

Fractional thermodynamic model for surface temperature patterns of brass

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Abstract

This study investigates thermal distribution and heat flux behaviour in a CuPuZn brass alloy using a fractional-order heat conduction model. The analysis extends to one-, two-, and three-dimensional brass slabs to examine temperature distribution and thermal response.

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The proposed framework provides predictive insights for improving thermal management and mechanical performance in metallurgical applications.

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1. Introduction

Brass is one of the most widely used engineering alloys because of its excellent thermal conductivity, corrosion resistance, mechanical strength, and ease of fabrication. Owing to these properties, brass components are extensively employed in heat exchangers, electrical connectors, bearings, marine equipment, and various metallurgical processes where thermal behaviour plays a significant role. Accurate prediction of temperature distribution during heating and cooling processes is therefore essential for improving thermal efficiency, structural reliability, and material performance.

Several experimental investigations have examined the influence of temperature on the mechanical and metallurgical properties of brass alloys. Klosak et al. [1] demonstrated that increasing temperature considerably affects the impact resistance and deformation characteristics of brass materials. Puspitasari et al. [2] reported that pouring temperature significantly influences the hardness and casting quality of brass specimens, while Abbas [3] analysed the effects of temperature on the dezincification behaviour of 70-30 brass in acidic environments.

Fractional calculus has recently emerged as an effective mathematical framework for modelling heat conduction in heterogeneous materials because fractional operators naturally incorporate memory and nonlocal effects that cannot be represented by classical diffusion models. Choudhary, Kumar, and Singh [4] developed a fractional heat conduction model for semi-infinite solids, whereas Khan et al. [5] employed the Natural Transform Decomposition Method to obtain analytical solutions of fractional heat and wave equations. Johansyah et al. [6] further demonstrated the efficiency of decomposition-based integral transform techniques for solving fractional differential equations.

Recent studies demonstrated that fractional differential equations provide an efficient mathematical framework for modelling heat transfer and nonlocal thermal diffusion processes [7], while generalized Mittag-Leffler functions continue to play a fundamental role in the analytical treatment of fractional models [8]. Motivated by these developments, the present work develops a fractional thermodynamic model for analysing temperature distribution and heat flux in brass slabs under one-, two-, and three-dimensional configurations.

2. Structure of Mathematical Modelling

The study analysed temperature effects on dezincification and corrosion in 70-30 brass exposed to $0.5M\ H_2SO_4$. Zinc and copper losses were quantified at

varying temperatures and time intervals using atomic absorption spectroscopy [3]. This thermodynamic framework focused on modeling temperature-dependent failure modes (520K–620K initial range) under dynamic conditions. Two computational models (Model-1 and Model-2) evaluated thermal response patterns in rectangular brass specimens, correlating temperature gradients with corrosion behavior. Key objectives included identifying critical temperature thresholds and deducing degradation mechanisms through fractional thermodynamic analysis.

Following the fractional heat conduction model developed by Choudhary, Kumar, and Singh [4], we consider the one-dimensional fractional-order heat conduction equation:

$$D_t^{(\alpha,\beta)} V(x,t) = \lambda \frac{\partial^2 V(x,t)}{\partial x^2}; \quad 0 \leq \alpha \leq 1, \beta \leq 1, t > 0, x \in \mathbb{R} \quad (1)$$

Here, λ represents the metal's thermal diffusivity, assumed constant. The normalized temperature $V(x,t)$ is defined as $\frac{T-T_0}{T_s-T_0}$, where $T_0 > 0$ is the initial surface temperature and $T_s > 0$ denotes the elevated temperature. The initial and boundary conditions are:

$$\begin{cases} \text{I. C.} & : V(x,0) = 0 \text{ for all } x \\ \text{B. C.} & : V(0,t) = 1 \text{ for } t > 0 \text{ and } V(x,t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ for } t > 0 \end{cases} \quad (2)$$

The Fourier sine transform is applied to equation (1) across the spatial variable x to analyze the thermal response, we obtained:

$$D_t^{(\alpha,\beta)} V_s(n,t) = n\lambda \sqrt{\frac{2}{\pi}} - n^2 \lambda V_s(n,t)$$

Equation (1) was solved using Laplace and inverse Laplace transforms, yielding a solution expressed through the Wright function $W(\alpha, \beta; z)$. This analytical approach ensured precise derivation of the temperature-dependent corrosion behavior under the specified thermodynamic framework,

$$V(x,t) = W\left(-\frac{\alpha}{2}, 1; \frac{-x}{\sqrt{\lambda t^\alpha}}\right) \quad (3)$$

Equation (3) characterizes the temperature distribution in a metal under thermal fluctuations, where heat propagates from regions of higher to lower temperature.

To quantify the thermal energy transfer rate in solid metals induced by such temperature gradients, the following theorem is proposed:

Theorem 1: Consider the one-dimensional fractional-order heat conduction equation (1), subject to the conditions given in equation (2), then the temperature distribution on the solid surface is expressed as:

$$V(x,t) = 1 - \frac{\left(\frac{x}{\sqrt{\lambda t^\alpha}}\right)}{\Gamma\left(\frac{-\alpha}{2}+1\right)1!} + \frac{\left(\frac{x}{\sqrt{\lambda t^\alpha}}\right)^2}{\Gamma(-\alpha+1)2!} - \dots = W\left(-\frac{\alpha}{2}, 1; \frac{-x}{\sqrt{\lambda t^\alpha}}\right)$$

where $W(\cdot)$ denotes the Wright function. The resultant heat flux at the surface ($x = 0$) is:

$$\Phi_q = \frac{C(T_s - T_0)}{\sqrt{\lambda t}^\alpha \Gamma\left(1 - \left(\frac{\alpha}{2}\right)\right)}; \quad c > 0, \alpha > 0, \lambda > 0$$

where C is the thermal conductivity and λ is the thermal diffusivity of the metal.

Proof: The theorem is established using Fourier's law for thermal conduction on a metallic surface. The heat flux is defined as:

$$\Phi_q = -C \left[\frac{\partial T}{\partial x} \right] \quad (4)$$

Applying initial conditions at $x = 0$ and substituting the derived temperature solution from equation (3) into equation (4), the surface heat flux simplifies to:

$$\Phi_q = C(T_s - T_0) \left(\frac{1}{\sqrt{\lambda t}^\alpha} \right) \left(\Gamma\left(\frac{-\alpha}{2} + 1\right) \right)^{-1} \quad (5)$$

Here, Γ denotes the gamma function. This result aligns with the theorem's assertion, confirming the relationship between thermal parameters and time-dependent heat flux under the given conditions.

2.1 Numerical Simulation of Brass Slab Thermal Behaviour:

Based on the thermal conductivity characteristics reported by Gu, Li, and Bao [9], a numerical investigation was carried out on a 10 cm thick brass slab insulated with black silicone paint to minimize heat loss. An external heat source applied 620K to one surface ($T_s = 620K$), while the initial slab temperature (T_0) was 520K at $t = 0$. The brass properties included thermal conductivity $C = 109 \text{ W/mK}$ and diffusivity $\lambda = 34.2 \times 10^{-6} \text{ m}^2/\text{s}$. Simulations spanned a 30-second heating period, incorporating fractional order values $\alpha \in [0.5, 2.0]$ derived from prior theoretical results. Heat flux magnitudes ranged between 0 and 650813 W/m² across varying α , with thermal response trends illustrated by a rectilinear curve in Figure 1. This analysis linked fractional-order dynamics to heat flux variability under transient conditions.

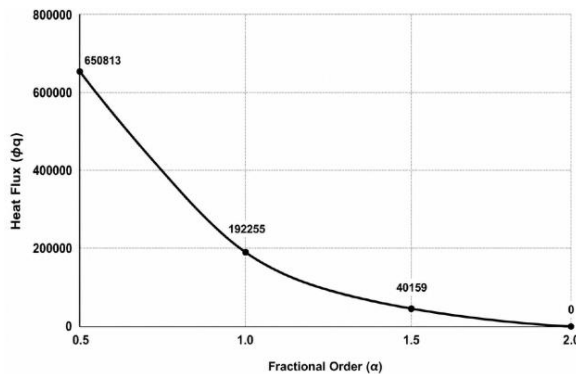


Figure 1
Heat flux v/s Fractional order on brass slab

2.2 Bivariate Analysis of Heat Flux and Fractional Order in Brass:

This study examines the relationship between heat flux (ϕ_q) and fractional order (α) in metallic brass slab. Brass, an alloy, exhibits lower thermal conductivity compared to pure metals (e.g., copper, iron) due to reduced free electron mobility [10]. To establish thermal connectivity between ϕ_q and α , a one-dimensional heat conduction framework was employed, with α as the independent variable and ϕ_q as the dependent parameter. Results revealed a strong inverse correlation ($-1 \leq r \leq -0.75$), indicating higher fractional orders correspond to diminished heat flux. Scatter plots with error bars (Figure 2) further validate this trend, emphasizing brass's unique thermal behavior under fractional thermodynamic conditions.

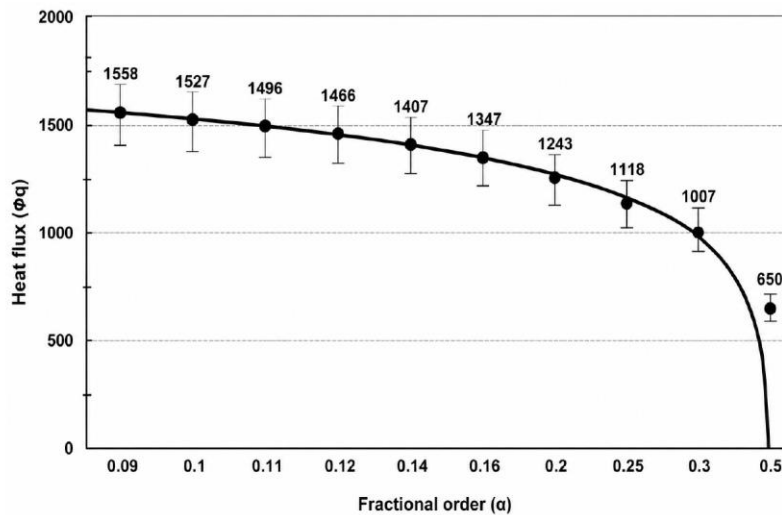


Figure 2
Fitted curve for brass slab

3. Model Implementation Framework

The analysis employs a dual-model approach, structured as Model-1 and Model-2 to systematically evaluate. This two-step framework ensures comprehensive investigation of critical variables while maintaining methodological clarity.

3.1 Thermal Modelling for Metallic Brass Slab (Model-1):

This study derives the temperature distribution in alpha brass by solving the two-dimensional heat conduction equation. The analytical framework establishes a numerical solution to predict thermal behaviour, enabling precise quantification of heat transfer dynamics under specified boundary conditions. Consistent with the thermo-metallurgical characteristics reported by Rojas-Rodríguez et al. [11], the proposed model incorporates the material-specific properties of alpha brass to

validate the temperature profiles and support applications in metallurgical processing and thermal management.

Solution of the Two-Dimensional Fractional Heat Equation: Here we examine heat conduction in a rectangular solid metallic slab using a two-dimensional fractional-order heat equation. The mathematical framework employs fractional calculus to model thermal behaviour, addressing non-local and memory-dependent heat transfer phenomena in the system,

$$\frac{\partial^\alpha V(x,y,t)}{\partial t^\alpha} = \lambda \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right), \quad x > 0, y > 0, t > 0 \quad (6)$$

The thermal diffusivity (λ) of the solid metal, treated as a constant parameter in this study, governs heat propagation within the material. The function $V(x,y,t)$ represents the temperature distribution on the solid's surface at any time $t > 0$.

Case 1: The study begins with the initial condition at $t = 0$:

$$V(x,y,0) = (1-x)e^y \quad \text{for } x > 0, y > 0 \quad (7)$$

Applying the Natural transform [5] to the governing 2D heat equation (6) yields:

$$\frac{s^\alpha}{u^\alpha} N^+[V(x,y,t)] - \frac{s^{(\alpha-1)}}{u^\alpha} V(x,y,0) = N^+ \left[\lambda \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right]$$

Using a decomposition technique [6], the solution is expressed as a series:

$$V(x,y,t) = (1-x)e^y \left[1 + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\lambda^3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\lambda^4 t^{4\alpha}}{\Gamma(4\alpha+1)} + \dots \right] \quad (8)$$

For $\alpha = 1$, this series converges to the closed-form solution:

$$V(x,y,t) = (1-x)e^{(y+\lambda t)} \quad (9)$$

equation (9) represents the exact temperature distribution on the surface of a metallic brass slab for $t > 0$, validating the decomposition method's accuracy for the 2D heat equation (6). This compact solution aids in predicting thermal behaviour under dynamic conditions.

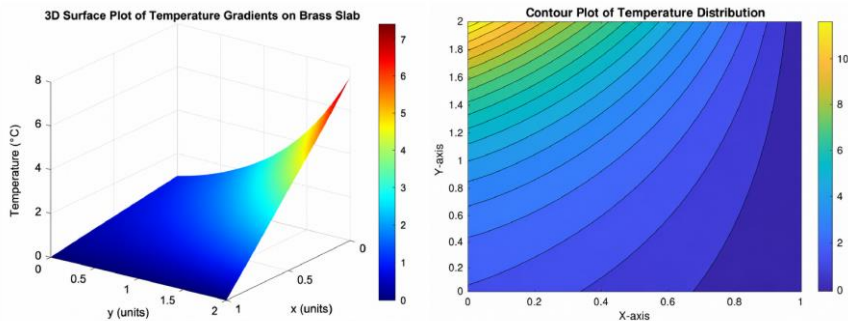


Figure 3
Temperature distribution and contour profile of the two-dimensional brass slab (Case-1)

Numerical Simulation for a 2D Brass Slab: A numerical study evaluated temperature distribution in a rectangular brass slab under external heating (1-minute duration). The slab dimensions were defined as $0 \leq x \leq 1$ and $0 \leq y \leq 2$ units. Using a thermal diffusivity value of $\lambda = 34.2 \times 10^{-6} \text{ m}^2/\text{s}$, temperature variations ($^{\circ}\text{C}$) across the slab's surface were computed. Figure 3 illustrates the resulting 3D surface plot of temperature gradients in the XY-plane and V-contour graph to highlight localized thermal patterns. Assumptions included uniform material properties and steady boundary conditions to simplify the energy flow analysis.

Case 2: This section presents an alternative solution to equation (6) using a second initial condition at $t = 0$:

$$V(x, y, 0) = e^{(2x+y)}, \quad x > 0, y > 0 \quad (10)$$

Applying the Natural transform to equation (6) yields:

$$N^+[V(x, y, t)] = \frac{V(x, y, 0)}{s} + \frac{u^\alpha}{s^\alpha} N^+ \left[\lambda \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \right]$$

For a specific α , the inverse Natural transform produces a series-form solution (eq. 11). When $\alpha = 1$, the solution simplifies to:

$$V(x, y, t) = e^{(2x+y)} \left[1 + 5t + \frac{(5\lambda t)^2}{2!} + \frac{(5\lambda t)^3}{3!} + \frac{(5\lambda t)^4}{4!} + \dots \right] \quad (11)$$

which converges compactly to:

$$V(x, y, t) = e^{(2x+y+5\lambda t)}, \quad x > 0, y > 0 \quad (12)$$

equation (12) represents the final alternative closed-form solution of equation (6), validating the convergence of the series expansion under the given initial conditions.

Numerical Simulation of a 2D Brass Slab: Temperature distributions across a two-dimensional brass slab were numerically simulated within a range $0 \leq x \leq 2$ and $0 \leq y \leq 2$. The resulting temperature profiles as a surface graph and their corresponding V-contour wireframe plots, highlighting thermal gradients within the brass specimen are visualized in Figure 4.

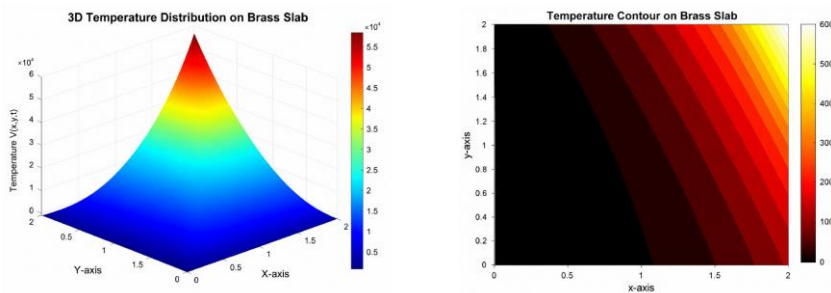


Figure 4
Temperature distribution and contour profile of the two-dimensional brass slab (Case-2)

3.2 3D Heat Conduction Analysis (Model-2):

This model solves the 3D fractional heat equation to determine temperature distribution in alpha brass. The governing equation for a metallic surface with spatial variables $x, y, z > 0$ is:

$$\frac{\partial^\alpha V(x, y, z, t)}{\partial t^\alpha} = \lambda \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right); \quad x > 0; y > 0; z > 0 \quad (13)$$

where $V(x, y, z, t)$ represents temperature and λ is thermal diffusivity. Two initial conditions were analyzed at $t = 0$.

Case 1: Exponential-Sine Initial Condition: At $t = 0$, the temperature distribution was defined as:

$$V_0(x, y, z, 0) = e^{(x+y)} \sin z; \quad (x, y, z > 0) \quad (14)$$

Using the Natural Transform Decomposition Technique, the solution of equation (13) for $\alpha = 1$ (classical heat equation) was derived as:

$$V(x, y, z, t) = e^{(x+y+\lambda t)} \sin z; \quad (x, y, z > 0) \quad (15)$$

The obtained analytical solution is consistent with the fractional thermal modelling approach reported by Verma and Menaria [12], thereby describing the three-dimensional heat conduction dynamics under the specified conditions.

Numerical Simulation of 3D Brass Slab: The numerical simulation modeled a three-dimensional brass slab to visualize temperature distribution (equation 15). The slab's dimensions were defined along the x -axis (length), y -axis (breadth), and z -axis (thickness). In this theoretical framework, the slab was externally heated for one minute. The resulting temperature grids across the brass surface, illustrated in Figure 5, demonstrate spatial thermal gradients under simulated heat exposure.

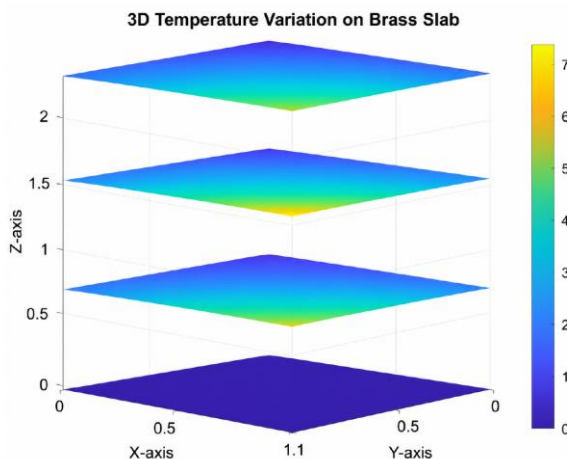


Figure 5
3D Temperature Variation on Brass Slab

Case 2 Algebraic-Exponential Initial Condition: To derive an alternative solution for equation (13), with the initial condition at $t = 0$:

$$V(x, y, z, t) = (1 - x)e^{(y+z)}; (x, y, z \geq 0) \quad (16)$$

Applying the Natural transform and decomposition technique to equation (13), the solution in series form is obtained:

$$V(x, y, z, t) = (1 - x)e^{(y+z)} \left[1 + 2\lambda t + \frac{(2\lambda t)^2}{2!} + \frac{(2\lambda t)^3}{3!} + \dots \right]$$

For $\alpha = 1$, this simplifies to the compact expression:

$$V(x, y, z, t) = (1 - x)e^{(y+z+2\lambda t)} \quad (17)$$

To analyse temperature variations in the brass slab, the assumptions from Case-1 are retained. Using $\alpha = 1$ and thermal diffusivity $\lambda = 34.2 \times 10^{-6} \text{ m}^2/\text{s}$, temperature distributions are graphically represented in Figure 6. This illustrates the dynamic thermal response under the specified conditions.

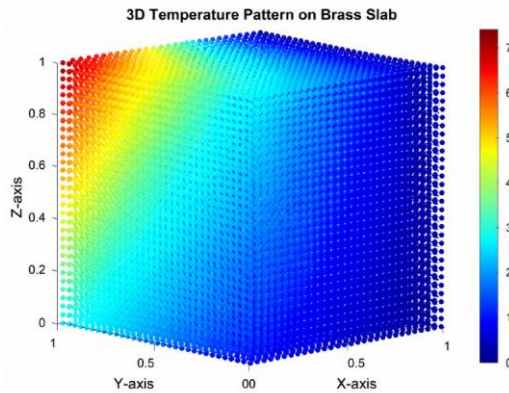


Figure 6
3D Temperature Pattern on Brass Slab

4. Conclusion

This study developed a numerical model to simulate thermal behaviour in brass specimens using a fractional-order one-dimensional heat conduction equation. A strong inverse correlation (-0.993) was observed between heat flux and fractional order. Analysis revealed a logarithmic relationship for heat flux: $\phi_q = 974 - 266 \ln \alpha$; $\alpha > 0$, with scattered data aligning closely to the derived curve. Model validation against experimental data confirmed its accuracy in predicting temperature responses across brass alloys. Case-specific results showed surface temperature growths of 7.389°C (Case-1) and 54.646°C (Case-2). The 3D thermal profile further illustrated temperature distribution dynamics. Findings emphasize that controlled temperature fluctuations enhance mechanical properties and metal strength during heat treatment, supporting optimized industrial metallurgical processes.

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