

A study of fractional kinetic equations associated with k -incomplete second Appell hypergeometric matrix functions

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Abstract

The present work introduces the k -incomplete second Appell hypergeometric matrix functions for positive real values of k . It establishes various generalized properties related to integral representations and differential relations. Furthermore, as an application of the newly introduced k -incomplete second Appell hypergeometric matrix functions, several new analytical results associated with fractional kinetic equations are established. Moreover, the obtained results may serve as a useful foundation for further investigations in fractional calculus, matrix analysis, and their applications in mathematical physics.

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1. Introduction

Let $\mathbb{C}^{S \times S}$ denote the vector space of $S \times S$ matrices over the complex field. For any ρ belonging to the set $\mathbb{C}$ and for any matrix $\mathcal{L}$ in $\mathbb{C}^{S \times S}$, $\sigma(\mathcal{L})$ represents the spectrum of the matrix $\mathcal{L}$ and $\sigma(\mathcal{L})$ is defined in [1] as the set of all eigenvalues of matrix $\mathcal{L}$:

$$\mu(\mathcal{L}) = \max\{\operatorname{Re}(\rho) : \rho \in \sigma(\mathcal{L})\}, \text{ and } \vartheta(\mathcal{L}) = \min\{\operatorname{Re}(\rho) : \rho \in \sigma(\mathcal{L})\}, \quad (1)$$

where the spectral abscissa of matrix $\mathcal{L}$ is denoted by $\mu(\mathcal{L})$ and $\mu(-\mathcal{L}) = -\vartheta(\mathcal{L})$. The matrix $\mathcal{L}$ in $\mathbb{C}^{S \times S}$ is called positive stable precisely when $\vartheta(\mathcal{L}) > 0$.

In the space $\mathbb{C}^{S \times S}$, 0 and I represent the null and identity matrix, respectively. Let matrix $\mathcal{L}$ in $\mathbb{C}^{S \times S}$, then by using the established properties of

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matrix functional calculus [2], if in the complex plane, Δ is an open set with $\mu(L) \subset \Delta$ and if $\varphi(\rho)$ and $\Theta(\rho)$ are analytic functions of a complex variable ρ on the set Δ , it follows that $\varphi(\rho)\Theta(\rho) = \Theta(\rho)\varphi(\rho)$. Again in [3], if the matrix $\mathcal{A}$ in $\mathbb{C}^{S \times S}$, with $\sigma(\mathcal{A}) \subset \Delta$, and if $\mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}$, then $\varphi(\mathcal{L})\Theta(\mathcal{A}) = \Theta(\mathcal{A})\varphi(\mathcal{L})$.

Definition 1.1: For $k > 0$, Mubeen, Rahman, and Arshad [4] defined the integral representation of the k -gamma and the k -beta matrix function denoted by $\Gamma_k(\mathcal{J})$ and $B_k(\mathcal{J}, \mathcal{G})$, for any two positive stable matrices $\mathcal{J}$ and $\mathcal{G}$ in $\mathbb{C}^{S \times S}$ as follows:

$$\Gamma_k(\mathcal{J}) = \int_0^\infty t^{\mathcal{J}-I} e^{-\frac{t}{k}} dt, \quad (2)$$

and

$$B_k(\mathcal{J}, \mathcal{G}) = \frac{1}{k} \int_0^1 t^{\frac{\mathcal{J}}{k}-I} (1-t)^{\frac{\mathcal{G}}{k}-I} dt. \quad (3)$$

Also, the inverse k -gamma function is represented by $\Gamma_k^{-1}(\rho)$ is an entire function of the variable ρ . According to Mubeen, Arshad, and Rahman [5], for any matrix $\mathcal{J}$ in $\mathbb{C}^{S \times S}$, we have

The matrix $\mathcal{J} + \ell kI$ remains invertible for all integer $\ell \geq 0$ and $k > 0$. (4)

In the framework of matrix functional calculus, $\Gamma_k(\mathcal{J})$ is an invertible matrix function defined by

$$\mathcal{J}(\mathcal{J} + kI)(\mathcal{J} + 2kI) \dots (\mathcal{J} + (\ell - 1)kI) \Gamma_k^{-1}(\mathcal{J} + \ell kI) = \Gamma_k^{-1}(\mathcal{J}),$$

where $\ell \geq 0$ and $k > 0$.

As stated in [4], let $(\mathcal{J})_{\ell,k}$ is expressed in the form:

$$(\mathcal{J})_{\ell,k} = \mathcal{J}(\mathcal{J} + kI)(\mathcal{J} + 2kI) \dots (\mathcal{J} + (\ell - 1)kI); \ell \geq 1 \quad (5)$$

Also,

$$(\mathcal{J})_{\ell,k} = \Gamma_k(\mathcal{J} + \ell kI) \Gamma_k^{-1}(\mathcal{J}), \ell \geq 1 \text{ and } k > 0. \quad (6)$$

Consider commuting positive stable matrices $\mathcal{J}$ and $\mathcal{G}$ in $\mathbb{C}^{S \times S}$. Under these conditions, the matrices $\mathcal{J} + \ell kI$, $\mathcal{G} + \ell kI$, and $\mathcal{J} + \mathcal{G} + \ell kI$ are invertible defined in [4]:

$$B_k(\mathcal{J}, \mathcal{G}) = \Gamma_k(\mathcal{J}) \Gamma_k(\mathcal{G}) \Gamma_k^{-1}(\mathcal{J} + \mathcal{G}). \quad (7)$$

When $k = 1$, the function $\Gamma_k(\mathcal{J})$ and $B_k(\mathcal{J}, \mathcal{G})$ reduce to the function $\Gamma(\mathcal{J})$ and $B(\mathcal{J}, \mathcal{G})$, respectively as given in [1, 6].

Definition 1.2: [1] Let $\mathcal{J}, \mathcal{G}$ and $\mathcal{Q}$ be matrices in $\mathbb{C}^{S \times S}$, with the property that $\mathcal{Q} + \ell kI$ is invertible for all nonnegative integers ℓ and $|\rho| \leq 1$. Under the above condition, the Gauss hypergeometric matrix function:

$${}_2F_1(\mathcal{J}, \mathcal{G}; \mathcal{Q}; \rho) = \sum_{\ell \geq 0} \frac{(\mathcal{J})_{\ell} (\mathcal{G})_{\ell} (\mathcal{Q})_{\ell}^{-1}}{\ell!} \rho^{\ell}. \quad (8)$$

Definition 1.3: [7] With the property that for all $\ell \geq 0$ and let $\mathcal{J}, \mathcal{G}$ and $\mathcal{Q}$ be matrices in $\mathbb{C}^{S \times S}$, $\mathcal{Q} + \ell kI$ is invertible. Then for $k \in \mathbb{R}^+$, the generalized k -hypergeometric matrix function:

$${}_2F_{1,k}(\mathcal{J}, \mathcal{G}; \mathcal{Q}; \rho) = \sum_{\ell \geq 0} \frac{(\mathcal{J})_{\ell,k} (\mathcal{G})_{\ell,k} \rho^\ell}{(\mathcal{Q})_{\ell,k} \ell!}. \quad (9)$$

2. Definitions and Preliminaries

2.1 Incomplete k -Gamma Matrix Function

Let $\mathcal{J}$ be a positive stable matrix in $\mathbb{C}^{S \times S}$ and assume that $k \in \mathbb{R}^+$. For $y \in \mathbb{R}^+$, the function $\gamma_k(\mathcal{J}, y)$ and its complement $\Gamma_k(\mathcal{J}, y)$ are defined as follows [8]:

$$\gamma_k(\mathcal{J}, y) = \int_0^y t^{\mathcal{J}-I} e^{-\frac{t^k}{k}} dt \quad (10)$$

and

$$\Gamma_k(\mathcal{J}, y) = \int_y^\infty t^{\mathcal{J}-I} e^{-\frac{t^k}{k}} dt, \quad (11)$$

Respectively, satisfy the relation:

$$\gamma_k(\mathcal{J}, y) + \Gamma_k(\mathcal{J}, y) = \Gamma_k(\mathcal{J}) \quad (12)$$

Remark 1: By setting $k = 1$, $\gamma_k(\mathcal{J}, y)$ and $\Gamma_k(\mathcal{J}, y)$ in Equations (10) and (11) simplifies to the function $\gamma(\mathcal{J}, y)$ and its complement $\Gamma(\mathcal{J}, y)$ as given in [9, 10], and are expressed as follows:

$$\gamma(\mathcal{J}, y) = \int_0^y t^{\mathcal{J}-I} e^{-t} dt$$

and

$$\Gamma(\mathcal{J}, y) = \int_0^\infty t^{\mathcal{J}-I} e^{-t} dt,$$

respectively, satisfy the relation:

$$\gamma(\mathcal{J}, y) + \Gamma(\mathcal{J}, y) = \Gamma(\mathcal{J}).$$

2.2 Incomplete k -Pochhammer Matrix Symbols

Let $\mathcal{J}$ be a matrix in $\mathbb{C}^{S \times S}$ and suppose that $k \in \mathbb{R}^+$. Then for $y \in \mathbb{R}^+$, the matrix symbols $(\mathcal{J}; y)_{\ell,k}$ and $[\mathcal{J}; y]_{\ell,k}$ are expressed as [8]:

$$(\mathcal{J}; y)_{\ell,k} = \gamma_k(\mathcal{J} + \ell k I, y) \Gamma_k^{-1}(\mathcal{J}) \quad (13)$$

and

$$[\mathcal{J}; y]_{\ell,k} = \Gamma_k(\mathcal{J} + \ell k I, y) \Gamma_k^{-1}(\mathcal{J}) \quad (14)$$

respectively, satisfy the following representation given by:

$$(\mathcal{J}; y)_{\ell,k} + [\mathcal{J}; y]_{\ell,k} = (\mathcal{J})_{\ell,k}.$$

2.3 k -Second Appell Hypergeometric Matrix Function

In [11], suppose $k \in \mathbb{R}^+$, $\ell, r \in \mathbb{N}^+$, $\omega, \nu \in \mathbb{C}$, and positive stable matrices in $\mathbb{C}^{S \times S}$ are $\mathcal{J}, \mathcal{G}, \mathcal{G}', \mathcal{Q}, \mathcal{Q}'$ where $\mathcal{Q}$ and $\mathcal{Q}'$ are invertible. Therefore, $F_{2,k}$ is defined as:

$$F_{2,k}(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J})_{\ell+r,k} (\mathcal{G})_{\ell,k} (\mathcal{G}')_{r,k} \omega^\ell \nu^r}{(\mathcal{Q})_{\ell,k} (\mathcal{Q}')_{r,k} \ell! r!}, \quad (15)$$

where $|\omega| < \frac{1}{k}$, $|\nu| < \frac{1}{k}$.

Remark 2: For $k = 1$, the function $F_{2,k}(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu)$ reduces to $F_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu)$ defined in [12, 13] as:

$$F_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J})_{\ell+r} (\mathcal{G})_{\ell} (\mathcal{G}')_r}{(\mathcal{Q})_{\ell} (\mathcal{Q}')_r} \frac{\omega^{\ell} \nu^r}{\ell! r!}, \quad (16)$$

where $|\omega| < 1$, $|\nu| < 1$.

3. k -Incomplete Second Appell Hypergeometric Matrix Function

Suppose $k, y \in \mathbb{R}^+$, $\ell, r \in \mathbb{N}^+$, $\omega, \nu \in \mathbb{C}$, and positive stable matrices in $\mathbb{C}^{S \times S}$ are $\mathcal{J}, \mathcal{G}, \mathcal{G}', \mathcal{Q}, \mathcal{Q}'$ where $\mathcal{Q}$ and $\mathcal{Q}'$ are invertible. Therefore k -incomplete second Appell hypergeometric matrix functions is defined in the form:

$$\gamma_{2,k}((\mathcal{J}; y), \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J}; y)_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \frac{\omega^{\ell} \nu^r}{\ell! r!}, \quad (17)$$

$$\Gamma_{2,k}([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \frac{\omega^{\ell} \nu^r}{\ell! r!}, \quad (18)$$

where $|\omega| < \frac{1}{k}$, $|\nu| < \frac{1}{k}$.

We get the following relation by using the equations (15), (17), and (18):

$$\begin{aligned} \gamma_{2,k}((\mathcal{J}; y), \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) + \Gamma_{2,k}([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = F_{2,k}(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu), \end{aligned}$$

Remark 3: If we take $k = 1$, it is defined in [14] as:

$$\gamma_2((\mathcal{J}; y), \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J}; y)_{\ell+r} (\mathcal{G})_{\ell} (\mathcal{G}')_r}{(\mathcal{Q})_{\ell} (\mathcal{Q}')_r} \frac{\omega^{\ell} \nu^r}{\ell! r!}, \quad (19)$$

$$\Gamma_2([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r} (\mathcal{G})_{\ell} (\mathcal{G}')_r}{(\mathcal{Q})_{\ell} (\mathcal{Q}')_r} \frac{\omega^{\ell} \nu^r}{\ell! r!}, \quad (20)$$

We get the following relation by using the equations (16), (19), and (20):

$$\begin{aligned} \gamma_2((\mathcal{J}; y), \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) + \Gamma_2([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = F_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \end{aligned} \quad (21)$$

3.1 Integral Expressions

Theorem 3.1: An integral representation for k -incomplete second Appell hypergeometric matrix functions is given by:

$$\begin{aligned} \Gamma_{2,k}([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \\ \Gamma_k^{-1}(\mathcal{J}) \times \left[\int_y^{\infty} t^{\mathcal{J}-I} e^{-\frac{t^k}{k}} {}_1F_{1,k}(\mathcal{G}; \mathcal{Q}; \omega t^k) {}_1F_{1,k}(\mathcal{G}'; \mathcal{Q}'; \nu t^k) dt \right] \end{aligned} \quad (22)$$

Proof: Using the integral representations in equations (11) and (14) for $[\mathcal{J}; y]_{\ell+r, k}$ in equation (18), the following result is derived:

$$\Gamma_{2,k}([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \Gamma_k^{-1}(\mathcal{J}) \sum_{\ell, r \geq 0} \left(\int_y^\infty e^{\frac{-t^k}{k}} t^{\mathcal{J} + (\ell + r)kl - I} dt \right) \\ \times \frac{(\mathcal{G})_{\ell, k}(\mathcal{G}')_{r, k} \omega^\ell \nu^r}{(\mathcal{Q})_{\ell, k}(\mathcal{Q}')_{r, k} \ell! r!},$$

Therefore, the above expression reduces to:

$$\Gamma_{2,k}([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = \Gamma_k^{-1}(\mathcal{J}) \left[\int_y^\infty e^{\frac{-t^k}{k}} t^{\mathcal{J} - I} \left(\sum_{\ell \geq 0} \frac{(\mathcal{G})_{\ell, k} (\omega t^k)^\ell}{(\mathcal{Q})_{\ell, k} \ell!} \right) \left(\sum_{r \geq 0} \frac{(\mathcal{G}')_{r, k} (\nu t^k)^r}{(\mathcal{Q}')_{r, k} r!} \right) dt \right], \\ \Gamma_{2,k}([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) = \\ \Gamma_k^{-1}(\mathcal{J}) \times \left[\int_y^\infty t^{\mathcal{J} - I} e^{\frac{-t^k}{k}} {}_1F_{1,k}(\mathcal{G}; \mathcal{Q}; \omega t^k) {}_1F_{1,k}(\mathcal{G}'; \mathcal{Q}'; \nu t^k) dt \right]$$

Therefore, the proof is concluded.

Remark 4: Setting $y = 0$, we obtain the following relation:

$$\Gamma_{2,k}(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = \Gamma_k^{-1}(\mathcal{J}) \times \left[\int_0^\infty t^{\mathcal{J} - I} e^{\frac{-t^k}{k}} {}_1F_{1,k}(\mathcal{G}; \mathcal{Q}; \omega t^k) {}_1F_{1,k}(\mathcal{G}'; \mathcal{Q}'; \nu t^k) dt \right]$$

which is given in [11].

Remark 5: Setting $k = 1$, we obtain the following relation:

$$\Gamma_2([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = \Gamma^{-1}(\mathcal{J}) \times \left[\int_y^\infty t^{\mathcal{J} - I} e^{-t} {}_1F_1(\mathcal{G}; \mathcal{Q}; \omega t) {}_1F_1(\mathcal{G}'; \mathcal{Q}'; \nu t) dt \right]$$

which is given in [12].

Remark 6: By setting $y = 0$ and $k = 1$, we obtain the following relation:

$$\Gamma_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \\ = \Gamma^{-1}(\mathcal{J}) \times \left[\int_0^\infty t^{\mathcal{J} - I} e^{-t} {}_1F_1(\mathcal{G}; \mathcal{Q}; \omega t) {}_1F_1(\mathcal{G}'; \mathcal{Q}'; \nu t) dt \right].$$

which is given in [15].

3.2 Differential Formulas

Theorem 3.2: The k -incomplete Second Appell hypergeometric matrix functions admits the following differential formula:

$$D_\omega^n [\Gamma_{2,k}([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu)] = (\mathcal{J})_{n,k} (\mathcal{G})_{n,k} [(\mathcal{Q})_{n,k}]^{-1} \\ [\Gamma_{2,k}([J + nkI; y], \mathcal{G} + nkI, \mathcal{G}'; \mathcal{Q} + nkI, \mathcal{Q}'; \omega, \nu)] \quad (23)$$

$$D_\nu^n [\Gamma_{2,k}([J; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu)] = (\mathcal{J})_{n,k} (\mathcal{G}')_{n,k} [(\mathcal{Q}')_{n,k}]^{-1} \\ [\Gamma_{2,k}([J + nkI; y], \mathcal{G}, \mathcal{G}' + nkI; \mathcal{Q}, \mathcal{Q}' + nkI; \omega, \nu)] \quad (24)$$

Proof: Applying differentiation with respect to ω of equation (18) and by using the Pochhammer matrix identity $[\mathcal{J}; y]_{\ell+r,k} = (\mathcal{J})_{\ell,k} [\mathcal{J} + k\ell I; y]_{r,k}$, we deduce the following expression

$$\begin{aligned} \frac{d}{d\omega} \{ \Gamma_{2,k}([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \nu) \} &= \sum_{\ell=1, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r,k} (\mathcal{G})_{\ell,k} (\mathcal{G}')_{r,k} \omega^{\ell-1} \nu^r}{(\mathcal{Q})_{\ell,k} (\ell r)_{r,k} (\ell-1)! r!} \\ &= \sum_{\ell=0, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r+1,k} (\mathcal{G})_{\ell+1,k} (\mathcal{G}')_{r,k} \omega^{\ell} \nu^r}{(\mathcal{G})_{\ell+1,k} (\mathcal{G}')_{r,k} \ell! r!} \\ &= (\mathcal{J})_{1,k} (\mathcal{G})_{1,k} (\mathcal{Q})_{1,k}^{-1} \sum_{\ell=0, r=0}^{\infty} \frac{[\mathcal{J}+kI; y]_{\ell+r,k} (\mathcal{G}+kI)_{\ell,k} (\mathcal{G}')_{r,k} \omega^{\ell} \nu^r}{(\mathcal{Q}+kI)_{\ell,k} (\mathcal{Q}')_{r,k} \ell! r!} \\ &= (\mathcal{J})_{1,k} (\mathcal{G})_{1,k} (\mathcal{Q})_{1,k}^{-1} [\Gamma_{2,k}([\mathcal{J} + kI; y], \mathcal{G} + kI, \mathcal{G}'; \mathcal{Q} + kI, \mathcal{Q}'; \omega, \nu)] \end{aligned}$$

The above expression yields equation (23) when $n = 1$. The general form is then proved by induction on n .

4. Applications

According to Haubold and Mathai [16], the fractional kinetic equations can be expressed in the following form:

$$\mathcal{K}(\mathfrak{t}) - \mathcal{K}_0 = -\mathcal{P}^u {}_0D_{\mathfrak{t}}^{-u} \mathcal{K}(\mathfrak{t}); \mathcal{P}, \mathfrak{t} \in \mathbb{R}^+, Re(u) > 0, \quad (25)$$

with ${}_0D_{\mathfrak{t}}^{-u}$ is defined by

$${}_0D_{\mathfrak{t}}^{-u} \varphi(t) = \frac{1}{\Gamma(u)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \omega)^{u-1} \varphi(\omega) d\omega; \mathfrak{t} \in \mathbb{R}^+, Re(u) > 0. \quad (26)$$

The solution associated with equation (25) is obtained as

$$\mathcal{K}(\mathfrak{t}) = \mathcal{K}_0 \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(u\ell+1)} (\mathcal{P}\mathfrak{t})^{u\ell} = \mathcal{K}_0 E_u(-\mathcal{P}^u \mathfrak{t}^u), \quad (27)$$

where $E_u(-\mathcal{P}^u \mathfrak{t}^u)$ is the Mittag-Leffler function with one parameter u while the two-parameter form is defined in [17, 18] as:

$$E_{u,s}(\mathfrak{t}) = \sum_{\ell=0}^{\infty} \frac{(\mathfrak{t})^{\ell}}{\Gamma(u\ell+s)}; Re(u) > 0, Re(s) > 0.$$

In [19] expressed the fractional kinetic equations as follows:

$$\mathcal{K}(\mathfrak{t}) - \mathcal{K}_0 h(\mathfrak{t}) = -\mathcal{P}^u {}_0D_{\mathfrak{t}}^{-u} \mathcal{K}(\mathfrak{t}); Re(u) > 0, \quad (28)$$

where $\mathcal{K}(\mathfrak{t})$ gives the population density at time $\mathfrak{t}$, with $\mathcal{K}_0 = \mathcal{K}(\mathfrak{t} = 0)$. Here $\mathcal{P}$ is the constant.

Theorem 4.1: Let $u > 0$ and $\mathfrak{t} \in \mathbb{C}$, the following is a solution of the generalized fractional kinetic matrix equation

$$\mathcal{K}(\mathfrak{t}) I - \mathcal{K}_0 \Gamma_{2,k}([\mathcal{J}; y], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \mathfrak{t}, \nu) = -B^u {}_0D_{\mathfrak{t}}^{-u} \mathcal{K}(\mathfrak{t}) \quad (29)$$

is obtained as

$$\mathcal{K}(\mathfrak{t}) I = \mathcal{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r,k} (\mathcal{G})_{\ell,k} (\mathcal{G}')_{r,k}}{(\mathcal{Q})_{\ell,k} (\mathcal{Q}')_{r,k}} \mathfrak{t}^{\ell} \frac{\nu^r}{r!} \times E_{u, \ell+1}(-B^u \mathfrak{t}^u), \quad (30)$$

where $E_{u, \ell+1}(-B^u \mathfrak{t}^u)$ is expressed as:

$$E_{u,\ell+1}(-B^u \mathfrak{t}^u) = \sum_{l=0}^{\infty} (-1)^l B^{ul} \frac{\mathfrak{t}^{ul}}{\Gamma(ul+\ell+1)}.$$

Proof: The Riemann–Liouville fractional integral operator can be expressed as [20, 21]:

$$\mathfrak{L} [{}_0D_{\mathfrak{t}}^{-u} \varphi(\mathfrak{t})] (\mathfrak{h}) = \mathfrak{h}^{-u} \varphi^*(\mathfrak{h}) \quad (31)$$

The Laplace transform corresponding to $\varphi(\mathfrak{t})$ is expressed as $\varphi^*(\mathfrak{h})$:

$$\varphi^*(\mathfrak{h}) = \mathfrak{L} [\varphi(\mathfrak{t})] (\mathfrak{h}) = \int_0^{\infty} e^{-\mathfrak{h}\mathfrak{t}} \varphi(\mathfrak{t}) d\mathfrak{t}, \operatorname{Re}(\mathfrak{h}) > 0. \quad (32)$$

Taking the Laplace transform of equation (29) on both sides yields

$$\mathfrak{L}[\mathfrak{K}(\mathfrak{t})I](\mathfrak{h}) = \mathfrak{K}_0 \mathfrak{L}[\Gamma_{2,k}([\mathcal{J}; \mathcal{Y}], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \mathfrak{t}, \mathfrak{v})](\mathfrak{h}) - B^u \mathfrak{L}[{}_0D_{\mathfrak{t}}^{-u} \mathcal{K}(\mathfrak{t})](\mathfrak{h}),$$

$$\mathfrak{K}^*(\mathfrak{h})[I + B^u \mathfrak{h}^{-u}] = \mathfrak{K}_0 \sum_{\ell, r \geq 0} \frac{[\mathcal{J}; \mathcal{Y}]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \frac{v^r}{r!}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \mathfrak{h}^{-(\ell+1)}$$

Utilizing the series expansion formula, the following expression is obtained

$$\mathfrak{K}^*(h) I = \mathfrak{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; \mathcal{Y}]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \frac{v^r}{r!}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \mathfrak{h}^{-(\ell+1)} \times \sum_{l=0}^{\infty} (-1)^l \left[\left(\frac{\mathfrak{h}}{B} \right)^{-u} \right]^l$$

Applying the inverse Laplace transform to both sides while using the identity:

$$\mathfrak{L}^{-1}\{\mathfrak{h}^{-u}\} = \frac{\mathfrak{t}^{u-1}}{\Gamma(u)}, \operatorname{Re}(u) > 0.$$

Thus, we have

$$\mathfrak{L}^{-1}\mathfrak{K}^*(\mathfrak{h}) I =$$

$$\begin{aligned} & \mathfrak{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; \mathcal{Y}]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \frac{v^r}{r!}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \times \mathfrak{L}^{-1}\left\{ \sum_{l=0}^{\infty} (-1)^l B^{ul} \mathfrak{h}^{-(ul+\ell+1)} \right\}, \\ \mathfrak{K}(\mathfrak{t}) I &= \mathfrak{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; \mathcal{Y}]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \frac{v^r}{r!}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \mathfrak{t}^{\ell} \times \left\{ \sum_{l=0}^{\infty} (-1)^l B^{ul} \frac{\mathfrak{t}^{ul}}{\Gamma(ul+\ell+1)} \right\}, \\ \mathfrak{K}(\mathfrak{t}) I &= \mathfrak{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; \mathcal{Y}]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \frac{v^r}{r!}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \mathfrak{t}^{\ell} \times E_{u, \ell+1}(-B^u \mathfrak{t}^u). \end{aligned}$$

Accordingly, the theorem has been proved.

Corollary 1: Put $k = 1$ and $y = 0$, the following result follows

$$\mathfrak{K}(\mathfrak{t}) I - \mathfrak{K}_0 \mathfrak{F}_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \mathfrak{t}, \mathfrak{v}) = -B^u {}_0D_{\mathfrak{t}}^{-u} \mathfrak{K}(\mathfrak{t})$$

has a solution in the form:

$$\mathfrak{K}(\mathfrak{t}) I = \mathfrak{K}_0 \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J})_{\ell+r} (\mathcal{G})_{\ell} (\mathcal{G}')_r \frac{v^r}{r!}}{(\mathcal{Q})_{\ell} (\mathcal{Q}')_r} \mathfrak{t}^m \times E_{u, \ell+1}(-B^u \mathfrak{t}^u).$$

Theorem 4.2: Let $u > 0$ and $\mathfrak{t} \in \mathbb{C}$, the following is a solution of the generalized fractional kinetic matrix equation

$$\mathfrak{K}(\mathfrak{t}) I - \mathfrak{K}_0 \Gamma_{2,k}([\mathcal{J}; \mathcal{Y}], \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \mathfrak{t}) = -B^u {}_0D_{\mathfrak{t}}^{-u} \mathfrak{K}(\mathfrak{t}) \quad (33)$$

is obtained as

$$\mathbb{K}(\mathfrak{t}) I = \mathbb{K}_0 \sum_{\ell, r=0}^{\infty} \frac{[\mathcal{J}; y]_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \omega^{\ell}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \frac{\omega^{\ell}}{\ell!} \mathfrak{t}^n \times E_{u, r+1}(-B^u \mathfrak{t}^u), \quad (34)$$

Corollary 2: Put $k = 1$ and $y = 0$, the following result follows

$$\mathbb{K}(\mathfrak{t}) I - \mathbb{K}_0 {}_2F_2(\mathcal{J}, \mathcal{G}, \mathcal{G}'; \mathcal{Q}, \mathcal{Q}'; \omega, \mathfrak{t}) = -B^v {}_0D_{\mathfrak{t}}^{-u} \mathbb{K}(\mathfrak{t})$$

has a solution of the form:

$$\mathbb{K}(\mathfrak{t}) I = \mathbb{K}_0 \sum_{\ell, r=0}^{\infty} \frac{(\mathcal{J})_{\ell+r, k} (\mathcal{G})_{\ell, k} (\mathcal{G}')_{r, k} \omega^{\ell}}{(\mathcal{Q})_{\ell, k} (\mathcal{Q}')_{r, k}} \frac{\omega^{\ell}}{\ell!} \mathfrak{t}^n \times E_{u, r+1}(-B^u \mathfrak{t}^u).$$

5. Conclusion

The k -incomplete Second Appell hypergeometric matrix functions generalize the classical hypergeometric functions and possess significant applications in several areas such as mathematics, physics, and engineering [22, 23, 24]. These functions have significant applications in solving differential equations, computing integrals, and describing complex processes. In this work, various generalized properties related to integral representations and differential relations is established. In addition, new results related to their use in fractional kinetic equations are derived. The findings presented here are new and are expected to support further developments and research in this domain.

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