

Class of $\mathcal{HUF}$ by fractional derivative operator

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Abstract

This research study, we explore a family of CV map both univalent and sensepreserve within $\mathbb{Z}$, and which are expressed in $h(\tilde{v}) = \ell(\tilde{v}) + (\kappa(\tilde{v}))$ whereas $\ell(\tilde{v})$ with $\kappa(\tilde{v})$ are analytic throughout $\mathbb{Z}$.

A central aspect newly introduced class defined through a generalized fractional differential operator. By setting in this operator, we define the function class as $SHP_{\alpha}(\tilde{\rho}_1, \dots, \tilde{\rho})$ belongs to family of $\mathcal{HUF}$. Alongwith we derive sufficient conditions on the coefficients for functions in this class, and further these conditions become necessary when the coefficients

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are negative. Additionally, we examine key function-theoretic features of this class, including estimates for distortion and the characterization of its extreme points.

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1. Introduction

Harmonic mappings play an essential role in a variety of scientific and engineering domains, including operational mathematics research and interconnected areas of applied part, as discussed in [1, 2, 3, 8, 9, 10, 12, 14]. The early and influential work by author Clunie and Sheil-Small [4] laid the groundwork for the theory of harmonic mappings, which has since attracted considerable interest within the field of geometric function theory for more [5, 6]. A succinct and insightful account of this theory is provided in the work of Ahuja [13]. Let $\mathbb{Z}$ and S_H gives the set of harmonic complex-valued map $\tilde{h}(\tilde{b})$ that are sense-preserving and fall within the class $\mathcal{UF}$ in $\mathbb{Z}$, subject to the normalization $\tilde{h}(0) = 0$ and $\tilde{h}_{\tilde{b}}(0) = 1$. All map $\tilde{h}(\tilde{b}) \in S_H$,

$\tilde{h}(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ whereas,

$$\ell(\tilde{b}) = \tilde{b} + \sum_{\wp=2}^{\infty} \nu_{\wp} \tilde{b}^{\wp}, \quad \kappa(\tilde{b}) = \sum_{\wp=1}^{\infty} \tilde{\psi}_{\wp} \tilde{b}^{\wp}, |\tilde{\psi}_1| < 1$$

and both $\ell(\tilde{b})$ and $\kappa(\tilde{b})$ both analytic throughout the $\mathbb{Z}$.

2. Different Subclasses

Different classes for class S_H , (see for example, [1, 3, 4, 6, 7, 8, 9, 10, 11]). For the function $\tilde{h}(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ defined in (1.1), we introduce $Z_{\alpha}^{m, \tilde{\rho}_1} \tilde{h}(\tilde{b})$, be first proposed by Li and Ponnusamy [9]. Operator is formulated using the Gamma function and applies to $\tilde{h}(\tilde{b}) \in A$, A function $\tilde{h}(\tilde{b})$ in $\mathbb{Z}$ must satisfy the condition $|\ell(\tilde{b})| > |\kappa'(\tilde{b})|$ in $\mathbb{Z}$. [15, 16] Several significant geometric subclasses of S_H were subsequently by Clunie and Sheil-Small [4]. Considering functions $\tilde{h}(\tilde{b})$ expressed by the series $\tilde{h}(\tilde{b}) = \tilde{b} + \sum_{\wp=2}^{\infty} \nu_{\wp} \tilde{b}^{\wp}$, the operator $Z_{\alpha}^{m, \tilde{\rho}_1} \tilde{h}(\tilde{b})$ is formally defined in the following manner:

$$Z_{\alpha}^{m, \tilde{\rho}_1} \tilde{h}(\tilde{b}) = \tilde{b} + \sum_{\wp=2}^{\infty} \varphi_{\wp, m}(\tilde{\rho}_1, \alpha) \nu_{\wp} \tilde{b}^{\wp}, \quad \tilde{\rho}_1 \in [0, 1) \quad (1)$$

where $m \in N_0 = \{0, 1, 2, \dots\}$ and $[(1 - \tilde{\rho}_1 + \wp)!]^m \varphi_{\wp, m}(\tilde{\rho}_1, \alpha) = [(1 + \alpha(\wp - 1))(\wp + 1)! (2 - \tilde{\rho}_1)!]^m$.

Noor, Malik, and Bukhari [8] have given an fractional differential operator for the complete linear multiplier $Z_{\alpha}^{m, \tilde{\rho}_1}$ to harmonic-map $\tilde{h}(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ as follow

$$Z_{\alpha}^{m, \tilde{\rho}_1} \tilde{h}(\tilde{b}) = \sum_{\wp=1}^{\infty} \varphi_{\wp, m}(\tilde{\rho}_1, \alpha) (\nu_{\wp} \tilde{b}^{\wp} + \overline{\tilde{\psi}_{\wp} \tilde{b}^{\wp}}), m \in N_0 \quad (2)$$

$$[(1 - \tilde{\rho}_1 + \wp)!]^m \varphi_{\wp, m}(\tilde{\rho}_1, \alpha) = [(1 + \alpha(\wp - 1))(\wp + 1)!(2 - \tilde{\rho}_1)!]^m. \quad (3)$$

whenever the co-analytic fold $\kappa(\tilde{b}) = 0$, then we obtain fractional differential operator for the complete linear multiplier $Z_\alpha^{m, \tilde{\rho}_1} \hbar(\tilde{b})$. $SHP_\alpha(\tilde{\rho}_1, \dots, \wp)$ above proved coefficient is also necessary for the functions inside class $THP_\alpha(\tilde{\rho}_1, \dots, \wp)$.

3. The Class $SHP_\alpha(\tilde{\rho}_1, \dots, \wp)$

Definition 3.1: Consider $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ is the harmonic function, whereas $\ell(\tilde{b})$ and $\kappa(\tilde{b})$ are presented in (1.1). Then $\hbar(\tilde{b}) \in SHP_\alpha(\tilde{\rho}_1, \dots, \wp)$ if it holds

$$\Re\left\{\frac{\ell(\tilde{b})Z_\alpha^{m, \tilde{\rho}_1(1-\tilde{\rho}_1)+(-1)^n(1-\tilde{\rho}_1)Z_\alpha^{m, \tilde{\rho}_1}\kappa(\tilde{b})}{\tilde{b}} + [\ell(\tilde{b})Z_\alpha^{m, \tilde{\rho}_1} + (-1)^n\overline{Z_\alpha^{m, \tilde{\rho}_1}\kappa(\tilde{b})}]\tilde{\rho}_1\right\} \geq \tilde{v}_2, \quad (4)$$

where $Z_\alpha^{m, \tilde{\rho}_1} \hbar(\tilde{b})$ is defined by (2).

Definition 3.2: Consider $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ is a harmonic map, whereas $\tilde{b} \in \mathbb{Z}$

$$\ell(\tilde{b}) = \tilde{b} - \sum_{\wp=2}^{\infty} |v_\wp| \tilde{b}^{\wp}, \quad \kappa(\tilde{b}) = -\sum_{\wp=1}^{\infty} |\tilde{\psi}_\wp| \tilde{b}^{\wp}, \quad |\tilde{\psi}_1| < 1. \quad (5)$$

Then $\hbar(\tilde{b}) \in THP_\alpha(\tilde{\rho}_1, \dots, \wp)$, the $SHP_\alpha(\tilde{\rho}_1, \dots, \wp)$, For $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ given by (1.1), $Z_\alpha^{m, \tilde{\rho}_1}$ as

$$Z_\alpha^{m, \tilde{\rho}_1} \hbar(\tilde{b}) = \sum_{\wp=1}^{\infty} \beta_{\wp, m}(\tilde{\rho}_1, \alpha) (v_\wp \tilde{b}^{\wp} + \overline{\tilde{\psi}_\wp \tilde{b}^{\wp}}), \quad m \in \mathbb{N}_0 \quad (6)$$

where

$$[(1 - \tilde{\rho}_1 + \wp)!]^m \beta_{\wp, m}(\tilde{\rho}_1, \alpha) = [(1 + \alpha(\wp - 1))(\wp + 1)!(2 - \tilde{\rho}_1)!]^m.$$

For $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ as

$$Z_\alpha^{n, m} \hbar(\tilde{b}) = Z_\alpha^{n, m} \ell(\tilde{b}) + (-1)^n \overline{Z_\alpha^{n, m} \kappa(\tilde{b})}, \quad (7)$$

$$Z_\alpha^{n, m} \ell(\tilde{b}) = \tilde{b} + \sum_{\wp=2}^{\infty} \beta_{\wp, n}(m, \alpha) v_\wp \tilde{b}^{\wp}, \quad Z_\alpha^{n, m} \kappa(\tilde{b}) =$$

$$\tilde{b} + \sum_{\wp=1}^{\infty} \beta_{\wp, n}(m, \alpha) \tilde{\psi}_\wp \tilde{b}^{\wp}, \quad |\tilde{\psi}_1| < 1$$

$$[(1 - \tilde{\rho}_1 + \wp)!]^n \beta_{\wp, m}(\tilde{\rho}_1, \alpha) = [(1 + \alpha(\wp - 1))(\wp + 1)!(2 - \tilde{\rho}_1)!]^n. \quad (8)$$

Theorem 3.1: Suppose $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$ let

$$2 - \tilde{v}_2 \geq \sum_{\wp=1}^{\infty} \beta_{\wp, m}(m, \alpha) (\wp \tilde{\rho}_1 + 1 - \tilde{\rho}_1) (|v_\wp| + |\tilde{\psi}_\wp|) \quad (9)$$

where $v_1 = 1$, $\alpha > 0$, with $\tilde{\rho}_1 > 0$, $\alpha + \tilde{\rho}_1 \geq 1$, $0 \leq \tilde{v}_2 < 1$. Then $\hbar(\tilde{b})$ is harmonic map preserving in $\mathbb{Z}$ and $\hbar(\tilde{b}) \in SHP_\alpha(\tilde{\rho}_1, \dots, \wp)$.

Proof: For $1 \geq |\tilde{b}_2| \geq |\tilde{b}_1|$ in (2.1)

$$|\hbar(\tilde{b}_1) - \hbar(\tilde{b}_2)| + |g(\tilde{b}_1) - g(\tilde{b}_2)| \geq |\hbar(\tilde{b}_1) - \hbar(\tilde{b}_2)| \quad (10)$$

$$\geq (|\tilde{b}_1 - \tilde{b}_2| - \sum_{\wp=2}^{\infty} \wp |v_{\wp}| |\tilde{b}_1|^{\wp-1} |\tilde{b}_1 - \tilde{b}_2| - \sum_{\wp=1}^{\infty} |\tilde{b}_1 - \tilde{b}_2| |\wp| |\tilde{\psi}_{\wp}| |\tilde{b}_2|^{\wp-1}) \quad (11)$$

$$\begin{aligned} &= (|\tilde{b}_1 - \tilde{b}_2| - |\tilde{b}_1 - \tilde{b}_2| \sum_{\wp=2}^{\infty} (\wp (|v_{\wp}| + |\tilde{\psi}_{\wp}|) + |\tilde{\psi}_1|)) |\tilde{b}_2| < 1 \\ &\geq |\tilde{b}_1 - \tilde{b}_2| (1 - \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \tilde{\rho}_1) (1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1) (|v_{\wp}| + |\tilde{\psi}_{\wp}|) + |\tilde{\psi}_1|) \quad (12) \\ &\geq |\tilde{b}_1 - \tilde{b}_2| (1 - [1 - \tilde{v}_2 - |\tilde{\psi}_1| + |\tilde{\psi}_1|]) = |\tilde{b}_1 - \tilde{b}_2| \tilde{v}_2 \geq 0 \end{aligned}$$

Here $\hbar(\tilde{b})$ is univalent inside $\mathbb{Z}$. Now To prove $\hbar(\tilde{b})$ be sense-preserving inside $\mathbb{Z}$.

$$|\ell'(\tilde{b})| \geq 1 - \sum_{\wp=2}^{\infty} \wp |v_{\wp}| |\tilde{b}|^{\wp-1} > (-\sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) (1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1) |v_{\wp}|) + 1 \quad (13)$$

$$\begin{aligned} &\geq \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) (|\tilde{\psi}_{\wp}| - |\tilde{\psi}_{\wp}| \tilde{\rho}_1 + |\tilde{\psi}_{\wp}| \wp \tilde{\rho}_1) + \tilde{v}_2 \\ &\geq \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) (|\tilde{\psi}_{\wp}| - |\tilde{\psi}_{\wp}| \tilde{\rho}_1 + |\tilde{\psi}_{\wp}| \wp \tilde{\rho}_1) \geq |\kappa'(\tilde{b})| \quad (14) \end{aligned}$$

$\hbar(\tilde{b})$ sense - preserving inside $\mathbb{Z}$. Now we claim $\hbar(\tilde{b}) \in SHP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ using $\text{Re}\{w\} \geq \tilde{v}_2$

$$\begin{aligned} |w + 1 - \tilde{v}_2| &= |1 - \tilde{v}_2 + \frac{Z_{\alpha}^{n,m} \ell(\tilde{b}) + (-1)^n \overline{Z_{\alpha}^{n,m} \kappa(\tilde{b})}}{\tilde{\#}} (1 - \tilde{\rho}_1) + \\ &\quad \tilde{\rho}_1 (Z_{\alpha}^{n,m} \ell(\tilde{b}) + (-1)^n \overline{Z_{\alpha}^{n,m} \kappa(\tilde{b})})| \quad (15) \end{aligned}$$

$$\begin{aligned} &= |1 - \tilde{v}_2 + (1 - \tilde{\rho}_1) \{1 + \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) v_{\wp} \tilde{b}^{\wp-1} + \\ &\quad (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1}\} + \{\tilde{\rho}_1 + \tilde{\rho}_1 \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) \wp v_{\wp} \tilde{b}^{\wp-1} + \\ &\quad \tilde{\rho}_1 (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \wp \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1}\} \tilde{b}^{\tilde{\wp}-1} + \\ &\quad (-1)^n \tilde{\rho}_1 \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \wp \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1}| \\ &= |2 + \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) v_{\wp} \tilde{b}^{\wp-1} (1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1) - \tilde{v}_2 + (-1)^n \\ &\quad \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \wp \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1} (1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1)| \\ &|1 - w + \tilde{v}_2| = |1 - (1 - \tilde{\rho}_1) \frac{[Z_{\alpha}^{n,m} \ell(\tilde{b}) + (-1)^n \overline{Z_{\alpha}^{n,m} \kappa(\tilde{b})}]}{\tilde{\#}} - \\ &\quad \tilde{\rho}_1 [Z_{\alpha}^{n,m} \ell(\tilde{b}) + (-1)^n \overline{Z_{\alpha}^{n,m} \kappa(\tilde{b})}] + \tilde{v}_2| \\ &= |1 - \tilde{v}_2 + (1 - \tilde{\rho}_1) \{1 + \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) v_{\wp} \tilde{b}^{\wp-1} + \\ &\quad (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1}\} - \tilde{\rho}_1 \\ &\quad \{1 + \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) \wp v_{\wp} \tilde{b}^{\wp-1} + (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \wp \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1}\} \\ &= |1 - \tilde{v}_2 + 1 + \tilde{\rho}_1 - \tilde{\rho}_1 - \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha) \wp v_{\wp} \tilde{b}^{\wp-1} [1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1] \\ &\quad - (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha) \wp \tilde{\psi}_{\wp} \tilde{b}^{\tilde{\wp}-1} [1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1] \tilde{v}_2 - \end{aligned}$$

$$\begin{aligned} & \sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1) v_{\wp} \tilde{b}^{\wp-1} \\ & - (-1)^n \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp \tilde{\rho}_1) \tilde{\psi}_{\wp} \tilde{b}^{\wp-1} ||1 - \tilde{v}_2 + w| - |1 + \tilde{v}_2 - w| \end{aligned} \quad (16)$$

$$\begin{aligned} & = |2 - \tilde{v}_2| - |\tilde{\psi}_1| - |\sum_{\wp=2}^{\infty} \dots| - |\tilde{v}_2| - |\tilde{\psi}_1| - |\sum_{\wp=2}^{\infty} \dots| \\ & = 2 - \tilde{v}_2 - |\tilde{\psi}_1| - |\tilde{v}_2| - |\tilde{\psi}_1| - 2|\sum_{\wp=2}^{\infty} \dots| \\ & = 2 - 2\tilde{v}_2 - 2|\tilde{\psi}_1| + 2 + 2|\tilde{\psi}_1| - 2[\sum_{\wp=2}^{\infty} \dots (-1 + |\tilde{\psi}_1|)] \\ & = 4 - 2 \sum_{\wp=1}^{\infty} \dots - 2\tilde{v}_2 \geq 0 \end{aligned} \quad (17)$$

$\sum_{\wp=2}^{\infty} |\tilde{b}_{\wp}| + \sum_{\wp=1}^{\infty} |y_{\wp}| = 1$ gives-coefficient-bound $SHP\alpha(\tilde{\rho}_1, \dots, \wp)$ because

$$\begin{aligned} & \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha)(\wp \tilde{\rho}_1 + 1 - \tilde{\rho}_1)(|v_{\wp}| + |\tilde{\psi}_{\wp}|) \\ & = 1 + (1 - \tilde{v}_2)(\sum_{\wp=2}^{\infty} |\tilde{b}_{\wp}| + \sum_{\wp=1}^{\infty} |y_{\wp}|) = 2 - \tilde{v}_2 \end{aligned}$$

Corollary 3.1: If $\hbar(\tilde{b}) \in THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ then

$$\omega: |\omega| < \frac{(1+\alpha)^n(1-|\tilde{\psi}_1|)(1+\tilde{\rho}_1)-(2-m)^n(1-\tilde{v}_2-|\tilde{\psi}_1|)}{(1+\alpha)^n(1+\tilde{\rho}_1)} \leq \hbar(v). \quad (18)$$

It follows class $THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ be the uniformly-bounded function, then we say normal.

Theorem 3.2: Let $\hbar(\tilde{b}) = \ell(\tilde{b}) + \overline{\kappa(\tilde{b})}$, as $\ell(\tilde{b})$ with $\kappa(\tilde{b})$ written as $\ell(\tilde{b}) = \tilde{b} - \sum_{\wp=2}^{\infty} v_{\wp} \tilde{b}^{\wp} \kappa(\tilde{b}) = (-1)^n \sum_{\wp=1}^{\infty} \tilde{\psi}_{\wp} \tilde{b}^{\wp}$ Then $\hbar(\tilde{b}) \in THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ if.

$$\hbar(\tilde{b}) = \sum_{\wp=1}^{\infty} (\alpha_{\wp} \ell_{\wp}(\tilde{b}) + p_{\wp} \kappa_{\wp}(\tilde{b})) \tilde{b} \in \mathbb{Z}.$$

$$\text{where } \ell_1(\tilde{b}) = \tilde{b}, \ell_{\wp}(\tilde{b}) = \tilde{b} - \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)} \tilde{b}^{\wp}, \wp = 2, 3, \dots$$

$$\kappa_{\wp}(\tilde{b}) = \tilde{b} - \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)} \tilde{b}^{-\wp} \wp = 1, 2, 3, \dots$$

$$\sum_{\wp=1}^{\infty} (\alpha_{\wp} + p_{\wp}) = 1 \alpha_{\wp} \geq 0 p_{\wp} \geq 0$$

with, extreme value of $THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ be $\{\ell_{\wp}\}$ and $\{\kappa_{\wp}\}$.

Proof: Let $\hbar(\tilde{b}) = \sum_{\wp=1}^{\infty} [\alpha_{\wp} \ell_{\wp}(\tilde{b}) + p_{\wp} \kappa_{\wp}(\tilde{b})]$

$$\hbar(\tilde{b}) = \sum_{\wp=1}^{\infty} [\alpha_{\wp} (\tilde{b} - \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)} \tilde{b}^{\wp}) + p_{\wp} (\frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)} \tilde{b}^{-\wp})]$$

$$\hbar(\tilde{b}) = \sum_{\wp=1}^{\infty} (\alpha_{\wp} + p_{\wp}) \tilde{b} - \sum_{\wp=2}^{\infty} \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)} \alpha_{\wp} \tilde{b}^{\wp} +$$

$$(-1)^n \sum_{\wp=1}^{\infty} \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m, \alpha)(1-\tilde{\rho}_1+\wp \tilde{\rho}_1)}$$

$$\hbar(\tilde{b}) = (-1)^n \sum_{\wp=1}^{\infty} \tilde{\psi}_{\wp} \tilde{b}^{\wp} + \tilde{b} - \sum_{\wp=2}^{\infty} v_{\wp} \tilde{b}^{\wp}$$

where $a_1 = 1$.

$$v_{\wp} = \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m,\alpha)(\wp\alpha+1-\tilde{\rho}_1)}\alpha_{\wp} \text{ and } \tilde{\psi}_{\wp} = \frac{1-\tilde{v}_2}{\beta_{\wp,n}(m,\alpha)(\wp\alpha+1-\tilde{\rho}_1)}p_{\wp}$$

since,

$$\begin{aligned} \sum_{\wp=2}^{\infty} \frac{\beta_{\wp,n}(m,\alpha)(\wp\tilde{\rho}_1+1-\tilde{\rho}_1)}{1-\tilde{v}_2} |v_{\wp}| + \sum_{\wp=1}^{\infty} \frac{\beta_{\wp,n}(m,\alpha)(\wp\tilde{\rho}_1+1-\tilde{\rho}_1)}{1-\tilde{v}_2} |\tilde{\psi}_{\wp}| \\ = \sum_{\wp=1}^{\infty} (\alpha_{\wp} + p_{\wp}) - \alpha_1 = 1 - \alpha_1 \leq 1 \end{aligned}$$

$$\begin{aligned} \therefore \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m,\alpha)(\wp\tilde{\rho}_1+1-\tilde{\rho}_1) \frac{1}{1-\tilde{v}_2} |v_{\wp}| + \\ \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m,\alpha)(\wp\tilde{\rho}_1+1-\tilde{\rho}_1) \frac{1}{1-\tilde{v}_2} |\tilde{\psi}_{\wp}| \\ \therefore \sum_{\wp=1}^{\infty} (|v_{\wp}| + |\tilde{\psi}_{\wp}|) \frac{\beta_{\wp,n}(m,\alpha)(1-\tilde{\rho}_1+\wp\tilde{\rho}_1)}{1-\tilde{v}_2} \leq \frac{1-\tilde{v}_2+1}{1-\tilde{v}_2} \\ \therefore \sum_{\wp=1}^{\infty} (|v_{\wp}| + |\tilde{\psi}_{\wp}|) \beta_{\wp,n}(m,\alpha)(1-\tilde{\rho}_1) \leq 2 - \tilde{v}_2 \Rightarrow \therefore \tilde{h}(\tilde{b}) \in THP_{\alpha}(\tilde{\rho}_1, \dots, \wp). \end{aligned}$$

Conversely, let $\tilde{h}(\tilde{b}) \in THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$.

$$\alpha_{\wp} = \frac{(\beta_{\wp,n}(m,\alpha)-\beta_{\wp,n}(m,\alpha)\tilde{\rho}_1+\beta_{\wp,n}(m,\alpha)\wp\alpha)}{1-\tilde{v}_2} |v_{\wp}|, \quad (\wp = 2, 3, \dots) \quad (19)$$

$$p_{\wp} = \frac{(\beta_{\wp,n}(m,\alpha)-\beta_{\wp,n}(m,\alpha)\tilde{\rho}_1+\beta_{\wp,n}(m,\alpha)\wp\alpha)}{1-\tilde{v}_2} |\tilde{\psi}_{\wp}|, \quad (\wp = 1, 2, 3, \dots) \quad (20)$$

$$\alpha_1 = 1 - \sum_{\wp=2}^{\infty} \alpha_{\wp} - \sum_{\wp=1}^{\infty} p_{\wp}.$$

using Theorem 3.1, $0 \leq \alpha_{\wp} \leq 1$, $(\wp = 2, 3, \dots)$, $0 \leq p_{\wp} \leq 1$, $(\wp = 1, 2, \dots)$, and $\alpha_1 \geq 0$.

We say $\tilde{h}(\tilde{b}) = \sum_{\wp=1}^{\infty} (\alpha_{\wp} \ell_{\wp}(\tilde{b}) + p_{\wp} \kappa_{\wp}(\tilde{b}))$ as required.

Theorem 3.3: Each member of $THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ ($\tilde{\rho}_1 \geq 0, \alpha + \tilde{\rho}_1 \geq 1$, $\tilde{v}_2 \in [0, 1)$) maps $\mathbb{Z}$ to the starlike-domain.

Proof: lets take $\tilde{h}(\tilde{b}) \in THP_{\alpha}(\tilde{\rho}_1, \dots, \wp)$ then,

$$Re\left\{\frac{\tilde{b}\ell'(\tilde{b}) - \overline{\tilde{b}\kappa'(\tilde{b})}}{\ell(\tilde{b}) + \kappa(\tilde{b})}\right\} > 0$$

with $Re(\omega) > 0$ iff $|1 + \omega| - |1 - \omega|$

Consider,

$$\begin{aligned} |\ell(\tilde{b}) + \overline{\kappa(\tilde{b})} - \tilde{b}\ell'(\tilde{b}) - \overline{\tilde{b}\kappa'(\tilde{b})}| |\tilde{b} - \sum_{\wp=2}^{\infty} v_{\wp} \tilde{b}^{\wp} + (-1)^n \sum_{\wp=1}^{\infty} \tilde{\psi}_{\wp} \tilde{b}^{\wp} \\ + \tilde{b}(1 - \sum_{\wp=2}^{\infty} v_{\wp} \wp \tilde{b}^{\wp-1})| \\ - \tilde{b}((-1)^n \sum_{\wp=1}^{\infty} \wp \tilde{\psi}_{\wp} \tilde{b}^{\wp-1} |2\tilde{b} - \sum_{\wp=2}^{\infty} (1 + \wp)|v_{\wp}| \tilde{b}^{\wp} - \sum_{\wp=1}^{\infty} (\wp - 1)|\tilde{\psi}_{\wp}| \tilde{b}^{\wp}| \end{aligned}$$

Consider $|\tilde{b}\kappa'(\tilde{b}) + \ell(\tilde{b}) + \overline{\kappa(\tilde{b})} - \tilde{b}\ell'(\tilde{b})|$,

$$|\sum_{\wp=2}^{\infty} (\wp - 1)|v_{\wp}| \tilde{b}^{\wp} - \sum_{\wp=1}^{\infty} |\tilde{\psi}_{\wp}| \tilde{b}^{\wp} (\wp + 1)|$$

Using above two inequality,

$$|2\tilde{b} - \sum_{\wp=2}^{\infty} (1 + \wp)|v_{\wp}| \tilde{b}^{\wp} - \sum_{\wp=1}^{\infty} (\wp - 1)|\tilde{\psi}_{\wp}| \tilde{b}^{\wp}|$$

$$\begin{aligned}
& -|\sum_{\wp=2}^{\infty} (\wp-1)|v_{\wp}|\tilde{b}^{\wp} - \sum_{\wp=1}^{\infty} (\wp+1)|\tilde{\psi}_{\wp}|\tilde{b}^{\wp}| \\
& \geq 2|\tilde{b}| - |\sum_{\wp=2}^{\infty} (1+\wp)|v_{\wp}|\tilde{b}^{\wp} + \sum_{\wp=1}^{\infty} (\wp-1)|\tilde{\psi}_{\wp}|\tilde{b}^{\wp}| \\
& \quad -|\sum_{\wp=2}^{\infty} (\wp-1)|v_{\wp}|\tilde{b}^{\wp} + \sum_{\wp=1}^{\infty} (\wp+1)|\tilde{\psi}_{\wp}|\tilde{b}^{\wp}| \\
& \geq 2|\tilde{b}|\{1 - (\sum_{\wp=2}^{\infty} \wp|v_{\wp}||x|^{\wp-1} + \sum_{\wp=1}^{\infty} \wp|\tilde{\psi}_{\wp}|\tilde{b}^{\wp-1})\} \geq 2|\tilde{b}| \\
& \quad \{1 - (\sum_{\wp=2}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp\tilde{\rho}_1)|v_{\wp}| + \\
& \quad \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp\tilde{\rho}_1)|\wp\wp|)\} \\
& \geq 2|\tilde{b}|\{1 - (\sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp\tilde{\rho}_1)|v_{\wp}| \\
& \quad + \sum_{\wp=1}^{\infty} d_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp\tilde{\rho}_1)|\wp\wp|)\} \\
& \geq 2|\tilde{b}|\{1 - (-1 + \sum_{\wp=1}^{\infty} \beta_{\wp,n}(m, \alpha)(1 - \tilde{\rho}_1 + \wp\tilde{\rho}_1)(|v_{\wp}| + |\wp\wp|))\} \\
& = 2|\tilde{b}|\{1 - (1 - \tilde{v}_2)\} \geq |\tilde{b}|\{-2\tilde{v}_2\}
\end{aligned}$$

4. Conclusion

In our research study notion from the class $SHP_{\alpha}(\tilde{\rho}_1, \dots, \wp) \mathcal{HUF}$ with the help of a fractional differential operator studied. We added some properties for the extreme points, distortions bounds, covering theorems for this subclass.

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