

Convex function : Advances in quantum integral inequalities

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Abstract

In this research work, we study q -calculus focusing on the development of quantum estimates involving q -differentials and q -integrals. By constructing identities based on these quantum tools, we analyze the structure and behavior of convex functions within q -calculus.

We present generalized formulations of various q -inequalities, offering a foundation for further investigation of different types of integral inequalities in the quantum setting.

Our approach highlights the utility of convex function properties in q -analysis and contributes to the advancement of inequality theory in quantum integral.

In our research work, based on quantum analog of Hermite-Hadamard inequality, the quantum analog of some trapezoid, midpoint, Bullen and Simpson type inequalities are

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established. Moreover, by taking the special selection of κ and σ and taking limit $q \rightarrow 1^-$ some new results are obtained, and some old results are verified.

Subject Classification (2020): 05A15, 05C38.

Keywords: Inequalities and $\bar{q}$ -calculus, Riemann integral ($\bar{R}$ -integral), $\bar{q}$ -integral, Convex function, $\bar{q}$ -differential, Hermite-Hadamard inequality.

1. Introduction

A well-known result for convex functions on I is the Hermite-Hadamard inequality [11]. Mapping $\mathcal{O}: I \rightarrow \mathbb{R}$, any points $\mu_1, \mu_2 \in I$ satisfying $\mu_1 < \mu_2$, the two-sided estimate holds:

$$\frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} \geq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \geq \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) \quad (1)$$

Another Bullen's inequality, which $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ is a convex function defined,

$$\frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \geq \frac{1}{4} \left[\mathcal{O}(\mu_1) + 2\mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) + \mathcal{O}(\mu_2) \right], \quad (2)$$

Furthermore [1-3, 5], consider the case where $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ possesses a continuous fourth derivative. If the supremum of the absolute value of the fourth derivative is finite, denoted by $\|\mathcal{O}^{(4)}\|_\infty < \infty$, then the subsequent error bound is valid:

$$\left| \left[\frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{6} + \frac{2}{3} \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) \right] - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \right| \leq \frac{(\mu_2 - \mu_1)^4}{2880} \|\mathcal{O}^{(4)}\|_\infty, \quad (3)$$

called as Simpson's inequality.

Lemma 1.1: [[10], Lemma 2.1] Consider a map $\mathcal{O}: I \subset \mathbb{R} \rightarrow \mathbb{R}$ be the differentiable inside I° , $\mu_1, \mu_2 \in I$ alongwith $\mu_1 < \mu_2$. Whenever $\mathcal{O}' \in L[\mu_1, \mu_2]$, $\kappa, \sigma \in \mathbb{R}$, gives

$$\begin{aligned} & \frac{1}{4} (\mu_2 - \mu_1) \int_0^1 \left[\mathcal{O}' \left(\ell \mu_1 + (1 - \ell) \frac{\mu_1 + \mu_2}{2} \right) (1 - \kappa - \ell) + \mathcal{O}' \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell) \mu_2 \right) (\sigma - \ell) \right] d\ell \\ & = \frac{2 - \kappa - \sigma}{2} \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell + \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} \end{aligned} \quad (4)$$

This work is dedicated to filling that gap by developing quantum analogues of these core identities and inequalities. Our approach is grounded in a recently established quantum Hermite-Hadamard type inequality from (refer also to [4]). As a special case, when $q \rightarrow 1^-$, our quantum results yield their classical counterparts, including the trapezoid, midpoint, Bullen, and Simpson type inequalities [6-12].

2. Background and Preliminaries

This section outlines the fundamental principles of quantum derivatives and q -integrals defined over a general finite interval $[\mu_1, \mu_2]$. Throughout the subsequent discussion, the parameter q is a fixed constant with having $0 < q < 1$.

Definition 2.1: [4] Suppose the $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ be the mapping then q -derivatives of $\mathcal{O}$ inside $[\mu_1, \mu_2]$ can be written as

$$\lim_{\ell \rightarrow \mu_1^+} \mathcal{D}_q \mathcal{O}(\ell) = \mu_1 \mathcal{D}_q \mathcal{O}(\mu_1) \quad (5)$$

$$\frac{\mathcal{O}(\ell) - \mathcal{O}(q\ell + (1-q)\mu_1)}{(1-q)(\ell - \mu_1)} = \mu_1 \mathcal{D}_q \mathcal{O}(\ell), \quad \ell \neq \mu_1 \quad (6)$$

So, When $\mu_1 = 0$ inside (5), implies ${}_0 \mathcal{D}_q \mathcal{O}(\ell) = \mathcal{D}_q \mathcal{O}(\ell)$ whereas $\mathcal{D}_q \mathcal{O}(\ell)$ called as quantum derivative for function $\mathcal{O}$ defined by

$$\lim_{\ell \rightarrow 0} \mathcal{D}_q \mathcal{O}(\ell) = \mathcal{D}_q \mathcal{O}(0), \quad \frac{\mathcal{O}(\ell) - \mathcal{O}(q\ell)}{(1-q)\ell} = \mathcal{D}_q \mathcal{O}(\ell), \quad \ell \neq 0 \quad (7)$$

Lemma 2.1: [4, 8] Suppose $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ is differentiable map. we write

$$\lim_{q \rightarrow 1^-} \mu_1^+ \mathcal{D}_q \mathcal{O}(\ell) = \frac{d\mathcal{O}(\ell)}{d\ell} \quad (8)$$

Lemma 2.2: [4, 8] Suppose $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ be a mapping. Whenever $\int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell$, we say

$$\int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell = \lim_{q \rightarrow 1^-} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \mu_1^+ d_q \ell \text{ Hold..} \quad (9)$$

Lemma 2.3: [2] Consider $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ be a differentiable map. Then we write

$$\lim_{q \rightarrow 1^-} \mu_2^- \mathcal{D}_q \mathcal{O}(\ell) = \frac{d\mathcal{O}(\ell)}{d\ell} \quad (10)$$

Lemma 2.4: [2] Consider $\mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$ be a mapping. whenever $\int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell$ holds, So

$$\lim_{q \rightarrow 1^-} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \mu_2^- d_q \ell = \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \quad (11)$$

Lemma 2.5: [5] $\mathcal{T}, \mathcal{O}: [\mu_1, \mu_2] \rightarrow \mathbb{R}$, be continuous function with having following inequality,

$$\begin{aligned} & \int_0^1 \mathcal{T}(\ell) \mu_1^+ \mathcal{D}_q \mathcal{O}(\ell \mu_2 + (1-\ell)\mu_1) {}_0^+ d_q \ell \\ &= \frac{\mathcal{T}(\ell) \mathcal{O}(\ell \mu_2 + (1-\ell)\mu_1)}{\mu_2 - \mu_1} \Big|_0^1 - \frac{1}{\mu_2 - \mu_1} \int_0^1 {}_0^+ \mathcal{D}_q \mathcal{T}(\ell) \mathcal{O}(q\ell \mu_2 + (1-q\ell)\mu_1) {}_0^+ d_q \ell. \end{aligned} \quad (12)$$

3. Main Results

In the following fold, we show for identities which used to get some generalized q -integral inequalities.

Lemma 3.1: Let $\mu_2 \in [0, 1]$ with $q \in (0, 1)$, implies

$$\mathcal{G}(q, \mu_2) := \int_0^1 |q\tau - \mu_2| 0^+ d_q \tau = \begin{cases} \frac{2\mu_2^2([2]_q - 1) - q[2]_q \mu_2 + q^2}{q[2]_q}, & 0 \leq \frac{\mu_2}{q} < 1 \\ \frac{[2]_2 \mu_2 - q}{[2]_q}, & 1 \leq \frac{\mu_2}{q} \end{cases}, \quad (13)$$

$$\mathcal{N}(q, \mu_2) := \int_0^1 |q\tau - \mu_2| \frac{\tau}{2} 0^+ d_q \tau = \begin{cases} \frac{\mu_2^3(2[3]_q - 2[2]_q) - q^2 \mu_2[3]_q + q^3[2]_q}{2q^2[2]_q[3]_q}, & 0 \leq \frac{\mu_2}{q} < 1 \\ \frac{\mu_2[3]_q - q[2]_q}{2[2]_q[3]_q}, & 1 \leq \frac{\mu_2}{q} \end{cases}, \quad (14)$$

$$\chi(q, \mu_2) = \int_0^1 |q\tau - \mu_2| \left(1 - \frac{\tau}{2}\right)_0^+ d_q \tau$$

$$= \begin{cases} \frac{4q\mu_2^2[3]_q([2]_q - 1) - 2q^2\mu_2[2]_q[3]_q + 2q^3[3]_q - \mu_2^3(2[3]_q - 2[2]_q) + q^2\mu_2[3]_q - q^3[2]_q}{2q^2[2]_q[3]_q}, & 0 \leq \frac{\mu_2}{q} < 1 \\ \frac{\mu_2[3]_q(2[2]_q - 1) + q([2]_q - 2[3]_q)}{2[2]_q[3]_q}, & 1 \leq \frac{\mu_2}{q} \end{cases}. \quad (15)$$

Proof: with the help of definition from left quantum integral, we can porve easily.

4. q -Integral inequalities

Theorem 4.1: Suppose Continuous map $\mathcal{O}: [\mu_1, \mu_2] \subset \mathbb{R} \rightarrow \mathbb{R}$,

$\mu_1 + \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_1 \right)$ with $\mu_2 - \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_2 \right)$ be left quantum integrable inside interval $[0, 1]$, with $0 \leq \kappa, \sigma \leq 1$. whenever $|\mu_1 + \mathcal{D}_q \mathcal{O}|$ with $|\mu_2 - \mathcal{D}_q \mathcal{O}|$ be convex map inside $[\mu_1, \mu_2]$, so we write

$$\left| \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} + \frac{2 - \kappa - \sigma}{2} \mathcal{O} \left(\frac{\mu_1 + \mu_2}{2} \right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right|$$

$$\leq \frac{\mu_2 - \mu_1}{4} \left[\frac{|\mathcal{N}(q, \kappa)| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) + \chi(q, \kappa) |\mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1)|}{|\mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1)| + \chi(q, \sigma) |\mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2)|} \right] \quad (16)$$

whereas $\mathcal{N}(q, \kappa), \mathcal{N}(q, \sigma), \chi(q, \kappa)$ with $\chi(q, \sigma)$ be given in to the Lemma (3.1).

Proof: Lets apply Lemma (3.1) with convex property $|\mu_1^+ \mathcal{D}_q \mathcal{O}|$ and $|\mu_2^- \mathcal{D}_q \mathcal{O}|$, implies

$$\begin{aligned}
 &= \left| \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} + \frac{2-\kappa-\sigma}{2} \mathcal{O}\left(\frac{\mu_1+\mu_2}{2}\right) - \frac{1}{\mu_2-\mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \\
 &\leq \frac{\mu_2 - \mu_1}{4} \left[\int_0^1 |q\ell - \kappa| \left| \mu_1^+ \mathcal{D}_q \mathcal{O}\left(\ell \left(\frac{\mu_1+\mu_2}{2}\right) + (1-\ell)\mu_1\right) \right| 0^+ d_q \ell \right. \\
 &\quad \left. + \int_0^1 |\sigma - q\ell| \left| \mu_2^- \mathcal{D}_q \mathcal{O}\left(\ell \left(\frac{\mu_1+\mu_2}{2}\right) + (1-\ell)\mu_2\right) \right| 0^+ d_q \ell \right] \\
 &= \frac{\mu_2 - \mu_1}{4} \left[\int_0^1 |q\ell - \kappa| \left| \mu_1^+ \mathcal{D}_q \mathcal{O}\left(\frac{\ell}{2}\mu_2 + \left(1-\frac{\ell}{2}\right)\mu_1\right) \right| 0^+ d_q \ell \right. \\
 &\quad \left. + \int_0^1 |\sigma - q\ell| \left| \mu_2^- \mathcal{D}_q \mathcal{O}\left(\frac{\ell}{2}\mu_1 + \left(1-\frac{\ell}{2}\right)\mu_2\right) \right| 0^+ d_q \ell \right]
 \end{aligned} \tag{17}$$

take inside (16), we say (17). Hence proof complete.

Using 4.1, we can add more result with some values of κ, σ alongwith $q \rightarrow 1^-$.

Corollary 4.1: Lets take $\kappa = \sigma = 0, 0.5, 1$ with $\frac{1}{3}$ inside 4.1, using quantum trapezoid, midpoint, and Bullen, and Simpson inequalities: $\left| \frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right|$

$$\begin{aligned}
 &\leq \frac{\mu_2 - \mu_1}{4} \left[\mathcal{N}(q, 1) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right| + \chi(q, 1) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right| \right. \\
 &\quad \left. + \mathcal{N}(q, 1) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right| + \chi(q, 1) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right| \right] \\
 &\left| \mathcal{O}\left(\frac{\mu_1+\mu_2}{2}\right) - \frac{1}{\mu_2+\mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right|
 \end{aligned} \tag{18}$$

$$\begin{aligned}
 &\leq \frac{\mu_2 + \mu_1}{4} \left[\mathcal{N}(q, 0) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right| + \chi(q, 0) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right| \right. \\
 &\quad \left. + \mathcal{N}(q, 1) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right| + \chi(q, 0) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right| \right] \\
 &\leq \left| \frac{1}{4} \left[\mathcal{O}(\mu_1) + 2\mathcal{O}\left(\frac{\mu_1+\mu_2}{2}\right) + \mathcal{O}(\mu_2) \right] - \frac{1}{\mu_2+\mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right|
 \end{aligned} \tag{19}$$

$$\begin{aligned}
 &\leq \frac{\mu_2 + \mu_1}{4} \left[\mathcal{N}\left(q, \frac{1}{2}\right) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right| + \chi\left(q, \frac{1}{2}\right) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right| \right. \\
 &\quad \left. \mathcal{N}\left(q, \frac{1}{2}\right) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right| + \chi\left(q, \frac{1}{2}\right) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right| \right]
 \end{aligned} \tag{20}$$

$$\begin{aligned}
 &\left| \frac{1}{3} \left[\frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} + 2\mathcal{O}\left(\frac{\mu_1+\mu_2}{2}\right) \right] - \frac{1}{\mu_2+\mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right|
 \end{aligned} \tag{21}$$

$$\begin{aligned}
 &\leq \frac{\mu_2 + \mu_1}{4} \left[\mathcal{N}\left(q, \frac{1}{2}\right) \left[\left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right| \right. \right. \\
 &\quad \left. \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right| + \chi\left(q, \frac{1}{3}\right) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right| \right. \right. \\
 &\quad \left. \left. \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right| \right] \right]
 \end{aligned}$$

with the help (20) and (19), we write [4, in 4.1], respectively.

Corollary 4.2: Consider if $\mathcal{O}$ be differentiable inside $[\mu_1, \mu_2]$ with $q \rightarrow 1^-$ respectively in (30) - (33), so property trapezoid, or midpoint, or Bullen or Simpson inequalities respectively:

$$\left| \frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \right| \leq \frac{\mu_2 - \mu_1}{8} [|\mathcal{O}'(\mu_1)| + |\mathcal{O}'(\mu_2)|]$$

$$\left| \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \right| \leq \frac{\mu_2 - \mu_1}{8} [|\mathcal{O}'(\mu_1)| + |\mathcal{O}'(\mu_2)|] \quad (22)$$

$$\left| \frac{1}{4} \left[\mathcal{O}(\mu_1) + 2\mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) + \mathcal{O}(\mu_2) \right] - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \right| \quad (23)$$

$$\leq \frac{\mu_2 - \mu_1}{16} [|\mathcal{O}'(\mu_1)| + |\mathcal{O}'(\mu_2)|]$$

$$\left| \frac{1}{3} \left[\frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} + 2\mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) \right] - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) d\ell \right| \quad (24)$$

$$\leq \frac{5(\mu_2 - \mu_1)}{72} [|\mathcal{O}'(\mu_2)| + |\mathcal{O}'(\mu_1)|]$$

Theorem 4.2: Suppose continuous map $\mathcal{O}: [\mu_1, \mu_2] \subset \mathbb{R} \rightarrow \mathbb{R}$,

$\mu_1 + \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_1 \right) \quad \mu_2 - \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_2 \right)$ hold property left quantum integrable inside $[0, 1]$, so $0 \leq \kappa, \sigma \leq 1$. If $\left| \mu_1^+ \mathcal{D}_q \mathcal{O} \right|^r$ and $\left| \mu_2^- \mathcal{D}_q \mathcal{O} \right|^r$ be the convex functions inside $[\mu_1, \mu_2]$, whereas $r \geq 1$,

$$\left| \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} + \frac{2 - \kappa - \sigma}{2} \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \quad (25)$$

$$\leq \frac{\mu_2 - \mu_1}{4} \left[\mathcal{G}^{1 - \frac{1}{r}}(q, \kappa) (\mathcal{N}(q, \kappa) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r + \chi(q, \kappa) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r)^{\frac{1}{r}} \right. \\ \left. + \mathcal{G}^{1 - \frac{1}{r}}(q, \sigma) (\mathcal{N}(q, \sigma) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r + \chi(q, \sigma) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r)^{\frac{1}{r}} \right],$$

whereas $\mathcal{G}(q, \kappa), \mathcal{G}(q, \sigma), \mathcal{N}(q, \kappa), \mathcal{N}(q, \sigma), \chi(q, \kappa)$ and $\chi(q, \sigma)$ as per in 3.1

Proof: As we know $\left| \mu_1^+ \mathcal{D}_q \mathcal{O} \right|^r$ and $\left| \mu_2^- \mathcal{D}_q \mathcal{O} \right|^r$ convex functions, using,

$$\left| \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} + \frac{2 - \kappa - \sigma}{2} \mathcal{O}\left(\frac{\mu_1 + \mu_2}{2}\right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \quad (26)$$

$$\leq \frac{\mu_2 - \mu_1}{4} \left[\int_0^1 |q\ell - \kappa| \left| \mu_1^+ \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_1 \right) \right| 0^+ d_q \ell \right. \\ \left. + \int_0^1 |\sigma - q\ell| \left| \mu_2^- \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell)\mu_2 \right) \right| 0^+ d_q \ell \right]$$

$$= \frac{\mu_2 - \mu_1}{4} \left[\int_0^1 |q\ell - \kappa| \left| \mu_1^+ \mathcal{D}_q \mathcal{O} \left(\frac{\ell}{2} \mu_2 + \left(1 - \frac{\ell}{2}\right) \mu_1 \right) \right| 0^+ d_q \ell \right. \\ \left. + \int_0^1 |\sigma - q\ell| \left| \mu_2^- \mathcal{D}_q \mathcal{O} \left(\frac{\ell}{2} \mu_1 + \left(1 - \frac{\ell}{2}\right) \mu_2 \right) \right| 0^+ d_q \ell \right]$$

$$\begin{aligned}
& + \left(\int_0^1 |\sigma - q\ell| 0^+ d_q \ell \right)^{1-\frac{1}{r}} \left(\int_0^1 |\sigma - q\ell| \left| \mu_2^- \mathcal{D}_q \mathcal{O} \left(\frac{\ell}{2} \mu_1 + \left(1 - \frac{\ell}{2}\right) \mu_2 \right) \right|^r 0^+ d_q \ell \right)^{\frac{1}{r}} \\
& \leq \frac{\mu_2 - \mu_1}{4} \left[\left(\int_0^1 |q\ell - \kappa| 0^+ d_q \ell \right)^{1-\frac{1}{r}} \left(\left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r \int_0^1 |q\ell - \kappa| \frac{\ell}{2} 0^+ d_q \ell \right. \right. \\
& \quad \left. \left. + \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r \int_0^1 |q\ell - \kappa| \left(1 - \frac{\ell}{2}\right) 0^+ d_q \ell \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \left(\int_0^1 |\sigma - q\ell| 0^+ d_q \ell \right)^{1-\frac{1}{r}} \left(\left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r \int_0^1 |\sigma - q\ell| \frac{\ell}{2} 0^+ d_q \ell \right. \right. \\
& \quad \left. \left. + \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r \int_0^1 |\sigma - q\ell| \left(1 - \frac{\ell}{2}\right) 0^+ d_q \ell \right)^{\frac{1}{r}} \right)
\end{aligned}$$

by applying 3.1 to (24), we have (26). which hold.

We take result from 4.2, by putting some values κ, σ and $q \rightarrow 1^-$.

Corollary 4.3: Consider $\kappa = \sigma = 1, 0, \frac{1}{2}$ and $\frac{1}{3}$ inside 4.2, we use quantum trapezoid, midpoint, Bullen and Simpson type inequalities,

$$\begin{aligned}
& \left| \frac{\mathcal{O}(\mu_1) + \mathcal{O}(\mu_2)}{2} - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \\
& \leq \frac{\mu_2 - \mu_1}{4} \mathcal{G}^{1-\frac{1}{r}}(q, 1) \left[\left(\mathcal{N}(q, 1) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r + \chi(q, 1) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \left(\mathcal{N}(q, 1) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r + \chi(q, 1) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r \right)^{\frac{1}{r}} \right] \quad (27)
\end{aligned}$$

$$\left| \mathcal{O} \left(\frac{\mu_1 + \mu_2}{2} \right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \quad (28)$$

$$\begin{aligned}
& \leq \frac{\mu_2 - \mu_1}{4} \mathcal{G}^{1-\frac{1}{r}}(q, 0) \left[\left(\mathcal{N}(q, 0) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r + \chi(q, 0) \left| \mu_1^+ \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r \right)^{\frac{1}{r}} \right. \\
& \quad \left. + \left(\mathcal{N}(q, 0) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_1) \right|^r + \chi(q, 0) \left| \mu_2^- \mathcal{D}_q \mathcal{O}(\mu_2) \right|^r \right)^{\frac{1}{r}} \right] \\
& \leq \left| \frac{1}{4} \left[\mathcal{O}(\mu_1) + 2\mathcal{O} \left(\frac{\mu_1 + \mu_2}{2} \right) + \mathcal{O}(\mu_2) \right] - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \quad (29)
\end{aligned}$$

Lets take (27) with (28), then [4, Theorem 5.2 and Theorem 4.2],

Theorem 4.3: Suppose Continuous function $\mathcal{O}: [\mu_1, \mu_2] \subset \mathbb{R} \rightarrow \mathbb{R}$, $\mu_1^+ \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell) \mu_1 \right)$ with $\mu_2^- \mathcal{D}_q \mathcal{O} \left(\ell \left(\frac{\mu_1 + \mu_2}{2} \right) + (1 - \ell) \mu_2 \right)$ be left quantum integrable inside $[0, 1]$, and $0 \leq \kappa, \sigma \leq 1$. If $\left| \mu_1^+ \mathcal{D}_q \mathcal{O} \right|^r$ and $\left| \mu_2^- \mathcal{D}_q \mathcal{O} \right|^r$ be convex functions inside $[\mu_1, \mu_2]$, whereas $\frac{1}{r} + \frac{1}{p} = 1$ and $r > 1$,

$$\left| \frac{\kappa \mathcal{O}(\mu_1) + \sigma \mathcal{O}(\mu_2)}{2} + \frac{2 - \kappa - \sigma}{2} \mathcal{O} \left(\frac{\mu_1 + \mu_2}{2} \right) - \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \mathcal{O}(\ell) \frac{\mu_1^+ + \mu_2^-}{2} d_q \ell \right| \quad (30)$$

$$\leq \frac{\mu_2 - \mu_1}{4} \left[M^{\frac{1}{p}}(q, \kappa) \left(\frac{|\mu_1^+ \mathcal{D}_q^0(\mu_2)|^r + 3|\mu_1^+ \mathcal{D}_q^0(\mu_1)|^r}{4} \right)^{\frac{1}{r}} + M^{\frac{1}{p}}(q, \sigma) \left(\frac{|\mu_2^- \mathcal{D}_q^0(\mu_1)|^r + 3|\mu_2^- \mathcal{D}_q^0(\mu_2)|^r}{4} \right)^{\frac{1}{r}} \right]$$

5. Conclusion

This research work has established integral inequalities within the quantum calculus. By helping fundamental properties of convex functions and q -identities, we have developed a approach to quantum integral analysis. This work is the formulation of a generalized quantum identity from which specific q -trapezoid, midpoint, Bullen, and Simpson-type inequalities are systematically derived. The generality of our approach is shows by the fact that selecting specific parameters, κ and σ , within our main results yields these distinct and well-known inequality types. Furthermore, our quantum extensions are confirmed by the fact that, in the limit as $q \rightarrow 1^-$, our results successfully reduce to both new findings in classical calculus and corroborate established classical inequalities.

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Received February, 2026