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A novel modified Picard-Krasnoselskii iterative algorithm with optimized convergence

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Abstract

We present a novel fixed-point iterative algorithm, termed as Modified Picard-Krasnoselskii Hybrid Algorithm, abbreviated as PK_m algorithm (m denotes the modification level). This new process is developed by modifying the classical Picard-Krasnoselskii hybrid iteration, aiming to achieve improved convergence characteristics. The motivation for this development stems from the need to optimize the convergence of well-known iterative processes such as Picard, Mann, Noor, and Ishikawa iterations, as well as their hybrid variants including Picard-Noor, Picard-Ishikawa, and Picard-Mann processes. The proposed algorithm is rigorously analyzed within the framework of fixed-point theory, and its superiority is established in the sense of Berinde's [1] definition of faster convergence. Theoretical results are examined numerically, which clearly demonstrates that the proposed algorithm achieves faster convergence than its predecessors when applied to a suitable test function. The comparison is made based on the iterations count required and the accuracy of the approximate fixed point. Our findings confirm that the Modified Picard-Krasnoselskii Hybrid iterative algorithm not only retains the stability and applicability of traditional methods but also exhibits significantly improved convergence behavior. This contribution opens new avenues for the development of even more efficient iterative processes and their applications in nonlinear functional analysis and related computational problems.

Subject Classification: *47H10, 47J25, 65H10, 47H09.*

Keywords: *Contraction mapping, Iterative process, Fast convergence, Stability, Hybrid iteration algorithms.*

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1. Introduction

Various fixed point (FP) iterative processes have been developed like Picard, Mann, Ishikawa and Noor and a number of hybrid schemes i.e. Picard-Mann, Picard-Ishikawa, Picard-Noor were also introduced [1-5], [11-16]. These hybrid processes were found to converge faster than many iterative methods in Berinde's [1] sense (see [6], [7], [11], [14], [15]). We are establishing a new iterative algorithm namely Modified Picard-Krasnoselskii (P-Kr) Hybrid algorithm. We claim that its convergence rate is superior than many of the iterative methods. Stability and data dependency of the algorithm is also proved.

Throughout this paper, $\mathcal{A} \neq \emptyset$ denotes the closed & convex subset of a Banach space $\mathfrak{B}$.

A map $\mathcal{F}: \mathcal{A} \to \mathcal{A}$ is known be a contraction map if $\exists\ \theta \epsilon (0, 1)$ such that

$$\|\mathcal{F}a_1 - \mathcal{F}a_2\| \leq \theta \|a_1 - a_2\| \text{ for all } a_1, a_2 \in \mathcal{A} \tag{1}$$

Some well-known iterative methods are:

Picard Iteration Method [2]

$$\varphi_{n+1} = \mathcal{F}\varphi_n \tag{2}$$

Mann Iteration Method [3]

$$\varphi_{n+1} = (1 - a_n)\varphi_n + a_n\mathcal{F}\varphi_n \tag{3}$$

where $\{a_n\} \subset (0, 1)$.

Definition 1: [1] If $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ are real sequences with $\varphi_n \to \varphi$ and $\sigma_n \to \sigma$ such that $lim_{n\to\infty} \frac{|\varphi_n - \varphi|}{|\sigma_n - \sigma|} = 0$, then $\{\varphi_n\}_{n=1}^{\infty}$ converges to φ faster than $\{\sigma_n\}_{n=1}^{\infty}$ converges to σ.

Definition 2: [2] If $\{\varphi_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be fixed point iterative processes, where $\varphi_n \to \wp$ and $t_n \to \wp$. Let $\|\varphi_n - \wp\| \leq a_n$ and $\|t_n - \wp\| \leq b_n\ \forall\ n \in \aleph$, where $a_n > 0$, $b_n > 0$. If $\{a_n\}_{n=1}^{\infty}$ converges speedier than $\{b_n\}_{n=1}^{\infty}$, then $\{\varphi_n\}_{n=1}^{\infty}$ converges quicker than $\{t_n\}_{n=1}^{\infty}$ to $\wp$.

For any fixed φ_1 in $\mathfrak{B}$, some hybrid iterative processes are defined by the sequence $\{\varphi_n\}_{n=1}^{\infty}$ as follows:

Picard-Mann Hybrid Iterative Method [8]

$$\begin{aligned} \varphi_{n+1} &= \mathcal{F}\sigma_n \\ \sigma_n &= (1 - a_n)\varphi_n + a_n\mathcal{F}\varphi_n \end{aligned} \tag{6}$$

where $\{a_n\} \subset (0, 1)$.

Picard-Krasnoselskii Hybrid Iterative Method [9]

$$\begin{aligned} \varphi_{n+1} &= \mathcal{F}\sigma_n \\ \sigma_n &= (1 - \delta)\varphi_n + \delta\mathcal{F}\varphi_n \end{aligned} \tag{7}$$

where $\delta \epsilon (0, 1)$.

The above iterative methods motivated us to introduce a new iterative algorithm defined as [8]-[10], [16]:

For any fixed φ_1 in $\mathfrak{B}$, we define PK_m iterative algorithm by the sequence $\{\varphi_n\}_{n=1}^{\infty}$ as

$$\varphi_{n+1} = \mathcal{F}^n \sigma_n$$
$$\sigma_n = (1-\delta)\varphi_n + \delta\, \mathcal{F}\varphi_n \tag{10}$$

where $\delta \in (0,1)$ and n is a natural number greater than 1. In case $n = 1$, (10) becomes same as (7).

Lemma 1: *Let θ be such that $0 \leq \theta < 1$ and let $\{\varepsilon_n\}_{n=1}^{\infty}$ $(\varepsilon > 0)$ be a sequence, with $lim_{n\to\infty}\, \varepsilon_n = 0$, then $lim_{n\to\infty}\, \rho_n = 0$, for $\{\rho_n\}_{n=1}^{\infty}, \rho > 0$, satisfying $\rho_{n+1} \leq \theta\rho_n + \varepsilon_n$, $n \in N$.*

2. Convergence and Stability Analysis

Theorem 1: *Suppose $\mathcal{F}: \mathcal{A} \to \mathcal{A}$ holds (1). If the process PK_m generates the sequence $\{\varphi_n\}_{n=1}^{\infty}$ iteratively, then $\{\varphi_n\}$ approaches to a fixed point of $\mathscr{F}$.*

Proof: By Banach contraction theorem, $\exists!$ fixed point $\wp$ of $\mathscr{F}$. We claim that $\varphi_n \to \wp$ as $n \to \infty$. Using (10) we have:

$$\begin{aligned}\|\sigma_n - \wp\| &= \|(1-\delta)(\varphi_n - \wp) + \delta(F\varphi_n - \wp)\| \\ &\leq (1-\delta)\|\varphi_n - \wp\| + \delta\,\theta\|\varphi_n - \wp\| \\ &\leq (1-(1-\theta)\delta)\,\|\varphi_n - \wp\|\end{aligned}$$

So, $\|\varphi_{n+1} - \wp\| = \|F^2\sigma_n - \wp\| \leq \theta^2\|\sigma_n - \wp\| \leq \theta^2(1-(1-\theta)\lambda)\|\varphi_n - \wp\|$

By Lemma 1, $lim_{n\to\infty}\|\varphi_n - \wp\| = 0$.

So, φ_n approaches to $\wp$ as $n \to \infty$.

Theorem 2: *Let $\mathcal{F}: \mathcal{A} \to \mathcal{A}$ be a mapping on $\mathcal{A}$ and let $\mathscr{F}$ satisfies (1). If the process PK_m (10) generates the sequence $\{\varphi_n\}_{n=1}^{\infty}$ iteratively, then PK_m is $\mathscr{F}$-stable.*

Proof: Suppose PK_m generates the sequence $\varphi_{n+1} = T_{\mathcal{F},\varphi_n}$, converging to a unique $\wp^* \epsilon\, Fix(\mathcal{F})$ (by theorem (1)). Let $\{\tau_n\}_{n=1}^{\infty}$ be any sequence in $\mathcal{A}$ and $\varepsilon_n = \left\|\tau_{n+1} - T_{\mathcal{F},\varphi_n}\right\|$. We will show that $lim_{n\to\infty}\, \varepsilon_n = 0$ iff $lim_{n\to\infty}\, \tau_n = \wp^*$.

Let $lim_{n\to\infty}\, \varepsilon_n = 0$. We have:

$$\begin{aligned}\|\tau_{n+1} - \wp^*\| &\leq \left\|\tau_{n+1} - T_{\mathcal{F},\varphi_n}\right\| + \left\|T_{\mathcal{F},\varphi_n} - \wp^*\right\| \\ &\leq \varepsilon_n + \theta^2(1-(1-\theta)\delta)\,\|\tau_n - \wp^*\|\end{aligned}$$

Now $\theta\epsilon(0,1)$, $\delta\epsilon(0,1)$ for $n \,\epsilon\, \aleph$ and $lim_{n\to\infty}\, \varepsilon_n = 0$

By Lemma 1, we have $lim_{n\to\infty}\, \tau_n = \wp^*$.

Contrarily, consider $lim_{n\to\infty}\, \tau_n = \wp^*$.

$$\varepsilon_n = \left\|\tau_{n+1} - T_{\mathcal{F},\varphi_n}\right\|$$

$$\leq \|\tau_{n+1} - \wp^*\| + \left\|T_{F,\varphi_n} - \wp^*\right\|$$
$$\leq \|\tau_{n+1} - \wp^*\| + \theta^2(1-(1-\theta)\delta)\|t_n - \wp^*\|$$

This implies that $lim_{n\to\infty}\, \varepsilon_n = 0$. Therefore PK_m is $\mathscr{F}$-stable.

We now prove the fast convergence of PK_m than others processes.

Theorem 3: *Consider $\mathcal{F}: \mathcal{A} \to \mathcal{A}$ as a contraction map that holds condition (1) with a unique $\wp \in Fix(\mathcal{F})$. For $\varphi_1 \in \mathcal{A}$, consider the iterative sequences generated by (2), (3), (4), (5), (9) and (10) with $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ being real sequences in $(0,1)$ satisfying $0 < \lambda \leq a_n, b_n, c_n < 1$ for $\lambda > 0$ and for all $n \in \aleph$. Then $\{\varphi_n\}_{n=1}^{\infty}$ defined by PK_m (10) converges to $\wp$ faster than others.*

Proof: Using (10), we have:

$$\|\sigma_n - \wp\| = \|(1-a_n)(\varphi_n - \wp) + a_n(\mathcal{F}\varphi_n - \wp)\|$$
$$\leq (1-a_n)\|\varphi_n - \wp\| + a_n\,\theta\|\varphi_n - \wp\|$$
$$\leq (1-(1-\theta)\lambda)\,\|\varphi_n - \wp\|$$

So,

$$\|\varphi_{n+1} - \wp\| = \|\mathcal{F}^2\sigma_n - \wp\| \leq \theta\|\mathcal{F}\sigma_n - \wp\| \leq \theta^2\|\sigma_n - \wp\|$$
$$\leq \theta^2(1-(1-\theta)\lambda)\,\|\varphi_n - \wp\|$$

… … …

$$\leq (\theta^2(1-(1-\theta)\lambda))^n\,\|\varphi_1 - \wp\|$$

Let $a_n = (\theta^2(1-(1-\theta)\lambda))^n$

Using (2),

$$\|\varphi_{n+1} - \wp\| \leq \theta^n\|\varphi_1 - \wp\|$$
$$\text{Let } b_n = \theta^n\|\varphi_1 - \wp\|$$
$$\therefore \frac{a_n}{b_n} \to 0 \text{ as } n \to \infty.$$

So (10) converges faster than (2).

Now Using (3),

$$\|\varphi_{n+1} - \wp\| = \|(1-a_n)(\varphi_n - \wp) + a_n(\mathcal{F}\varphi_n - \wp)\|$$
$$\leq (1-a_n)\|\varphi_n - \wp\| + a_n\theta\,\|\varphi_n - \wp\|$$
$$\leq (1-(1-\theta)\lambda)\,\|\varphi_n - \wp\|$$

… … … .

$$\leq (1-(1-\theta)\lambda)^n\,\|\varphi_1 - \wp\|$$

Let $c_n = (1-(1-\theta)\lambda)^n\,\|\varphi_1 - \wp\|$

$$\therefore \frac{a_n}{c_n} \to 0 \text{ as } n \to \infty.$$

So (10) converges faster than (3).

Using (6),

$$\|\varphi_{n+1} - \wp\| = \|\mathcal{F}\sigma_n - \wp\| \leq \theta\ \|\sigma_n - \wp\|$$
$$\leq \theta\ \|(1 - a_n)(\varphi_n - \wp) + a_n(\mathcal{F}\varphi_n - \wp)\|$$
$$\leq \theta[(1 - a_n)\|\varphi_n - \wp\| + a_n\ \theta\|\varphi_n - \wp\|]$$
$$\leq \theta(1 - (1 - \theta)\lambda)\ \|\varphi_n - \wp\|$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$\leq \left(\theta(1 - (1 - \theta)\lambda)\right)^n \|\varphi_1 - \wp\|$$
$$\text{Let } d_n = [\theta(1 - (1 - \theta)\lambda)]^n\ \|\varphi_1 - \wp\|$$
$$\therefore \frac{a_n}{d_n} \to 0 \text{ as } n \to \infty.$$

So (10) converges faster than (6).

In support of above proof, we now provide numerical example:

Example 1: Let $\mathcal{A} = [1, 25] \subseteq B = R$ and $\mathscr{F}$: $\mathcal{A} \to \mathcal{A}$ be defined in by $\mathcal{F}(\varphi) = (\varphi + 24)^{\frac{1}{3}}$.

For all $\varphi \epsilon \mathcal{A}$. Choose $a_n = b_n = c_n = 0.5$ for each $n \epsilon N$ with initial guess $\varphi_1 = 20$.

Here, $\mathscr{F}$ is a contraction map with $Fix(\mathcal{F}) = \{3\}$. Table 1 shows the comparison of the convergence rates of the processes (10), (2), (3), (4) and (5), (9).

Table 1

Comparison of Convergence Rate

Step	New Process	Picard	Mann	Ishikawa	Picard-Mann	Picard-Noor
1	20.000000000000000	20.000000000000000	20.000000000000000	20.000000000000000	20.000000000000000	20.000000000000000
2	3.530348335326063	3.530348335326063	11.765174167663032	11.647366078775978	3.294732157551954	3.291054909357043
3	3.019515306601869	3.019515306601869	7.529953162607493	7.402258745997276	3.005648777148326	3.005481428702942
4	3.000722615061918	3.000722615061918	5.344566837886251	5.242191079888928	3.000108477124344	3.000103420107380
5	3.000026763282053	3.000026763282053	4.214501907950995	4.142303865930707	3.000002083235410	3.000001951333421
6	3.000000991232341	3.000000991232341	3.629412689181518	3.582035992828175	3.000000040007263	3.000000036817836
7	3.000000036712308	3.000000036712308	3.326272718554686	3.296584717587478	3.000000000768315	3.000000000694680
8	3.000000001359715	3.000000001359715	3.169154270863968	3.151134464421982	3.000000000014755	3.000000000013107
9	3.000000000050360	3.000000000050360	3.087703102955502	3.077016956785218	3.000000000000283	3.000000000000247
10	3.000000000001865	3.000000000001865	3.045473927645047	3.039247620859340	3.000000000000005	3.000000000000004
11	3.000000000000069	3.000000000000069	3.023578601269677	3.020000572572700	3.000000000000000	3.000000000000000
12	3.000000000000000	3.000000000000002	3.012225814357283	3.010192309404176	3.000000000000000	3.000000000000000
13	3.000000000000000	3.000000000000000	3.006339276984310	3.005194016417845	3.000000000000000	3.000000000000000
14	3.000000000000000	3.000000000000000	3.003287023324011	3.002646880393915	3.000000000000000	3.000000000000000
15	3.000000000000000	3.000000000000000	3.001704379994299	3.001348855597787	3.000000000000000	3.000000000000000

Above proved theorems and provided numerical example shows that our process is faster than (2), (3), (4), (5), (6), (9). So, there is some scope of further improvement in this area using some iterative of function.

Conclusion

This paper presents a novel Modified Picard-Krasnoselskii iterative algorithm PK_m with Optimized Convergence rate and considers the algorithm under the Banach space fixed-point theory. The algorithm enhances the convergence rate with an increase in the classical Picard -Krasnoselskii scheme by adding an extra modification parameter which preserves the simplicity and stability of the currently existing hybrid schemes. Convergence analysis shows that the sequence that occurs is convergent to the unique single-point contraction mapping with high probability, and algorithm is insensitive to perturbation of the data which proves the policy of the work. Comparative studies using the acceleration criterion of Berinde [1] show that the proposed scheme is able to arrive at solutions faster than many known iterative algorithms, such as Picard, Mann, Ishikawa, Noor, Picard-Mann and Picard-Noor. Experiments involving numbers prove the fact that the new algorithm has a better convergence rate and is more computationally efficient. The Modified Picard-Krasnoselskii hybrid algorithm is an important implementation innovation in solve designs of fixed-point problems. Its increased convergence and stability and smooth computation make it a useful aid to the solution of nonlinear equations in applied mathematics, such as optimization, variational inequalities, and other computational aspects of application. The algorithmic architecture also has extensions to previous include larger groups of operators, including nonexpansive or pseudo-contractive mappings, and use of realistic numerical models. The extensions have a bright potential of future research.

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