

Fixed point results in CAT (0) space using non-expansive mappings

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Abstract

This paper introduces new fixed point theorems for non-expansive mappings specifically within complete CAT(0) spaces, which are vital instances of geodesic metric spaces characterized by non-positive curvature. We specifically demonstrate the efficiency of the established SNIA iterative process in approximating fixed points. This is achieved using Δ -convergence, in CAT(0) Space. The results presented here enhance and integrate various known convergence theorems, establishing a more comprehensive context for analyzing iterative schemes in CAT(0) spaces.

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1. Introduction

Fixed point theory has become a highly dynamic field of mathematical research because of its applications across diverse domains like engineering, physics, economics, game theory, image processing, and many other scientific areas.

Let $R \neq \emptyset$ & $f: R \rightarrow R$. An element $\kappa \in R$ is fixed point f to κ results in κ itself, i.e; $f(\kappa) = \kappa$.

$$f(\kappa) = \kappa$$

Therefore, a fixed point is an element that the mapping does not move.

The uniqueness and existence is proven if there is a scalar $\lambda \in [0,1)$ such that

$$g(T\kappa, T\varpi) \leq \lambda g(\kappa, \varpi), \quad \forall \kappa, \varpi \in R,$$

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then T admits exactly one fixed point in R which is known as Banach contraction principle introduced in 1922 [1].

Following the introduction of metric spaces by Fréchet [2] in 1906, numerous generalizations and extensions of this structure have been developed. The types of spaces considered include (but are not limited to) partial metric spaces, b -metric spaces, G -metric spaces, and cone metric spaces. Such generalized frameworks have provided fertile ground for extending and unifying fixed point results for a wide variety of mappings [3, 4, 5, 9, 13, 14].

2. Preliminaries

Consider (R, g) as metric space and continuous mapping $T: [0, p] \rightarrow R$ with $T(0) = \mathfrak{u} \in R$ and $T(p) = \mathfrak{w} \in R$. Then T is *geodesic* if

$$g(T(\kappa), T(\mu)) = |\kappa - \mu|, \forall \kappa, \mu \in [0, p].$$

The set $[\mathfrak{u}, \mathfrak{w}]$ is referred to as a *geodesic segment*, and the space (R, g) is termed a *geodesic space*. Moreover, if each pair of different points in the space connected by exactly one geodesic, the space is called uniquely geodesic [6, 7, 8].

CAT(0) space is geodesic metric space (R, g) where $\Delta(r_1, r_2, r_3)$, obey the CAT(0) inequality when contrasted with their comparison triangle $\bar{\Delta}$ in the Euclidean plane $\mathbb{E}^2$ we have

$$g(\mathfrak{u}, \mathfrak{w}) \leq g_{\mathbb{E}^2}(\bar{\mathfrak{u}}, \bar{\mathfrak{w}}), \forall \mathfrak{u}, \mathfrak{w} \in \Delta, \bar{\mathfrak{u}}, \bar{\mathfrak{w}} \in \bar{\Delta}.$$

Equivalently, CAT(0) spaces satisfy the *Bruhat--Tits (CN) inequality*:

$$g(\mathfrak{u}, s_0)^2 \leq \frac{1}{2} g(\mathfrak{u}, s_1)^2 + \frac{1}{2} g(\mathfrak{u}, s_2)^2 - \frac{1}{4} g(s_1, s_2)^2,$$

where s_0 is the midpoint of $[s_1, s_2]$.

2.1 Δ -Convergence

Δ -convergence was first introduced by Lim [15] and later generalized to CAT(0) space by Kirk [6] and Suzuki [10]. We say that a sequence $\{R_n\} \subset R$ Δ -converges to $\kappa \in R$ ($\Delta\text{-}\lim R_n = \kappa$) if κ is the sole asymptotic center of every one of its subsequences. In addition, if $\mathfrak{w} \neq \kappa$, the following distance property must hold: the sequence approaches κ more closely than it approaches $\mathfrak{w}$.

$$\limsup_{n \rightarrow \infty} g(R_n, \kappa) \leq \limsup_{n \rightarrow \infty} g(R_n, \mathfrak{w}).$$

Lemma 2.1: [9] *Consider R be a CAT(0) space.*

1. *Any bounded sequence in R , there exists a subsequence of that sequence is Δ -convergent.*
2. *If $S \subset R$ is closed and convex, and $\{R_n\} \subset S$ is bounded, then its asymptotic center also lies in S .*
3. *If $G: S \rightarrow R$ is a nonexpansive mapping, then $\{R_n\}$ Δ -converges to $\mathfrak{u}$ with $g(R_n, GR_n) \rightarrow 0$.*

2.2 Nonexpansive Mappings

Mapping $T: C \rightarrow R$ is known as *nonexpansive* if

$$g(T\kappa, T\varpi) \leq g(\kappa, \varpi), \forall \kappa, \varpi \in C.$$

Suzuki [10] introduced a refined class of nonexpansive mappings called *Suzuki Non-expansive mapping* (SNEM). Later, Ullah et al. [11] extended this to *enriched Suzuki nonexpansive mappings* (ESNEM). For $\nu \geq 0$, $T: R \rightarrow R$ is ESN if

$$g(\nu(\kappa - \varpi) + T\kappa - T\varpi) \leq (\nu + 1)g(\kappa - \varpi), \forall \kappa, \varpi \in C \text{ with } \frac{1}{2}g(\kappa, T\kappa) \leq (\nu + 1)g(\kappa - \varpi).$$

2.3 Iterative Processes

Chauhan et al. [12] introduced the *SNIA iteration*, a three-step process with improved convergence properties:

$$\begin{aligned} \sigma_{n+1} &= T[(1 - \kappa_n - \mu_n)\tau_n + \kappa_n T\tau_n + \mu_n T\phi_n], \\ \tau_n &= T[(1 - a_n - b_n)\phi_n + a_n T\phi_n + e_n T\sigma_n], \\ \phi_n &= T[(1 - c_n)\sigma_n + c_n T\sigma_n], \end{aligned}$$

where $\{\sigma_n\}, \{\tau_n\}, \{\phi_n\}$ are sequences in S , and $\{a_n\}, \{b_n\}, \{c_n\}, \{\kappa_n\}, \{\mu_n\} \subset [0, 1]$.

3. Main Result

The SNIA iterative scheme is employed to prove new fixed points and Δ -convergence of enriched and standard Suzuki nonexpansive mappings operating in complete CAT(0) spaces.

Theorem 3.1: *If complete CAT(0) space (R, g) , if $S \neq \emptyset$ and bounded closed convex subset and $G: S \rightarrow S$ is Suzuki NEM, it is guaranteed that G possesses at least one fixed point; hence, $F(G) \neq \emptyset$.*

Proof: Since G is Suzuki non expansive mapping (SN) by definition 1.4 it follows that $\exists$ a constant $b \in [0, \infty)$ s.t

$$g\left(\frac{1}{\nu+1}Tr - \frac{1}{\nu+1}Ts\right) \leq g(\kappa - \varpi) \quad (1)$$

$\forall \kappa, \varpi \in S$ satisfying $\frac{1}{2}g(\kappa - T\kappa) \leq g(\kappa - \varpi)$.

By putting $\nu = \frac{1}{\kappa} - 1$ for $\nu > 0$ and $\kappa \in (0, 1)$ and equation (1) becomes

$$g(\kappa Tr - \kappa Ts) \quad (2)$$

Let $G_\kappa(\kappa) = \kappa Tr$ and $G_\kappa(\varpi) = \kappa Ts$ then

$$g[G_\kappa(\kappa) - G_\kappa(\varpi)] \leq g(\kappa - \varpi) \forall \kappa, \varpi \in S$$

The average operator G_κ is nonexpansive if follows that G_κ has one or more fixed points.

From remark 1.3 $F(G_\kappa) = F(G) \neq \emptyset$

Hence $F(G)$ is nonempty.

Theorem 3.2: Assume that R satisfies Opial's property, then the sequences $\{\sigma_{n+1}\}, \{\tau_n\}, \{\varphi_n\}$ developed by SNIA Iteration process Δ -converges to fixed point of the mapping $\mathcal{G}$.

Proof: By Lemma 2.1 and Lemma 2.2 we observe that $\{\sigma_{n+1}\}, \{\tau_n\}, \{\varphi_n\}$ are bounded sequences with

$$\begin{aligned}\lim_{n \rightarrow \infty} g(\sigma_{n+1}, \mathcal{G}\sigma_{n+1}) &= 0 \\ \lim_{n \rightarrow \infty} g(\tau_n, \mathcal{G}\tau_n) &= 0 \\ \lim_{n \rightarrow \infty} g(\varphi_n, \mathcal{G}\varphi_n) &= 0\end{aligned}$$

Let $\{R_n\}$ be the sequence consisting of $\{\sigma_{n+1}\}, \{\tau_n\}, \{\varphi_n\}$.

Let $W_n(\{R_n\}) = \cup A(\{\sigma_{n+1}\}\{\tau_n\}\vartheta\{\varphi_n\})$, where all subsequences $\{\sigma_{n+1}\}, \{\tau_n\}, \{\varphi_n\}$ over $\{R_n\}$.

In order to prove that Δ -convergence of $\{R_n\}$ to fixed point of $\mathcal{G}$, $W_w(\{R_n\}) \subset F(\mathcal{G})$, thereafter urge that $W_w(\{R_n\})$ is singleton set.

To show $W_w(\{R_n\}) \subset F(\mathcal{G})$ let $\varpi \in W_w(\{R_n\})$. Then $\exists$ a subsequence $\{a_n\}$ of $\{R_n\}$ s.t $A(\{a_n\}) = a$. By Lemma 1.1 (a) and Lemma 1.1(b) $\exists$ a subsequence $\{e_n\}$ of $\{a_n\}$ s.t

$$\Delta -\lim_{n \rightarrow \infty} e_n = v \in S$$

By Lemma 1.1(c) $v \in F(\mathcal{G})$.

By Lemma 1.2 $v = a$.

To prove that $W_w(\{R_n\})$ is singleton let $\{a_n\}$ be a subsequence of $\{R_n\}$. From Lemma 1.1(a) and Lemma 1.1(b) $\exists$ a subsequence $\{e_n\}$ of $\{a_n\}$ s.t

$$\Delta -\lim_{n \rightarrow \infty} e_n = v$$

Let $A\{a_n\} = \{a\}$ and $A(\{R_n\}) = \kappa$. Earlier we have shown that $a = v$ similarly $v = \kappa$ then by Lemma 2.1 $\{g(R_n, t)\}$ is convergent. By uniqueness

$$\begin{aligned}\lim_{n \rightarrow \infty} \sup g(e_n, en) &= \lim_{n \rightarrow \infty} \sup g(e_n, \kappa) \\ &\leq \sup_{n \rightarrow \infty} \sup g(R_n, \kappa) \\ &< \sup \lim_{n \rightarrow \infty} \sup g(R_n, v) \\ &= \lim_{n \rightarrow \infty} \sup g(e_n, v).\end{aligned}$$

Hence the theorem.

Theorem 3.3: Let (R, g) be complete $CAT(0)$ Space, $S \neq \emptyset$ and bounded closed convex subset, $\mathcal{G}: S \rightarrow S$ be enriched Suzuki NEM. Then $F(\mathcal{G})$ is nonempty.

Proof: Since $\mathcal{G}$ is enriched Suzuki non expansive mapping (ESN) by definition 1.5 it follows that $\exists$ a constant $b \in [0, \infty)$ s.t

$$g\left[\left(\frac{v}{v+1}\kappa + \frac{1}{v+1}Tr\right) - \left(\frac{v}{v+1}\varpi + \frac{1}{v+1}Ts\right)\right] \leq g(\kappa - \varpi) \quad (3)$$

$$\forall \kappa, \varpi \in S \text{ satisfying } \frac{1}{2}g(\kappa - T_\kappa\kappa) \leq g(\kappa - \varpi).$$

By putting $\nu = \frac{1}{\kappa} - 1$ for $\nu > 0$ and $\kappa \in (0,1)$ and equation (3) becomes

$$g[\{(1-\kappa)\mathfrak{x} + \kappa Tr\} - \{(1-\kappa)\mathfrak{w} + \kappa Ts\}] \leq g(\mathfrak{x} - \mathfrak{w}) \quad (4)$$

Let $\mathcal{G}_\kappa(\mathfrak{x}) = (1-\kappa)\mathfrak{x} + \kappa Tr$ and $\mathcal{G}_\kappa(\mathfrak{w}) = (1-\kappa)\mathfrak{w} + \kappa Ts$ then

$$g[\mathcal{G}_\kappa(\mathfrak{x}) - \mathcal{G}_\kappa(\mathfrak{w})] \leq g(\mathfrak{x} - \mathfrak{w}) \forall \mathfrak{x}, \mathfrak{w} \in S$$

The average operator $\mathcal{G}_\kappa$ is nonexpansive if follows that $\mathcal{G}_\kappa$ has one or more fixed points.

From remark 1.1 $F(\mathcal{G}_\kappa) = F(\mathcal{G}) \neq \emptyset$

Hence $F(\mathcal{G})$ is nonempty.

Now, SNIA iteration is used to demonstrate the Δ -convergence for enriched Suzuki NEM using Opial's property.

Theorem 3.4: *Assume that R satisfies Opial's property then the sequences $\{\sigma_{n+1}\}$, $\{\tau_n\}$, $\{\varphi_n\}$ developed by SNIA Iteration process Δ -converges to fixed point of the mapping $\mathcal{G}$.*

Proof: We consider a sequence $\{R_n\}$ consisting subsequences $\{\sigma_{n+1}\}$, $\{\tau_n\}$ and $\{\varphi_n\}$. The prove is same as done above in Theorem 2.4.

3.1 Application: IoT Clock Synchronization

One practical application of CAT(0) space is **clock synchronization in IoT networks**. In such systems, each device has a local clock offset, and consensus on a common reference time is essential for reliable communication and scheduling.

This can be modeled as a fixed point problem. For device i , define the averaging operator

$$(T\theta)_i = \frac{1}{\deg(i)+1}(\theta_i + \sum_{j \in N(i)} \theta_j),$$

where θ_i is the clock offset and $N(i)$ the set of neighbors. The operator T is nonexpansive in Euclidean space (a CAT(0) space). Its κ -averaged form

$$T_\kappa = (1-\kappa)I + \kappa T$$

is a κ -enriched nonexpansive mapping, fitting the framework of this paper.

By our convergence theorems, the SNIA iteration applied to T_κ converges (in the Δ -sense, equivalent to strong convergence here) i.e., the consensus state where all devices share the same time.

Numerical Example: Consider four devices with initial offsets

$$\theta^{(0)} = (105, 98, 101, 96) \text{ ms}$$

arranged in a ring topology. Iterative application of T_κ with $\kappa = 0.6$ yields convergence to the consensus offset of 100 ms. Thus, the devices achieve synchronization, demonstrating the practical utility of our theoretical results in distributed IoT systems.

4. Conclusion

Using the SNIA iterative scheme, we have derived new fixed point results for several categories of nonexpansive mappings within complete $CAT(0)$ spaces. Findings demonstrate that SNIA iteration is a powerful and unified technique for confirming existence and Δ -convergence of fixed points, especially for Suzuki non-expansive mappings. Furthermore, the application to IoT clock synchronization highlights the practical relevance of these theoretical findings, illustrating how abstract fixed point principles can effectively address real-world problems in distributed systems.

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