

A sixth-order iterative scheme based on multiplicative derivatives for solving nonlinear equations

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Abstract

Numerical root-finding methods of nonlinear equations (NLE) are important as it has a many applications in engineering and sciences. Therefore, this arises to the study of the designing and stability of iterative approaches to solve NLE. Several higher order iterative technique are available in literature to approximate the solution of NLE. To increase the efficiency and stability of iterative methods different approaches are used by earlier authors. Recently, authors have developed the iterative schemes with multiplicative derivative to solve the NLE. In this direction, we have constructed the sixth order iterative method with the concept of multiplicative derivative instead of ordinary derivative. Some numerical have been tested and compared with earlier methods.

Subject Classification: 65G99, 65H05.

Keywords: Order of convergence, Nonlinear equations, Multiplicative derivative, Iterative methods, Convergence analysis, Taylor series.

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1. Introduction

The new calculus based on multiplicative is constructed by Grossman and Katz [1] in the 1972. There are a variety of fields in which multiplicative calculus can be applied. The theoretical foundations of multiplicative calculus as well as its applications has been discussed in Bashirov, Kurpinar, and Ozapici [2]. On biomedical image analysis, Florack and Assen [3] applied multiplicative calculus. In their study of economic growth, Filip and Piatecki [4] used it. Multiplicative calculus has been utilized to develop multiplicative iterative methods by Riza, Ozapici, and Misirli. [5], and Misirli and Ozapici [6]. In contrast, the construction of multiplicative differential equations has been done by Bashirov, Misirli and Tandogdu [7]. A further extension to the multiplicative calculus was proposed by Bashirov and Norozpour [8] and Uzer [9], which included complex-valued functions of complicated variables, which were restricted to positive real-valued functions. In a recent study, the multiplicative derivative on a scale of the time and its general properties has been described by Goktas, Yilmaz, and Yar [10].

After reviewing the above discussion, it should be clear that multiplicative calculus plays a crucial role in computational engineering, applied mathematics, and sciences. In recent years, researchers have utilized multiplicative derivatives to construct new iterative approaches to solve NLE. Before attempting to apply the Approach of Multiplicative Calculus (MCA) to numerical techniques, the authors must have prior knowledge of iterative approaches. The distinctions between iterative schemes can be made on the memory basis (included or excluded), one or multipoint sub steps, and global or local convergence. One of the key distinctions of iterative approaches is global and local convergence. Methods with open and closed brackets are referred to as local and global, respectively. Although they are typically quicker than global methods, local convergent methods are not always convergent. There aren't many globally convergent techniques. In contrast, we have an abundance of methods that are convergent locally because these have quicker convergence and simple implementation on NLE [11-16]. Therefore, in this study the authors will focus on locally convergent methods.

The multiplicative calculus methodology has recently been adopted by academics like Ozyapici, Sensoy, and Karanfiller [17] for the creation of one-point iterative procedures. Additionally, Singh, Bhalla, and Behl [18] introduced a multiplicative variant of the Schroder method, achieving quadratic convergence, while Waseem et al. [19] proposed another quadratically convergent one-point iterative method based on multiplicative calculus. We are aware of the several issues one-point approaches have with their efficiency index and order of convergence. The most significant class of optimal multi-point iterative technique, attracted the attention of many academics as a result. It is difficult to find the multi-point iterative procedures using a MCA. The theoretical justification of a convergence order demanding a better efficiency index, extensive and challenging calculus work, and the optimal convergence order are some of the key causes of this.

In light of these considerations, we propose a novel MCA-based multi-point iterative method. The essential theorem suggests a comprehensive convergence investigation. We adopt four different metrics to compare our methods to the ones already in use, including (i) convergence order, (ii) the absolute error, (iii) Computational time, and (iv) the line graphs of errors, in order to fairly compare our methods with the ones already in use. We draw the conclusion that our new proposed scheme outperforms the previous approaches on the basis of four distinct comparisons.

The rest of content of paper are section 2 discussed the proposed technique and analysis of convergence. Section 3 depicts the numerical results. Lastly, conclusion is represented in section 4.

2. The Proposed Technique and Analysis of Convergence

This section will outline our suggested approach to utilizing the multiplicative calculus notion to discover a simple root. The definition for convergence order used for our approach is defined as follows.

A third order method developed by Unal, Cumhur, and Gokdogan [11] to solve NLE is written as following $y_q = x_q - \frac{\ln g(x_q)}{\ln g^*(x_q)}$,

$$x_{q+1} = x_q - \frac{2 \ln g(x_q)}{\ln g^*(x_q) + \ln g^*(y_q)}, q = 0, 1, 2, \dots$$

By adding one more step to the upper method, we have proposed the Sixth order iterative technique in multiplicative derivative represented as

$$\begin{aligned} y_q &= x_q - \frac{\ln g(x_q)}{\ln g^*(x_q)}, \\ z_q &= x_q - \frac{2 \ln g(x_q)}{\ln g^*(x_q) + \ln g^*(y_q)}, \\ x_{q+1} &= z_q - \frac{\log g^*(x_q) + \log g^*(y_q)}{3 \log g^*(y_q) - \log g^*(x_q)} \cdot \frac{\log g(z_q)}{\log g^*(x_q)}. \end{aligned} \quad (2.1)$$

Where, $g^*(x)$ is multiplicative derivative.

The following theorem, has been proved for convergence analysis.

Theorem 2.2: Assume an open interval D and let $t \in D$ be a multiplicative zero of a sufficiently differentiable function $g: D \subseteq \mathbb{R} \rightarrow \mathbb{R}^+$. Then, the multiplicative sixth-order method possesses convergence of sixth-order with the error given by

$$x_{q+1} = \frac{1}{4}(4b_2^5 - 8b_2^3b_3 - 5b_2b_3^2)e_q^6 + \mathcal{O}(e_q^7).$$

Proof: Suppose a equation $g(x) = 1$ has a simple root t and $e_q = x_q - t$ denotes the error at q^{th} iteration. By employing the multiplicative Taylor series for function $g(x)$, one can express as

$$g(x_q) = g(t + e_q) = g(t)(g^*(t))^{e_q} (g^{(2*)}(t))^{\frac{e_q^2}{2!}} (g^{(3*)}(t))^{\frac{e_q^3}{3!}} \mathcal{O}^*(e_q^4). \quad (2.2)$$

On implementing the natural logarithm on each side, one can obtain

$$\begin{aligned}
\ln g(x_q) &= \ln g(t) + \ln g^*(t)e_q + \ln g^{(2*)}(t)\frac{e_q^2}{2!} + \ln g^{(3*)}(t)\frac{e_q^3}{3!} + \mathcal{O}(e_q^4), \\
&= \ln g^*(t) \left(e_q + \frac{1}{2!} \frac{\ln g^{(2*)}(t)}{\ln g^*(t)} e_q^2 + \frac{1}{3!} \frac{\ln g^{(3*)}(t)}{\ln g^*(t)} e_q^3 + \mathcal{O}(e_q^4) \right), \\
&= \ln g^*(t)(e_q + b_2 e_q^2 + b_3 e_q^3 + \mathcal{O}(e_q^4)),
\end{aligned} \tag{2.3}$$

where $b_j = \frac{1}{j!} \frac{\ln g^{(j*)}(t)}{\ln g^*(t)}$.

On taking the multiplicative derivative of (2.2), we get

$$\begin{aligned}
\ln g^*(x_q) &= \ln g^*(t) + \ln g^{(2*)}(t)e_q + \ln g^{(3*)}(t)\frac{e_q^2}{2!} + \mathcal{O}(e_q^3), \\
&= \ln g^*(t) \left(1 + \frac{1}{2!} \frac{\ln g^{(2*)}(t)}{\ln g^*(t)} e_q + \frac{1}{3!} \frac{\ln g^{(3*)}(t)}{\ln g^*(t)} e_q^2 + \mathcal{O}(e_q^3) \right), \\
&= \ln g^*(t)(1 + 2b_2 e_q + 3b_3 e_q^2 + \mathcal{O}(e_q^3)).
\end{aligned} \tag{2.4}$$

By dividing (2.3) with (2.4), one can have

$$\frac{\ln g(x_q)}{\ln g^*(x_q)} = \frac{(e_q + b_2 e_q^2 + b_3 e_q^3 + \mathcal{O}(e_q^4))}{(1 + 2b_2 e_q + 3b_3 e_q^2 + \mathcal{O}(e_q^3))}. \tag{2.5}$$

The initial step of method (2.1) is derived by applying equation (2.5)

$$y_q = t + b_2 e_q^2 + 2(b_3 - b_2^2)e_q^3 + \mathcal{O}(e_q^4). \tag{2.6}$$

On expanding the multiplicative Taylor series upon $g^*(y_q)$ about t , one can derive the following

$$g^*(y_q) = (g^*(t))(g^{(2*)}(t))^{(y_q-t)}(g^{(3*)}(t))^{\frac{(y_q-t)^2}{2!}} \mathcal{O}^*(e_q^3).$$

On implementing the natural logarithm on each side, one can get

$$\begin{aligned}
\ln g^*(y_q) &= \ln g^*(t) + \ln g^{(2*)}(t)(y_q - t) + \ln g^{(3*)}(t)\frac{(y_q-t)^2}{2!} + \mathcal{O}^*(e_q^3), \\
&= \ln g^*(t) \left(1 + 2b_2 e_q^2 + 4(b_2 b_3 - b_2^3)e_q^3 + \mathcal{O}(e_q^4) \right).
\end{aligned} \tag{2.7}$$

Adding (2.4) and (2.7), one can get

$$\ln g^*(x_q) + \ln g^*(y_q) = 2\ln g^*(t) \left(1 + b_2 e_q + \left(b_2^2 + \frac{3}{2} b_3 \right) e_q^2 + \mathcal{O}(e_q^3) \right). \tag{2.8}$$

By using equation (2.3) and (2.8), we obtained the second step of scheme

$$\begin{aligned}
z_q &= x_q - \frac{2\ln g(x_q)}{\ln g^*(x_q) + \ln g^*(y_q)}, \\
z_q &= \left(b_2^2 + \frac{1}{2} b_3 \right) e_q^3 + \mathcal{O}(e_q^4).
\end{aligned} \tag{2.9}$$

Again expanding $g(z_q)$ about t and by utilizing the multiplicative Taylor expansion we obtain

$$g(z_q) = g(t)(g^*(t))^{(z_q-t)}(g^{(2*)}(t))^{\frac{(z_q-t)^2}{2!}}(g^{(3*)}(t))^{\frac{(z_q-t)^3}{3!}} \mathcal{O}^*(e_q^4).$$

By using the natural logarithm to each side, one can get

$$\ln g(z_q) = \ln g(t) + \ln g^*(t)(z_q - t) + \ln g^{(2*)}(t)\frac{(z_q-t)^2}{2!} + \mathcal{O}^*(e_q^3),$$

$$= \ln g^*(t) \left(b_2^2 + \frac{1}{2} b_3 \right) e_q^3 + \mathcal{O}^*(e_q^4). \quad (2.10)$$

Now using (2.4), (2.7), (2.8) and (2.10), we get the final error of scheme

$$\begin{aligned} x_{q+1} &= z_q - \frac{\log g^*(x_q) + \log g^*(y_q)}{3 \log g^*(y_q) - \log g^*(x_q)} \cdot \frac{\log g(z_q)}{\log g^*(x_q)}, \\ &= \frac{1}{4} (4b_2^5 - 8b_2^3 b_3 - 5b_2 b_3^2) e_q^6 + \mathcal{O}(e_q^7). \end{aligned} \quad (2.11)$$

Hence, the method (2.1) has sixth convergence order.

3. Experimental Work

In this section, numerical simulations are performed for some important and commonly used models in Real-world life. These models can be incorporated into the nonlinear equations so that approximate solutions can be found. As mentioned above, the exact solution may not be possible due to nonlinearity, so iterative methods must be used. Tables [1-2] show the results of iterative approaches, with all computations performed using software Mathematica and with the stopping condition $|x_{q+1} - x_q| < \epsilon$ and $\epsilon = 10^{-300}$.

The proposed method denoted by PM has been compared with some earlier existing iterative methods like Wang method denoted as WM, Neta method denoted by NM, and Behl method denoted as BM.

Furthermore, the approximated computational order of convergence (ACOC) is calculated using the formula

$$\rho \cong \frac{\ln \left| \frac{x_{q+1} - t}{x_q - t} \right|}{\ln \left| \frac{x_q - t}{x_{q-1} - t} \right|}. \quad (3.1)$$

Example 3.1: Chemical Reaction: A chemical reaction occurs when two chemicals interact in order to form a new chemical with a different composition. The properties of chemical's element or compound cause changes in composition during this process. Chemists are primarily concerned with these chemical changes. In [20], it is vital to note that "the two questions that have to be answered for a chemically reacting system are: (1) what changes are expected to occur and (2) how fast will they occur?" It is clear from this that mathematics play a crucial role in chemical investigations. In general, nonlinear functions are involved in this process, so numerical approaches must be used. As an example, suppose a chemical reaction takes place, and a particular ion's concentration at time x is written as a following NLE:

$$g(x) = 5e^{-3x} + e^{-5x}. \quad (3.2)$$

If we want to check when this concentration will be half of its value at time 0, we need to apply mathematics to figure it out. Finding a nonlinear equation's root will be similar to doing so if the initial assumption is that $g(0) = 2$.

$$5e^{-3x} + e^{-5x} - 1 = 0. \quad (3.3)$$

Chemists typically utilize standard numerical techniques to calculate the root of (3.3). However, in many cases, multiplicative base approaches should

still be considered due to their effectiveness in solving problems involving exponential functions. Table 1 provides a clear demonstration of the multiplicative numerical methods’ advantage in solving this issue.

Table 1
Chemical Reaction

Method	q	$ t_q - t_{q-1} $	$ g(t_q) $	ρ	C.P.U time
CM	2	1.864(−1)	7.992(−1)	5.914	0.265
	3	2.298(−4)	7.178(−4)		
	4	1.427(−21)	4.456(−21)		
AM	2	3.179(−1)	−7.062(−1)	5.648	0.266
	3	7.958(−2)	−2.195(−1)		
	4	1.548(−6)	4.834(−6)		
SM	2	d	d	d	d
	3				
	4				
PM	2	6.820(−8)	1.000	6.000	0.198
	3	2.387(−48)			
	4	4.386(−291)			

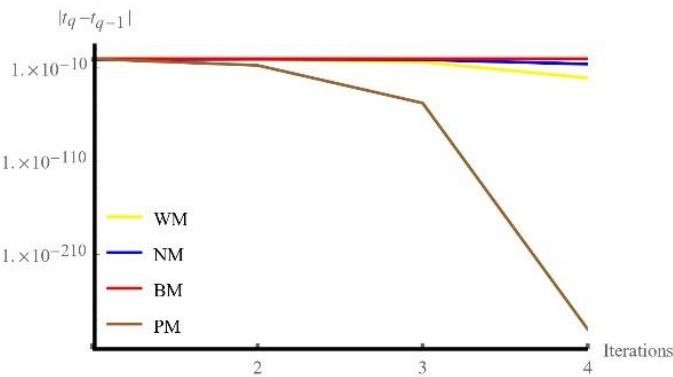


Figure 1
Graphical analysis of Example 3.1

The error behavior for Example 3.1 is shown in Figure 1. As seen, the proposed method reduces the error significantly faster than the compared existing methods.

Example 3.2: Planck’s radiation law problem: The solution to this example, which determines the density of energy of an isothermal blackbody, is provided by [13].

$$\psi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{\frac{hc}{\lambda BT}} - 1}, \tag{3.4}$$

where the blackbody's absolute temperature is represented by T , the radiation's wavelength denoted by λ , Boltzmann constant is termed as B , Planck's constant denoted as h , and the light's speed is denoted by c . The λ , which corresponds to the highest energy density $\psi(\lambda)$, that is we are interested in finding out.

Additionally, $\psi'(\lambda) = 0$ indicates that ψ reaches its maximum value when

$$\frac{\frac{ch}{\lambda BT} e^{\frac{ch}{\lambda BT}}}{e^{\frac{ch}{\lambda BT}} - 1} = 5. \quad (3.5)$$

If $x = \frac{ch}{\lambda BT}$, then (3.5) is satisfied when

$$g(x) = e^{-x} + \frac{x}{5} - 1 = 0. \quad (3.6)$$

As a result, the following formula can be used to determine the radiation's maximum wavelength λ from the solutions of g .

$$\lambda \approx \frac{ch}{xBT}, \quad (3.7)$$

where x is a solution of (3.7). Our desired root is $x = 4.94.9651142317442$.

Table 2 shows the numerical outcomes for the $g(x) = 0$ test equation. It can be seen that the proposed method has modest residual error numbers when compared to other methods.

Table 2
Planck's radiation law problem

Method	q	$ x_q - x_{q-1} $	$ g(x_q) $	ρ	C.P.U time
WM	2	4.325(-3)	8.349(-4)	6.000	0.234
	3	1.305(-20)	2.520(-21)		
	4	9.943(-126)	1.919(-126)		
NM	2	7.950(-4)	1.534(-4)	6.000	0.265
	3	1.654(-25)	3.193(-26)		
	4	1.346(-155)	2.695(-156)		
BM	2	8.874(-3)	1.713(-3)	5.996	0.25
	3	1.064(-18)	2.055(-19)		
	4	3.219(-114)	6.213(-115)		
PM	2	2.653(-4)	1.000	6.000	0.214
	3	2.320(-28)			
	4	1.037(-172)			

The error behavior for Example 3.2 is shown in Figure 2. As seen, the proposed method reduces the error significantly faster than the compared existing methods.

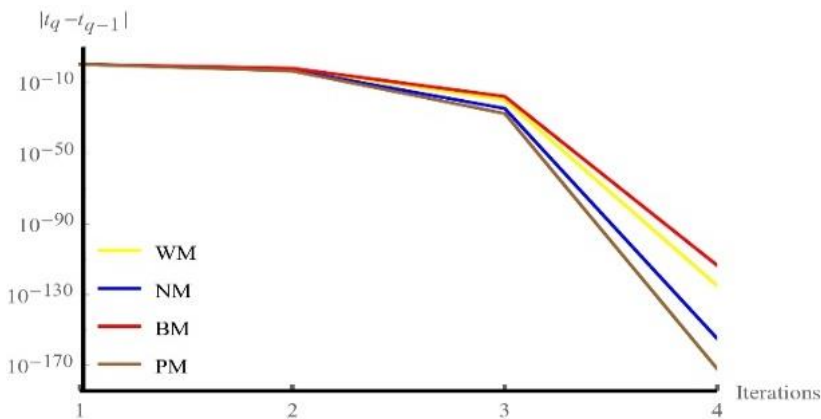


Figure 2
Graphical Analysis of Example 3.2

4. Conclusion

By using the concepts of multiplicative calculus, the authors developed a novel sixth-order iterative technique to solve NLE. We conducted a thorough convergence analysis to validate the proposed scheme. The performance of our method was compared against existing techniques, evaluating metrics such as convergence order, the absolute error between consecutive iterations, number of iterations, absolute error graphs, CPU computational time. Our findings demonstrate that the proposed method delivers superior approximations with reduced computational time and complexity. Additionally, it maintains a stable computational order of convergence (COC) across all tested examples. The primary disadvantage of this method is that $g^*(x)$ approaches or is close to 1, which may affect its performance in certain cases.

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