}} \right\rangle$, where $\tilde{\mu}_n \in \tilde{\mathfrak{Z}}, n \in \mathbb{N}, \tilde{\eta}_1 \in [0,1]$, and it needs to be established that this sequence converges to a $\mathcal{FP}$ $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1} \in \tilde{Z}^{\tilde{\mathfrak{Z}}}$,

$$\tilde{\eth}_{\tilde{Z}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1m}}\right) < \varepsilon, \text{for every } n, m \geq N \tag{3.2}$$

As a consequence of Corollary (4.1), Specifically, these include a sequence of support elements $< \tilde{\mu}_n > \in \tilde{\mathfrak{Z}}$ and a sequence of real numbers corresponding to $\tilde{\eta}_1$-levels, denoted as $< \tilde{\eta}_{1_n} > \in [0,1]$, such that:

$$\tilde{\delta}_{\tilde{\mathfrak{Z}}}(\tilde{\mu}_n, \tilde{\mu}_m) < \varepsilon \text{ and } \left|\tilde{\eta}_{1_n} - \tilde{\eta}_{1_m}\right| < \varepsilon, \text{for every } n, m \geq N$$

To establish the convergence of $< \tilde{\mu}_n >$ and $< \tilde{\eta}_{1_n} >$ to $\tilde{\mu}$ and $\tilde{\eta}_1$, respectively, we consider the following: Since $< \tilde{\mu}_n >$, $\left(\tilde{\mathfrak{Z}}, \tilde{\delta}_{\tilde{\mathfrak{Z}}}\right)$, which is a complete non- $\mathcal{FMS}$, it follows that $\langle \tilde{\mu}_n \rangle$ converges to some $\tilde{\mu} \in \tilde{\mathfrak{Z}}$. Additionally, $< \tilde{\eta}_{1_n} >$ constitutes a Cauchy sequence of real numbers in the interval $[0,1]$.

Given that the real number system is complete, this guarantees that $< \tilde{\eta}_{1_n} >$ converges to a value $\tilde{\eta}_1 \in [0,1]$. As a consequence of Remark (2.1), it follows that for all $n \geq N$, the inequality $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) < \varepsilon$ holds. This confirms that $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} >$ is a convergent sequence within $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\right)$. Hence, the space $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\right)$ satisfies the completeness condition for a $(\mathcal{FMS})$. Converse statement holds same manner.

4. Contraction of Fuzzy Mapping

This section explores the theory of contractive mappings within $\mathcal{FMS}$ before establishing and proving the Banach $\mathcal{FIP}$ theorem in this setting.

Definition 4.1: Suppose $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\right)$ be a $\mathcal{FMS}$, with map $\tilde{\aleph}: \tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}} \to \tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$. The mapping $\tilde{\aleph}$ is said to be a contractive fuzzy map $\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$ then a constant $k \in (0,1)$ we write,

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right), \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right)\right) \leq \tilde{\hbar}\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}\right) \tag{4.3}$$

any $\mathcal{FP}$s $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ and $\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}$ within $\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$, where $\tilde{\eta}_1, \tilde{\eta}_2 \in [0,1]$.

Let's create Corollary using Remark (2.1) and Theorem (3.1):

Corollary 4.1: Assume $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\right)$ is a $\mathcal{FMS}$, then $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \right\rangle, \tilde{\mu}_n \in \tilde{\mathfrak{Z}}, \tilde{\eta}_{1_n} \in [0,1], n \in \mathbb{N}$ be a cauchy sequence only when two of non-fuzzy Cauchy sequences are present, such that $< \tilde{\mu}_n > \in \tilde{\mathfrak{Z}}$ with monotonic $\mathbb{R} < \tilde{\eta}_{1_n} > \in [0,1]$.

We define our main Theorem in following section:

Theorem 4.1: *Let* $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}}\right)$ *be a complete* $\mathcal{FMS}$, $\tilde{\aleph}: \tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}} \to \tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$ *fuzzy map,* $\exists$ *fuzzy* $\mathcal{FIP}$ *of* $\tilde{\aleph}$ *in* $\tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$.

Proof: Suppose $\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}} \in \tilde{\mathcal{Z}}^{\tilde{\mathfrak{Z}}}$, and consider the iterative sequence generated by the fuzzy mapping $\tilde{\aleph}$ as follows:

$$\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right), \tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) = \tilde{\aleph}^2\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right), \tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}\right)$$
$$= \tilde{\aleph}^3\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right), \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} = \tilde{\aleph}^n\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right).$$

$\tilde{\aleph}^n$ represents the n^{th} composition of the fuzzy mapping $\tilde{\aleph}$. then the sequence of $\mathcal{FP}$s $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} >$, where $\tilde{\mu}_n \in \tilde{\Im}$, $\tilde{\eta}_{1_n} \in [0,1]$, and $n \in \mathbb{N}$.

Given that $\tilde{\aleph}$ is a contractive mapping, applying inequality (4.3) leads to the following results:

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}\right) = \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(f\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right), f\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right)\right) \leq \tilde{\hbar}\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right)$$
$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) = \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(f\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right), f\left(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}\right)\right) \leq \tilde{\hbar}\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}\right)$$
$$\leq \tilde{\hbar}^2\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) \cdots \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_{n-1}}^{\tilde{\eta}_{1_{n-1}}}\right) \leq \tilde{\hbar}^n\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right).$$

By, applying Triangle inequality from Definition (2.1),

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_{\tilde{\hbar}}}^{\tilde{\eta}_{1_{\tilde{\hbar}}}}\right) + \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_{\tilde{\hbar}}}^{\tilde{\eta}_{1_{\tilde{\hbar}}}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right)$$

By using equation (2.1), we write $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) = \left|\tilde{\eta}_{1_n} - \tilde{\eta}_{1_m}\right| + \tilde{\mathfrak{p}}(\tilde{\mu}_n, \tilde{\mu}_m)$ any non-fuzzy sequence $< \tilde{\mu}_n >$ and $< \tilde{\eta}_{1_n} >$ with the help triangle inequality:

$$\left|\tilde{\eta}_{1_n} - \tilde{\eta}_{1_{n+2}}\right| \leq \left|\tilde{\eta}_{1_n} - \tilde{\eta}_{1_{n+1}}\right| + \left|\tilde{\eta}_{1_{n+1}} - \tilde{\eta}_{1_{n+2}}\right|$$
$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right)$$
$$\leq \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_{n+1}}^{\tilde{\eta}_{1_{n+1}}}\right) + \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_{n+1}}^{\tilde{\eta}_{1_{n+1}}}, \tilde{\Xi}_{\tilde{\mu}_{n+2}}^{\tilde{\eta}_{1_{n+2}}}\right) + \cdots + \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_{m-1}}^{\tilde{\eta}_{1_{m-1}}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right)$$
$$\leq \left(\tilde{\hbar}^n + \tilde{\hbar}^{n+1} + \cdots + \tilde{\hbar}^{m-1}\right)\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) = \tilde{\hbar}^n\left(1 + \tilde{\hbar} + \cdots + \tilde{\hbar}^{m-1-n} + \cdots\right)$$
$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) = \frac{\tilde{\hbar}^n}{1-\tilde{\hbar}}\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right), \tilde{\eta}_1 \in [0,1]$$

for $\varepsilon > 0$, we consider N natural number, which gives $\frac{\tilde{\hbar}^n}{1-\tilde{\hbar}}\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) < \varepsilon$. $\forall n, m \geq N$, the inequality $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \varepsilon$ holds. Consequently, the sequence $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \right\rangle, n \in \mathbb{N}$ forms a $\tilde{\mathcal{Z}}^{\tilde{\Im}}$. Given that $\left(\tilde{\mathcal{Z}}^{\tilde{\Im}}, \tilde{\mathfrak{p}}_{\tilde{\mathcal{Z}}^{\tilde{\Im}}}\right)$ is a complete $\mathcal{FMS}$, it follows that $< \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} >$ converges to some $\mathcal{FP}$ $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1} \in \tilde{\mathcal{Z}}^{\tilde{\Im}}$. To establish that $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ is indeed a $\mathcal{FIP}$ of $\tilde{\aleph}$, we proceed as follows. Since the sequence $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}$ converges to $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ and the mapping $\tilde{\aleph}$ is continuous, it follows that $\tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}\right) \to \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right)$. This implies that $\tilde{\Xi}_{\tilde{\mu}_{n+1}}^{\tilde{\eta}_{1_{n+1}}} \to \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right)$. Since it has already been established that $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \to \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$, it follows that $\tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) = \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$. Thus, $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ is a $\mathcal{FIP}$ of the mapping $\tilde{\aleph}$.

This work introduces a second definition of contractive mapping for $\mathcal{FMS}$s, which will be utilized to establish the second main result of the article, which is another $\mathcal{FP}$.

Definition 4.2: Consider a complete $\mathcal{FMS}$, denoted as $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}, \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\right)$. A fuzzy mapping $f: \tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}} \to \tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}$ is said to be contractive on $\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}$ if there exists a constant $\tilde{\lambda} \in (0,1)$ such that, for every pair of $\mathcal{FP}$s $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ and $\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2} \in \tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}$, where $\tilde{\mu}, \tilde{\nu} \in \tilde{\mathfrak{I}}$ and $\tilde{\eta}_1, \tilde{\eta}_2 \in [0,1]$, then

$$\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(f(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}), f(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2})) \leq \tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, f(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1})), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}, f(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}))\} \qquad (4.4)$$

Next, we present the statement of the second fuzzy $\mathcal{FIP}$ theorem based on Definition (4.2), followed by its proof.

Theorem 4.2: *et's* $\left(\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}, \tilde{\partial}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}\right)$ *ibe a complete* $\mathcal{FMS}$ *and* $\tilde{\aleph}: \tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}} \to \tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}$ *is a contractive fuzzy mapping that satisfies inequality (4.4), then* $\tilde{\aleph}$ *possesses a fuzzy* $\mathcal{FIP}$ *in* $\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}$.

Proof: As we have,

$$\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}), \tilde{\aleph}(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2})) \leq \tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1})), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\nu}}^{\tilde{\eta}_2}))\}, \tilde{\lambda} \in (0,1)$$

Consider an initial $\mathcal{FP}$ $\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}} \in \tilde{\mathcal{Z}}^{\tilde{\mathfrak{u}}}$ and define an iterative sequence by setting $\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right)$. Proceeding recursively, we obtain $\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) = \tilde{\aleph}^2\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right)$. Similarly, the next term in the sequence is given by $\tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}} = \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}\right) = \tilde{\aleph}^3\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right)$, and in general, for any $n \in \mathbb{N}$, we have $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} = \tilde{\aleph}^n\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right)$. Now, we proceed as follows:

$$\begin{aligned}
\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}, \tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}}) &= \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}), \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}})) \\
&\leq \tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}})), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}))\} \\
&= \tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}), \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}})), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}), \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}))\} \\
&\leq \tilde{\lambda} Max\{\tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}), \mathfrak{d}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}))\}, \\
&\quad = \tilde{\lambda} Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}})), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}, \tilde{\aleph}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}))\}\}
\end{aligned}$$

$$\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}}, \tilde{\Xi}_{\tilde{\mu}_4}^{\tilde{\eta}_{1_4}}) \leq \tilde{\lambda}^3 Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}),$$

$$\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}, \tilde{\Xi}_{\tilde{\mu}_2}^{\tilde{\eta}_{1_2}}), .\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}}, \tilde{\Xi}_{\tilde{\mu}_3}^{\tilde{\eta}_{1_3}})\} \cdots \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_{n+1}}^{\tilde{\eta}_{1_{n+1}}}) \leq$$

$$\tilde{\lambda}^n Max\{\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}), \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{u}}}}(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}), \ldots, \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mathfrak{I}}}}(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}})\}$$

for every $m \geq n$,

$$\tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_k}^{\tilde{\eta}_{1_k}}\right) + \tilde{\mathfrak{d}}_{\tilde{\mathcal{Z}}^{\tilde{\mu}}}\left(\tilde{\Xi}_{\tilde{\mu}_k}^{\tilde{\eta}_{1_k}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right).$$

Now using inequality (4.5), we write $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right)$,

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \tilde{\lambda}^n \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right) + \tilde{\lambda}^{n+1} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right) + \cdots + \tilde{\lambda}^{m-1} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}^{0}}\right)$$

$$= \frac{\tilde{\lambda}^n}{1-\tilde{\lambda}} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right), \text{where } 0 \leq \tilde{\lambda} < 1$$

Thus, given any $\varepsilon > 0$, we choose a sufficiently large natural number N such that the inequality $\frac{\tilde{\lambda}^n}{1-\tilde{\lambda}} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}, \tilde{\Xi}_{\tilde{\mu}_0}^{\tilde{\eta}_{1_0}}\right) < \varepsilon$ holds for all $n \geq N$. Consequently, it follows that $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \varepsilon$ for all $m \geq n$. Hence, the sequence $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \right\rangle$ is a Cauchy sequence of $\mathcal{FP}$s in $\tilde{\mathcal{Z}}^{\tilde{\jmath}}$ for all $n \in \mathbb{N}$. Since $\left(\tilde{\mathcal{Z}}^{\tilde{\jmath}}, \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\right)$ is a complete $\mathcal{FMS}$, the sequence $\left\langle \tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \right\rangle$ converges to some $\mathcal{FP}$ $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1} \in \tilde{\mathcal{Z}}^{\tilde{\jmath}}$.

Now, to establish that $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ is indeed a $\mathcal{FIP}$ of $\tilde{\aleph}$, we use the fact that $\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}} \to \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ and that $\tilde{\aleph}$ is continuous. This implies that

$$\tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}\right) \to \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right),$$

which, in turn, means $\tilde{\Xi}_{\tilde{\mu}_{n+1}}^{\tilde{\eta}_{1_{n+1}}} \to \tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right)$ as $n \to \infty$. Since limits are unique, we conclude that $\tilde{\aleph}\left(\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}\right) = \tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$, confirming that $\tilde{\Xi}_{\tilde{\mu}}^{\tilde{\eta}_1}$ is indeed a $\mathcal{FIP}$ of $\tilde{\aleph}$. Likewise, inequality (4.5) gives $\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right)$, Then we write,

$$\tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_n}^{\tilde{\eta}_{1_n}}, \tilde{\Xi}_{\tilde{\mu}_m}^{\tilde{\eta}_{1_m}}\right) \leq \frac{\tilde{\lambda}^n}{1-\tilde{\lambda}} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right), 0 < \tilde{\lambda} < 1$$

Moreover, for any $\varepsilon > 0$, choose a sufficiently large natural number N such that the inequality $\frac{\tilde{\lambda}^n}{1-\tilde{\lambda}} \tilde{\delta}_{\tilde{\mathcal{Z}}^{\tilde{\jmath}}}\left(\tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}, \tilde{\Xi}_{\tilde{\mu}_1}^{\tilde{\eta}_{1_1}}\right) < \varepsilon$ holds for every $n \geq N$. Hence we proved.

5. Conclusions

Our Research work, with theoretical results on $\mathcal{FMS}$s. We also explore their relationship with non-fuzzy (crisp) $\mathcal{MS}$s. $\mathcal{FIP}$ theory in both fuzzy and non-$\mathcal{FMS}$s plays a crucial role, with contractive mappings being fundamental in establishing $\mathcal{FIP}$ results.

By redefining $\mathcal{FMS}$s using fuzzy scalars defined in [4] instead of fuzzy or real numbers, we offer a novel perspective. It has been demonstrated that any ordinary $\mathcal{MS}$ can generate a corresponding $\mathcal{FMS}$ that remains complete if the original space is complete. Consequently, proving $\mathcal{FIP}$ theorems depends on the nature for specific application.

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