

Fractal generation and its comparative analysis for sine functions using JN iteration and JI iteration with s-convexity

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Abstract

In this research, we utilize the Jungck-Ishikawa and Jungck-Noor iteration techniques in combination with s -convexity to derive the escape criteria for generating fractals of the complex-valued sine function $\sin \xi^m + a\xi^2 - b\xi + c$, assuming m is a positive integer exceeding two, and a, b, c belonging to the set of complex numbers. Leveraging the escape criteria, we generate Julia sets for both the iteration methods. The fractals produced exhibit star-shaped structures with $2m$ spokes along with intricate internal patterns. We also conduct a comparative analysis of the outcomes obtained from the aforementioned iterations, highlighting the distinct characteristics and aesthetic appeal of the generated fractals.

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1. Introduction and Foundational Concepts

The word fractal is famous now a days in current research scenario. It represents wide range of natural things like leaf patterns, tree branches, lightening, clouds, sea coast etc. It has applications in many field such as in biology, computational architectural design, image compression, image encryption, fluid mechanics etc. In mathematical terms, a fractal can be described as a subset of a space equipped with a metric which exhibits self-similarity across scales and whose fractal dimension exceeds the topological dimension.

In the first decade of the 20th century, French mathematicians Julia [1, 2] and Fatou [3] independently pioneered the study of complex dynamics through the iteration of complex functions, and led the theoretical foundations for the

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subsequent development of fractal geometry. This advancement has further expanded the research horizons, enabling scholars to explore novel dimensions and applications within this field. Over the past forty years, many researchers have explored a wide range of properties of these kind of sets and proposed several generalizations, we refer to see [4] and references therein.

The theory based on fixed-points holds a central place in Applied and Pure Mathematics (both) due to its wide-ranging practical domains. The Banach contraction theorem stands as one of the foundational results in this field, facilitating the convergence of diverse iterative methods to a fixed point. Fixed-point iterative methods offer approaches to finding fixed points of nonlinear mappings [5], which has played a vital role in the exploration of Julia and Mandelbrot sets. These sets represent well-known examples of fractals produced through the repeated iteration of complex-valued functions.

In 2004, Rani and Kumar [17] introduced the Mann iterative process to the study of generation of fractals, replacing the classical Picard iteration and pioneering the visualization of novel fractal structures. Current literature including recent studies on Ishikawa iteration [6], Noor iteration [7], Jungck–Ishikawa iteration [8], Jungck–Noor iteration [9], demonstrate their use not only in generating Mandelbrot and Julia sets but also in producing diverse classes of fractals. Besides the above, [10, 11] also provide a notable work in the arena of fractal theory.

We begin this article by presenting some fundamental definitions along with the iterative schemes stated earlier.

Definition 1.1: [12] Let $Q_c: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree greater than one. The filled Julia set J_{Q_c} is defined as the set of all points $\xi \in \mathbb{C}$ whose forward orbits under iteration of Q_c remain bounded, that is,

$$J_{Q_c} = \{\xi \in \mathbb{C} : \{Q_c^n(\xi)\}_{n=0}^\infty \text{ is bounded}\}.$$

The Julia set associated with Q_c is given by the boundary (most commonly known as topological boundary ∂J_{Q_c}) of the filled Julia set.

Definition 1.2 [13] Let S and T be two mappings such that S is one-to-one, while T is differentiable and of degree greater than or equal to 2. For any $\xi_0 \in \mathbb{C}$, the Jungck–Ishikawa iteration with s -convexity is defined as follows:

$$Su_n = (1 - \beta)^s S\xi_n + \beta^s T\xi_n; S\xi_{n+1} = (1 - \alpha)^s S\xi_n + \alpha^s Tu_n,$$

where $\alpha, \beta, s \in (0, 1]$ and $n = 0, 1, 2, \dots$

Definition 1.3: Let S and T be two mappings such that S is one-to-one, and T is differentiable with degree greater than or equal to 2. For any initial point $\xi_0 \in \mathbb{C}$, the Jungck–Noor iteration with s -convexity is defined as follows:

$$Sv_n = (1 - \gamma)^s S\xi_n + \gamma^s T\xi_n; Su_n = (1 - \beta)^s S\xi_n + \beta^s Tv_n; \\ S\xi_{n+1} = (1 - \alpha)^s S\xi_n + \alpha^s Tu_n,$$

where $\alpha, \beta, \gamma, s \in (0, 1]$ and $n = 0, 1, 2, \dots$

2. Escape criteria with s -convexity

Lets begin this section, by splitting the polynomial $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$, where $m \geq 3$ and a, b, c belonging to the set of complex numbers, using two self maps S and T as: $Q_c(\xi) = T\xi - S\xi$, where $S\xi = b\xi$ is injective and $T\xi = \sin \xi^m + a\xi^2 + c$.

Furthermore, using the expansion of $\sin \xi$ (by Taylor's series), we obtain

$$|\sin \xi| = |\xi| \left| 1 - \frac{\xi^2}{3!} + \frac{\xi^4}{5!} - \frac{\xi^6}{7!} + \dots \right|.$$

which implies that

$$|\sin \xi| \geq |\xi| |w_1|, \quad (1)$$

where $w_1 \in (0, 1]$ is a real number. In the similar way, we can write

$$|\sin u| \geq |u| |w_2|; \quad |\sin v| \geq |v| |w_3|, \quad (2)$$

where $w_2, w_3 \in (0, 1]$ are real numbers.

Lets begin with our main result.

Theorem 2.1: Let $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$ be a given function. Assume that

$$|\xi| \geq |c| > \frac{2(|b|+1)}{\alpha s(|w_2||c|^{m-2}-|a|)}, \quad (3)$$

$$|\xi| \geq |c| > \frac{2(|b|+1)}{\beta s(|w_3||c|^{m-2}-|a|)}, \quad (4)$$

$$|\xi| \geq |c| > \frac{2(|b|+1)}{\gamma s(|w_1||c|^{m-2}-|a|)}, \quad (5)$$

where $\alpha, \beta, \gamma, s, w_1, w_2, w_3 \in (0, 1]$ and a, b, c are complex numbers. Then for given iterative scheme in Definition 1.3 for the function $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$, we have $\lim_{n \rightarrow \infty} |\xi_n| = \infty$.

Proof: Define $S\xi = b\xi$ and $T\xi = \sin \xi^m + a\xi^2 + c$; $m > 2$ and $m \in \mathbb{N}$. We will prove the required result by using iteration given in Definition 1.3. On using the first step of iteration, we have

$$\begin{aligned} |Sv_0| &= |(1-\gamma)^s S\xi_0 + \gamma^s T\xi_0| = |\gamma^s (\sin \xi_0^m + a\xi_0^2 + c) + (1-\gamma)^s b\xi_0| \\ &\geq \gamma^s |(\sin \xi_0^m + a\xi_0^2 + c)| - (1-\gamma)^s |b\xi_0|. \end{aligned}$$

Since, for $s, \gamma \in (0, 1]$, it is true that $\gamma^s \geq s\gamma$ and $(1-\gamma)^s \geq 1-s\gamma$, we have

$$\begin{aligned} |Sv_0| &\geq s\gamma |(\sin \xi_0^m + a\xi_0^2 + c)| - (1-s\gamma) |b\xi_0| \geq s \\ &\gamma |\sin \xi_0^m| - s\gamma |a\xi_0^2| - s\gamma |c| - (1-s\gamma) |b\xi_0|. \end{aligned}$$

On using Eq. (1) and $|\xi_0| \geq |c|$ and the fact that $S\xi = b\xi$, we obtain

$$\begin{aligned} |bv_0| &= |Sv_0| \geq s\gamma |\xi_0^m| |w_1| - s\gamma |a\xi_0^2| - s\gamma |\xi_0| - |b\xi_0| + s\gamma |b\xi_0| \\ &\geq |\xi_0^2| s\gamma (|c|^{m-2} |w_1| - |a|) - |\xi_0| (1 + |b|) \\ &\geq |\xi_0| (1 + |b|) \left(\frac{s\gamma |\xi_0| (|c|^{m-2} |w_1| - |a|)}{1 + |b|} - 1 \right). \end{aligned} \quad (6)$$

Since $|v_0| \geq \frac{|b||v_0|}{1+|b|} = \frac{|bv_0|}{1+|b|}$. Thus Eq. (6) implies that

$$|v_0| \geq |\xi_0| \left(\frac{s\gamma|\xi_0|(|c^{m-2}||w_1|-|a|)}{1+|b|} - 1 \right)$$

By using Eq. (5), we have

$$|v_0| \geq |\xi_0|. \quad (7)$$

By applying the second iteration step from Definition 1.3, we get

$|Su_0| = |(1-\beta)^s S\xi_0 + \beta^s Tv_0| = |\beta^s(\sin v_0^m + av^2 + c) + (1-\beta)^s b\xi_0|$
and so, $|Su_0| \geq \beta^s |(\sin v_0^m + av^2 + c)| - (1-\beta)^s |b\xi_0|$. Since, for $s, \beta \in (0,1]$, it is true that $\beta^s \geq s\beta$ and $(1-\beta)^s \geq 1-s\beta$, give that

$$|Su_0| \geq s\beta |(\sin v_0^m + av^2 + c)| - (1-s\beta) |b\xi_0| \geq s\beta |\sin v_0^m| - (1-\beta s) |b\xi_0| - s\beta |av^2| - s\beta |c|.$$

On using Eq. (2) and $|\xi_0| \geq |c|$, we obtain

$$\begin{aligned} |Su_0| &\geq s\beta |v_0^m||w_3| - |b\xi_0| + \beta s |b_0\xi_0| - s\beta |a||v_0^2| - |\xi_0| [\text{Since } s, \beta \leq 1] \\ &\geq s\beta |v_0^m||w_3| - |b||\xi_0| - s\beta |a||v_0^2| - |\xi_0| \\ &= |v_0^2| s\beta (|v_0^{m-2}||w_3| - |a|) - |\xi_0|(1+|b|). \end{aligned}$$

Using Eq.(7) and $|\xi_0| \geq |c|$, we have

$$\begin{aligned} |bu_0| &= |Su_0| \geq |\xi_0^2| s\beta (|v_0^{m-2}||w_3| - |a|) - |\xi_0|(1+|b|) \\ &\geq |\xi_0|(1+|b|) \left(\frac{s\beta|\xi_0|(|c^{m-2}||w_3|-|a|)}{1+|b|} - 1 \right). \end{aligned} \quad (8)$$

Since, $|u_0| \geq \frac{|b||u_0|}{1+|b|} = \frac{|bu_0|}{1+|b|}$ implies from Eq. (8) that

$$|u_0| \geq |\xi_0| \left(\frac{s\beta|\xi_0|(|c^{m-2}||w_3|-|a|)}{1+|b|} - 1 \right).$$

By assumption in (4), it follows that

$$|u_0| \geq |\xi_0|. \quad (9)$$

Again, at last, for third step of iteration, we get

$$|S\xi_1| = |(1-\alpha)^s S\xi_0 + \alpha^s Tu_0| \geq \alpha^s |(\sin u_0^m + au_0^2 + c)| - (1-\alpha)^s |b\xi_0|.$$

Since, for $s, \alpha \in (0,1]$, it is true that $\alpha^s \geq s\alpha$ and $(1-\alpha)^s \geq 1-s\alpha$, we get

$$\begin{aligned} |S\xi_1| &= s\alpha |(\sin u_0^m + au_0^2 + c)| - (1-s\alpha) |b\xi_0| \geq s\alpha |u_0^m||w_2| \\ &\quad - |b\xi_0| + \alpha s |b\xi_0| - s\alpha |a||u_0^2| - |\xi_0|. \end{aligned}$$

$$\begin{aligned} \text{So, } |b\xi_1| &\geq s\alpha |\xi_0^m||w_2| - |b||\xi_0| - s\alpha |a||\xi_0^2| - |\xi_0| \\ &= |\xi_0^2| (s\alpha |\xi_0^{m-2}||w_2| - |a|) - |\xi_0|(1+|b|). \end{aligned}$$

Since $|\xi_1| \geq \frac{|b||\xi_1|}{|b|+1} = \frac{|b\xi_1|}{|b|+1}$. Therefore, on using above inequality, we obtain

$$|\xi_1| \geq |\xi_0| \left(\frac{s\alpha|w_2||c^{m-2}|-|a|}{1+|b|} - 1 \right).$$

By our assumption in Eq. (3), it follows that $|\xi_1| \geq |\xi_0|$. Also, there may exists some $\mu > 0$ such that so $|\xi_1| > (1+\mu)|\xi_0|$. Repeating all the steps

processed above, we obtain that $|\xi_2| > (1 + \mu)|\xi_1|$. This implies that $|\xi_2| > (1 + \mu)|\xi_1| > (1 + \mu)^2|\xi_0|$. Proceeding in the same manner, we can get $|\xi_n| > (1 + \mu)^n|\xi_0|$. Hence, $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$. So, orbit of $\xi_0 \rightarrow \infty$ with the assumed conditions.

We now establish the escape criterion based on the Jungck--Ishikawa fixed point iteration with s -convexity for the function $\sin(\xi^m) + a\xi^2 - b\xi + c$, where $m \geq 3$ and $a, b, c \in \mathbb{C}$.

Theorem 2.2: Let $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$ be a given function. Assume that

$$|\xi| \geq |c| > \frac{2(|b|+1)}{\alpha s(|w_3||c|^{m-2}-|a|)}, \quad (10)$$

$$|\xi| \geq |c| > \frac{2(|b|+1)}{\beta s(|w_2||c|^{m-2}-|a|)}, \quad (11)$$

where c is a complex number and $\alpha, \beta, s \in (0, 1]$. Then for given iteration in Definition 1.2 for the function $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$, we have $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: Define $S\xi = b\xi$ and $T\xi = \sin \xi^m + a\xi^2 + c$; $m = 0, 1, 2, \dots$. Also by using iteration given in Definition 1.2, we get that from the first step of iteration,

$$\begin{aligned} |Su_0| &= |(1 - \beta)^s S\xi_0 + \beta^s T\xi_0| \\ &= |\beta^s (\sin \xi_0^m + a\xi_0^2 + c) + (1 - \beta)^s b\xi_0| \\ &\geq \beta^s |(\sin \xi_0^m + a\xi_0^2 + c)| - (1 - \beta)^s |b\xi_0|. \end{aligned}$$

Since, for $s, \beta \in (0, 1]$, it is true that $\beta^s \geq s\beta$ and $(1 - \beta)^s \geq 1 - s\beta$, this gives

$$\begin{aligned} |Su_0| &= s\beta |(\sin \xi_0^m + a\xi_0^2 + c)| - (1 - s\beta) |b\xi_0| \geq s\beta |\xi_0^m| |w_1| \\ &\quad - s\beta |a\xi_0^2| - s\beta |\xi_0| - |b\xi_0| + s\beta |b\xi_0| \\ &\geq s\beta |\xi_0^m| |w_1| - s\beta |a\xi_0^2| - s\beta |\xi_0| - |b\xi_0| \\ &= s\beta (|\xi_0^m| |w_1| - |a\xi_0^2|) - (|\xi_0| + |b||\xi_0|). \end{aligned}$$

As $S\xi = b\xi$, so it implies that

$$\begin{aligned} |bu_0| &= |Su_0| \geq |\xi_0^2| s\beta (|\xi_0^{m-2}| |w_1| - |a|) - |\xi_0| (1 + |b|) \\ &\geq |\xi_0^2| s\beta (|c|^{m-2} |w_1| - |a|) - |\xi_0| (1 + |b|) \\ &= |\xi_0| (1 + |b|) \left(\frac{s\beta |\xi_0| (|c|^{m-2} |w_1| - |a|)}{1 + |b|} - 1 \right). \end{aligned} \quad (12)$$

Since $|u_0| \geq \frac{|b||u_0|}{1 + |b|} = \frac{|bu_0|}{1 + |b|}$. So, Eq. (12) implies

$$|u_0| \geq |\xi_0| \left(\frac{s\beta |\xi_0| (|c|^{m-2} |w_1| - |a|)}{1 + |b|} - 1 \right). \text{ By our assumption } |\xi| > \frac{2(|b|+1)}{s\beta (|c|^{m-2} |w_1| - |a|)}, \text{ it follows that } |u_0| \geq |\xi_0|.$$

For second step of iteration, we get

$$|S\xi_1| = |(1 - \alpha)^s S\xi_0 + \alpha^s T u_0| = |\alpha^s (\sin u_0^m + a u_0^2 + c) + (1 - \alpha)^s b\xi_0|$$

Since, for $s, \alpha \in (0, 1]$, it is true that $\alpha^s \geq s\alpha$ and $(1 - \alpha)^s \geq 1 - s\alpha$, we have

$$\begin{aligned}
|S\xi_1| &= s\alpha|(\sin u_0^m + au_0^2 + c)| - (1 - s\alpha)|b\xi_0| \\
&\geq s\alpha|\sin u_0^m| - (1 - \alpha s)|b\xi_0| - |au_0^2| - |c| \\
&\geq s\alpha|u_0^m||w_2| - |b\xi_0| + \alpha s|b\xi_0| - s\alpha|a||u_0^2| - |\xi_0| \quad [\text{Using (2)}] \\
&\geq s\alpha|u_0^m||w_2| - |b||\xi_0| - s\alpha|a||u_0^2| - |\xi_0|.
\end{aligned}$$

Since both s and $\beta \leq 1$, so we can get

$$\begin{aligned}
|b\xi_1| &= |S\xi_1| \geq |u_0^2|s\alpha(|u_0^{m-2}||w_2| - |a|) - |\xi_0|(1 + |b|) \\
&\geq |\xi_0^2|s\alpha(|\xi_0^{m-2}||w_2| - |a|) - |\xi_0|(1 + |b|) \\
&\geq |\xi_0|(1 + |b|) \left(\frac{s\alpha|\xi_0|(|c|^{m-2}||w_2| - |a|)}{1 + |b|} - 1 \right).
\end{aligned} \tag{13}$$

Since $|\xi_1| \geq \frac{|b||u_0|}{1 + |b|} = \frac{|b|u_0|}{1 + |b|}$, Eq. (13) implies

$$|\xi_1| \geq |\xi_0| \left(\frac{s\alpha|\xi|(|c|^{m-2}||w_2| - |a|)}{1 + |b|} - 1 \right)$$

By our assumption in (6) it follows that $|\xi_1| \geq |\xi_0|$. So there may exists some $\mu > 0$ such that we may take $|\xi_1| > (1 + \mu)|\xi_0|$. Similarly, $|\xi_2| > (1 + \mu)|\xi_1|$. This implies that $|\xi_2| > (1 + \mu)|\xi_1| > (1 + \mu)^2|\xi_0|$. Proceeding in the same manner, we get $|\xi_n| > (1 + \mu)^n|\xi_0|$.

Hence, $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$.

3. Graphical Analysis of Julia Sets: Jungck-Noor vs. Jungck-Ishikawa Iterations

In this section, some Julia set images were generated using escape time algorithm for Jungck-Noor iteration and Jungck-Ishikawa iteration along with s -convexity with Jet colormap and maximum iteration counts are set to 50. Each Figure contains six images out of which first three (in first row) are composed using Jungck-Ishikawa iteration and rest three (in second row) using Jungck-Noor iteration and a parameter is varied with three different values.

3.1 Visualisation of Julia set: Algorithm

We used the following algorithms used for visualisation of Julia sets in which requires are $Q_c(\xi) = \sin \xi^m + a\xi^2 - b\xi + c$, where $m = 3, 4, \dots$; $a, b, c \in \mathbb{C}$ are parameters; $A \subset \mathbb{C}$ is area; K is the maximum iteration counts; $\alpha, \beta, \gamma \in (0, 1]$ etc. are parameters used in Jungck-Noor and Jungck Ishikawa iterations; Colormap[0 ... $C - 1$]-colour map with C colours.

Algorithm 1 Julia set generation via Jungck-Noor iteration

Ensure: Julia set over the region A .

Compute the escape radius

$$R = \max \left\{ |c|, \frac{2(1+|b|)}{s\gamma(|w_1||c_0|^{m-2}-|a|)}, \frac{2(1+|b|)}{s\alpha(|w_2||c_0|^{m-2}-|a|)}, \frac{2(1+|b|)}{s\beta(|w_3||c_0|^{m-2}-|a|)} \right\}.$$

for each initial point $\xi_0 \in A$ **do**

Initialize $n = 0$.

while $n \leq K$ **do**

$$\begin{aligned} \text{Compute } v_n &= \frac{(1-\gamma)^s(b\xi_n) + \gamma^s(T\xi_n)}{b}, \quad u_n = \frac{(1-\beta)^s(b\xi_n) + \beta^s(Tv_n)}{b}, \\ \xi_{n+1} &= \frac{(1-\alpha)^s(b\xi_n) + \alpha^s(Tu_n)}{b}. \end{aligned}$$

if $|\xi_{n+1}| > R$ **then**

exit loop

end

Increment $n \leftarrow n + 1$.

end while

Assign color index $i = \lfloor (C-1)\log_2(1 + |\xi_{n+1}|) \rfloor$ and color ξ_0 using colormap $[i]$.

end for

Algorithm 2 Julia set generation via Jungck-Ishikawa Iteration

Ensure: Julia set for area A .

Compute the escape radius

$$R = \max \left\{ |c|, \frac{(2(1+|b|))}{(s\alpha(|w_2||c_0|^{m-2}-|a|))}, \frac{(2(1+|b|))}{(s\beta(|w_3||c_0|^{m-2}-|a|))} \right\}$$

for each initial point $\xi_0 \in A$ **do**

Initialize $n = 0$.

while $n \leq K$ **do**

Compute

$$\begin{aligned} u_n &= \frac{((1-\beta)^s(b\xi_n) + \beta^s(T\xi_n))}{b}, \\ \xi_{n+1} &= \frac{((1-\alpha)^s(b\xi_n) + \alpha^s(Tu_n))}{b}. \end{aligned}$$

if $|\xi_{n+1}| > R$ **then**

exit loop

end if

Increment $n \leftarrow n + 1$.

end while

Assign color index $i = \lfloor (C-1)\log_2(1 + |\xi_{n+1}|) \rfloor$ and color ξ_0 using colormap $[i]$.

end for

3.1.1 Visualisation

Here we will represent Julia set images generated using Jungck-Ishikawa and Jungck-Noor iteration along with s -convexity.

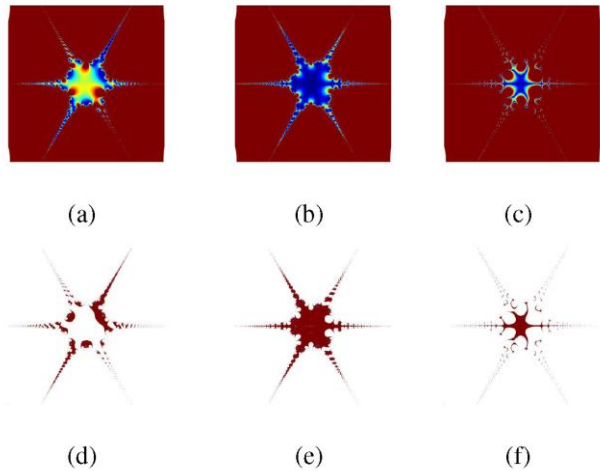


Figure 1
Julia set images for parameters described in Table 1 reflecting effect of change in value of c

Table 1
Detail of parameters for Figure 1

c	$m = 3, a = 0.34, b = 0.34, s = 0.9, \alpha = \beta = \gamma = 0.6, w_1 = w_2 = w_3 = 0.8$	
	Run-Time for JI iteration (in sec)	Run-Time for JN iteration (in sec)
-4i	2.426	103.646
10	2.565	108.503
3.4	2.615	108.425

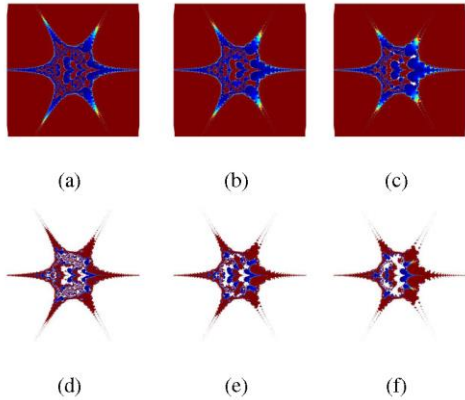


Figure 2
Julia set images for parameters described in Table 2 reflecting effect of change in value of s

Table 2
Detail of parameters for Figure 2

s	$m = 3, a = 11, b = 11, c = 11, \alpha = \beta = \gamma = 0.6, w_1 = w_2 = w_3 = 0.8$	
	Run-Time for JI iteration (in sec)	Run-Time for JN iteration (in sec)
0.9	3.151	101.960
0.5	2.729	75.972
0.1	3.096	100.582

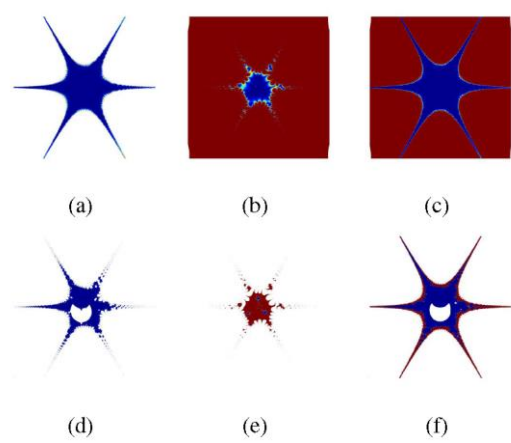
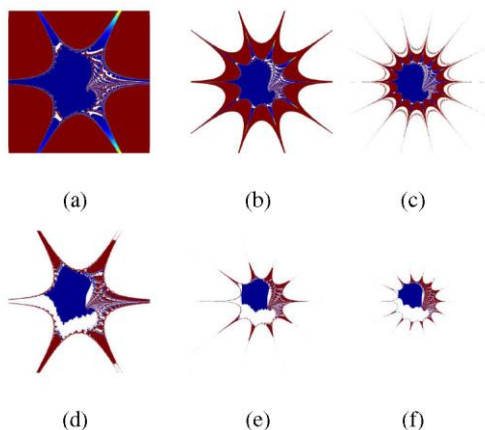


Figure 3
 Julia set images for parameters described in Table 3 reflecting effect of change in value of b

Table 3
Detail of parameters for Figure 3

b	$m = 3, a = 0.34 + 0.3i, c = 0.34i, \alpha = \beta = \gamma = 0.6, s = 0.9, w_1 = w_2 = w_3 = 0.8$	
	Run-Time for JI iteration (in sec)	Run-Time for JN iteration (in sec)
$1.2+0.8i$	66.615	100.314
$-0.09i$	2.470	101.564
2	4.066	99.200

**Figure 4**

Julia set images for parameters described in Table 4 reflecting effect of change in value of m

Table 4

Detail of parameters for Figure 4

m	$a = 501, b = 5000, c = 405i, \alpha = \beta = \gamma = 0.9, s = 0.001, w_1 = w_2 = w_3 = 0.5$	
	Run-Time for JI iteration (in sec)	Run-Time for JN iteration (in sec)
3	13.717	93.379
5	59.945	104.928
7	72.958	128.360

3.1.2 Result and Discussion

- (i) The star-shaped fractal is obtained in each of the Julia set image for both of the iterations in which the numbers of perks or vertices are two times of the value of parameter m .
- (ii) In Figure 1, change in the values of the parameter c has prominent effect on stars produced. Using Jungck-Ishikawa iteration, the star produced are multicolored with red background while that of produced using Jungck-Noor iteration are single colored. When the absolute value of c (Real or Complex) increased, the star reduced into discrete shapes for both the iteration process.
- (iii) Figure 2 reflects the influence of change in convexity parameter s . The stars shape obtained along with intricate inner structure filled with primarily appearing cardioid like shapes(which vary in shape and size) from which numerous filaments or bulbs like structure emanate. All these images are symmetric about x -axis. The stars produced using Jungck-Ishikawa iteration are blue coloured with multicoloured wings with red background. While in

case of Jungck-Noor iteration, the star are of red coloured with the internal orbit made of blue and white colours alternatively.

- (iv) Figure 3 shows, if b is sufficiently small, the spokes of star becomes discrete and less visible in both the iterations. In Jungck-Ishikawa iteration, the stars we got are solid blue coloured sometimes with red and sometimes white background. In case of Jungck-Noor iteration, the stunning fractals of star containing a moon within it were obtained.
- (v) In Figure 4 it is noticable that number of vertices of star is equal to 2 times the value of m . The increase in the value of m a more little bit, leaves a circular vision in the shape and the size of star get reduced. In case of Jungck-Ishikawa iteration, another larger star can be seen covering the original star. The internal structure of the star remained identical even with the change in value of m . But increment in the value of m also increases the run time of images.

Remark

From many works done in the literature [14], [15], [16], we can see that Jungck-Noor iteration converges faster than Jungck-Ishikawa iteration. But, we found that the execution time for generation of fractal taken by Jungck-Noor iteration is more than that of Jungck-Ishikawa iteration.

4. Conclusion

In this paper, we derived the escape condition for the generation of fractals using Jungck-Ishikawa and Jungck-Noor iterations with s -convexity for sine function $\sin \xi^m + a\xi^2 - b\xi + c$. Using this criteria, we generated the Julia sets for both the iterations. We found the following observations:-

- (i) The fractals generated (Julia set) are of star-shaped and the number of vertices of stars are double of the value of m and with the increase of m , the size of fractals is reduced.
- (ii) We compared the run-time results for both the iterations and found that the Jungck-Ishikawa iteration with s -convexity outperforms the Jungck-Noor iteration with s -convexity in terms of speed and computational efficiency but the fractal shapes generated using both iterations are equally applicable.
- (iii) Most of the Figures obtained are symmetric about the x -axis and some about y -axis.
- (iv) It is remarkable to note that we also found the shape of moon within a star (using Jungck-Noor iteration process) while the visualization process of Julia set.

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