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Unique fixed point results in 2-metric spaces with $\ddot{\alpha}$ - $\mathfrak{F}$ admissible mappings

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Abstract

Fixed point theory has developed as a significant area of nonlinear analysis because of its wide applications in both theoretical and applied mathematics. In this paper, we study $\ddot{\alpha}$-$\mathfrak{F}$ contractive mapping in the setting of complete 2-metric spaces. By employing admissibility conditions Together with suitable properties of the control function F, several f.p results are established for the considered mapping.

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1. Introduction

Fixed point methods are frequently used to determine the existence and uniqueness of solutions to numerous nonlinear problems. Banach [1] contraction principle, which guarantees the presence of a unique f.p for contraction mappings specified on complete metric spaces, served as the foundation for this theory. Due to its many applications in differential equations, mathematics, engineering, optimization theory, and economics, f.p theory has emerged as one of the most significant subfields of nonlinear analysis.

In order to obtain more broad f.p solutions, a number of mathematicians expanded the contraction principle in several directions following Banach [1] groundbreaking work. Specifically, generalized sloppy contractive conditions have been investigated in a variety of abstract spaces, including dislocated m.s, fuzzy m.s, b-m.s, and cone m.s. Among these advancements, Wardowski [9] presented the concept of $\mathfrak{F}$-contractions, which garnered significant interest because of its adaptability and suitability for nonlinear situations.

Fixed point results involving $\mathfrak{F}$ -contractive mappings under various admissibility assumptions were later studied by other scholars. The authors [8] proposed the notion of $\ddot{\alpha}$ -admissible mappings, which offered a useful framework for proving f.p theorems under weaker hypotheses. Many intriguing expansions in generalized metric structures have resulted from combining admissibility criteria with generalized $\ddot{\alpha} - \mathfrak{F} -$ contractions [3].

Gahler [4] proposed the notion of a 2-m.s. as a natural generalization of metric spaces where the distance function depends on three variables rather than two. Since its inception, numerous authors have used various contractive conditions to study f.p and common f.p theorems in 2-m.s. Rational contractions, integral-type contractions in complete 2-m.s., and generalized contractions using Hardy-Rogers type mappings have all been thoroughly studied in recent years [2].

Additionally, generalized f.p methods have been used to a number of real-world issues, including as dynamical systems, boundary value problems, and image compression. [5, 6, 7] The author proved fixed point theorems in 2-metric spaces.

Definition 1.1: Let $\mathfrak{Q} = \emptyset$ and let $\wp: \mathfrak{Q} \times \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}$ be a mapping that meets the following requirements:

1. For every pair of distinct elements $\breve{\Omega}, \widetilde{\Upsilon} \in \mathfrak{Q}, \exists\, \widehat{\mathfrak{H}} \in \mathfrak{Q} \ni$
$$\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}) \neq 0.$$
2. If any two among $\breve{\Omega}, \widetilde{\Upsilon}$ and $\widehat{\Lambda}$ are equal, then
$$\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}) = 0.$$
3. $\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}) = \wp(\widetilde{\Upsilon}, \widehat{\mathfrak{H}}, \breve{\Omega}) = \wp(\widehat{\mathfrak{H}}, \breve{\Omega}, \widetilde{\Upsilon}).$
4. $\forall \breve{\Omega}, \widetilde{\Upsilon}, \widehat{\Lambda}, \omega \in \mathfrak{Q},$
$$\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}) \leq \wp(\breve{\Omega}, \widetilde{\Upsilon}, \omega) + \wp(\breve{\Omega}, \omega, \widehat{\mathfrak{H}}) + \wp(\omega, \widetilde{\Upsilon}, \widehat{\mathfrak{H}})$$
Then $\wp$ is a 2-metric on $\mathfrak{Q}$, and $(\mathfrak{Q}, \wp)$ is a 2-m.s.

Definition 1.2: Let

$$\Xi: \mathfrak{Q} \rightarrow \mathfrak{Q}$$

be a self mapping and $(\mathfrak{Q}, \wp)$ be a 2-m.s. The mapping Ξ is reffered to as $\mathfrak{F}-$ contractions, if $\exists$ a constant $\grave{\tau} > 0 \ni$

$$\wp(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}}) > 0$$

implies

$$\grave{\tau} + \mathfrak{F}(\wp(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}})) \leq \mathfrak{F}(\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}))$$

for all $\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}, \in \mathfrak{Q}$, where

$$\mathfrak{F}: \mathbb{R}^+ \rightarrow \mathbb{R}$$

Satisfies the below conditions:

(1). $\mathfrak{F}$ is strictly growing.

(2). For each sequence $\{\ddot{\alpha}_n\} \subseteq \mathbb{R}^+$,
$\ddot{\alpha}_n \rightarrow 0$ if and only if $\mathfrak{F}(\ddot{\alpha}_n) \rightarrow -\infty$.

(3). $\exists\, k \in (0,1) \ni$

$$\lim_{\ddot{\alpha}_n \rightarrow 0} \ddot{\alpha}^k\, \mathfrak{F}(\ddot{\alpha}) = 0.$$

Definition 1.3: Let $(\mathfrak{Q}, \wp)$ is a 2-m.s and let

$$\ddot{\alpha}: \mathfrak{Q} \times \mathfrak{Q} \times \mathfrak{Q} \rightarrow [0, \infty)$$

be a function. A self map

$$\Xi: \mathfrak{Q} \rightarrow \mathfrak{Q}$$

is referred as $\ddot{\alpha}$- admissible if

$$\ddot{\alpha}(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}) \geq 1$$

Implies

$$\ddot{\alpha}(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}}) \geq 1$$

Foe all $\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}} \in \mathfrak{Q}$.

Definition 1.4: Let $(\mathfrak{Q}, \wp)$ is a 2- m.s and let

$$\Xi: \mathfrak{Q} \rightarrow \mathfrak{Q}$$

be a self mapping on $\mathfrak{Q}$. Then Ξ is known as $\ddot{\alpha} - \mathfrak{F}$ contraction, if $\exists$ a constant $\grave{\tau} > 0 \ni$

$$\wp(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}}) > 0$$

implies

$$\grave{\tau} + \ddot{\alpha}(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}})\mathfrak{F}(\wp(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}})) \leq \mathfrak{F}(\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}))$$

for all $\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}, \in \mathfrak{Q}$.

Theorem 1.5: *Assume $(\mathfrak{Q}, \wp)$ is a complete 2-m.s, and $\Xi: \mathfrak{Q} \rightarrow \mathfrak{Q}$ is a self mapping. Assume the following conditions hold:*

(1). Ξ is a modified extension $\ddot{\alpha}-\mathfrak{F}$ contraction, i.e.,

$$\wp(\Xi\breve{\Omega},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}})>0$$

implies

$$\grave{\tau}+\ddot{\alpha}(\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}})\mathfrak{F}(\wp(\Xi\breve{\Omega},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}}))\leq\mathfrak{F}(M(\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}}))$$

for all $\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}},\in\mathfrak{Q}$, where

$$M(\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}})=max\{\wp(\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}}),\wp(\breve{\Omega},\Xi\breve{\Omega},\widehat{\mathfrak{H}}),\wp(\widetilde{\Upsilon},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}}),\wp(\breve{\Omega},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}}),$$
$$\wp(\Xi\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}}),\frac{\wp(\breve{\Omega},\Xi\breve{\Omega},\widehat{\mathfrak{H}})+\wp(\widetilde{\Upsilon},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}})}{2},$$
$$\frac{\wp(\breve{\Omega},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}})+\wp(\Xi\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}})}{2}\}$$

(2). Ξ is $\ddot{\alpha}$- admissible if

$$\ddot{\alpha}(\breve{\Omega},\widetilde{\Upsilon},\widehat{\mathfrak{H}})\geq 1\Longrightarrow\ddot{\alpha}(\Xi\breve{\Omega},\Xi\widetilde{\Upsilon},\widehat{\mathfrak{H}})\geq 1.$$

(3). There exists $\breve{\Omega}_0\in\mathfrak{Q}\ni$

$$\ddot{\alpha}(\breve{\Omega}_0,\Xi\breve{\Omega}_0,\widehat{\mathfrak{H}})\geq 1$$

and

$$\wp(\breve{\Omega}_0,\Xi\breve{\Omega}_0,\widehat{\mathfrak{H}})\geq 1$$

for all $\widehat{\mathfrak{H}}\in\mathfrak{Q}$.

Them Ξ has a unique f.p in $\mathfrak{Q}$.

Proof: By the hypothesis, $\exists$ a point $\breve{\Omega}_0\in\mathfrak{Q}\ni$

$$\ddot{\alpha}(\breve{\Omega}_0,\Xi\breve{\Omega}_0,\widehat{\mathfrak{H}})\geq 1.$$

Now define a sequence $\{\breve{\Omega}_n\}$ by

$$\breve{\Omega}_{n+1}=\Xi\breve{\Omega}_n$$

for all $n\in\mathbb{N}_0$.

If for some $n\in\mathbb{N}$,

$$\breve{\Omega}_n=\Xi\breve{\Omega}_n$$

Then $\breve{\Omega}_n$ is a f.p of Ξ and the proof is complete.

So, assume that

$$\breve{\Omega}_n\neq\Xi\breve{\Omega}_n$$

Since $\ddot{\alpha}(\breve{\Omega}_0,\Xi\breve{\Omega}_0,\widehat{\mathfrak{H}})\geq 1$ and Ξ is $\ddot{\alpha}$- admissible we obtain

$$\ddot{\alpha}(\breve{\Omega}_0,\breve{\Omega}_1,\widehat{\mathfrak{H}})\geq 1\Longrightarrow\ddot{\alpha}(\breve{\Omega}_1,\breve{\Omega}_1,\widehat{\mathfrak{H}})\geq 1\Longrightarrow\cdots\Longrightarrow\ddot{\alpha}(\breve{\Omega}_n,\breve{\Omega}_{n+1},\widehat{\mathfrak{H}})\geq 1$$

Since $\wp(\Xi\breve{\Omega}_{n-1},\Xi\breve{\Omega}_n,\widehat{\mathfrak{H}})>0$ and Ξ is a modified generalized $\ddot{\alpha}-\mathfrak{F}$ contraction, for a certain $\grave{\tau}>0$, thus

$$\mathfrak{F}\left(\wp(\breve{\Omega}_n,\breve{\Omega}_{n+1},\widehat{\mathfrak{H}})\right)=\mathfrak{F}\left(\wp(\Xi\breve{\Omega}_{n-1},\Xi\breve{\Omega}_n,\widehat{\mathfrak{H}})\right)$$
$$\leq\grave{\tau}+\ddot{\alpha}(\breve{\Omega}_{n-1},\breve{\Omega}_n,\widehat{\mathfrak{H}})\,\mathfrak{F}\left(\wp(\Xi\breve{\Omega}_{n-1},\Xi\breve{\Omega}_n,\widehat{\mathfrak{H}})\right)$$
$$\leq\mathfrak{F}\left(M\left((\breve{\Omega}_{n-1},\breve{\Omega}_n,\widehat{\mathfrak{H}})\right)\right).$$

Since

$$\breve{\Omega}_n = \Xi\breve{\Omega}_{n-1} \text{ and } \breve{\Omega}_{n+1} = \Xi\breve{\Omega}_n$$

we get

$$M\left(\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right)\right) \leq max\{\wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right), \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\}$$

Suppose

$$max\{\wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right), \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\} = \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)$$

Then

$$\grave{\tau} + \ddot{\alpha}\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right) \mathfrak{F}\left(\wp\left(\Xi\breve{\Omega}_{n-1}, \Xi\breve{\Omega}_n, \widehat{\mathfrak{H}}\right)\right) \leq \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)$$

which is impossible.

Hence,

$$max\{\wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right), \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\} = \wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right)$$

Therefore,

$$\mathfrak{F}\left(\wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\right) \leq \mathfrak{F}\left(\wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right)\right)$$

Thus,

$$\wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right) \leq \wp\left(\breve{\Omega}_{n-1}, \breve{\Omega}_n, \widehat{\mathfrak{H}}\right).$$

Thus, the set $\{\wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\}$ forms a decreasing sequence of nonnegative real numbers.

We claim that

$$\lim_{n\to\infty} \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right) = 0.$$

Assume on the contrary that

$$\lim_{n\to\infty} \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right) = \delta > 0.$$

Then repeated use of the contraction condition yields

$$\mathfrak{F}(\delta) < F\left(\wp\left(\breve{\Omega}_0, \breve{\Omega}_1, \widehat{\mathfrak{H}}\right)\right) - n\grave{\tau}$$

Letting $n \to \infty$, we get

$$\mathfrak{F}(\delta) = -\infty$$

which is impossible.

Hence,

$$\lim_{n\to\infty} \wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right) = 0.$$

Next, we establish that the sequence $\{\breve{\Omega}_n\}$ is a Cauchy sequence.

Since $\exists\ k \in (0,1)$ such that

$$\lim_{\ddot{\alpha}_n\to 0^+} \ddot{\alpha}^k\, \mathfrak{F}(\ddot{\alpha}) = 0.$$

we obtain

$$n\left(\wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right)\right)^k \to 0.$$

Thus, $\exists\ n_0 \in \mathbb{N} \ni$

$$\wp\left(\breve{\Omega}_n, \breve{\Omega}_{n+1}, \widehat{\mathfrak{H}}\right) \leq \frac{1}{n^{\frac{1}{k}}}$$

for all $n \leq n_0$.

For all $m > n > n_0$,

$$\wp\left(\breve{\Omega}_n, \breve{\Omega}_m, \widehat{\mathfrak{H}}\right) \leq \sum_{j=n}^{m-1} \wp\left(\breve{\Omega}_j, \breve{\Omega}_{j+1}, \widehat{\mathfrak{H}}\right)$$
$$\leq \sum_{j=1}^{m-1} \frac{1}{j^{1/k}}$$

Since $\frac{1}{k} > 1$, the series

$$\sum_{j=1}^{\infty} \frac{1}{j^{1/k}}$$

is convergent.

Therefore,

$$\wp\left(\breve{\Omega}_n, \breve{\Omega}_m, \widehat{\mathfrak{H}}\right) \to 0 \text{ as } m, n \to \infty.$$

As a result, the sequence $\{\breve{\Omega}_n\}$ qualifies as a Cauchy sequence. Since $(\mathfrak{Q}, \wp)$ is complete, $\exists$ some $\breve{\Omega} \in \mathfrak{Q} \ni$

$$\breve{\Omega}_n \to \breve{\Omega}.$$

Now, we show that $\breve{\Omega}$ is a f.p of Ξ.

Since $\breve{\Omega}_n \to \breve{\Omega}$ and

$$\breve{\Omega}_{n+1} = \Xi\breve{\Omega}_n$$

by continuity of Ξ, we obtain

$$\Xi\breve{\Omega}_n \to \Xi\breve{\Omega}.$$

Taking limits

$$\breve{\Omega}_{n+1} = \Xi\breve{\Omega}_n$$

we get

$$\breve{\Omega} = \Xi\breve{\Omega}$$

Hence $\breve{\Omega}$ is a f.p of Ξ.

To prove uniqueness, assume that $\breve{\Omega}$ and $\widetilde{\Upsilon}$ are two distinct f.p of Ξ.

Then

$$\dot{\tau} + \ddot{\alpha}(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}})\mathfrak{F}(\wp(\Xi\breve{\Omega}, \Xi\widetilde{\Upsilon}, \widehat{\mathfrak{H}})) \leq \mathfrak{F}(\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}))$$

Since

$$\breve{\Omega} = \Xi\breve{\Omega} \text{ and } \widetilde{\Upsilon} = \Xi\widetilde{\Upsilon}$$

we obtain

$$\dot{\tau} + \ddot{\alpha}(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}})\mathfrak{F}(\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}})) \leq \mathfrak{F}(\wp(\breve{\Omega}, \widetilde{\Upsilon}, \widehat{\mathfrak{H}}))$$

which is impossible because $\dot{\tau} > 0$.

Hence the fixed point is unique.

2. Conclusion

In this paper, a new class of modified $\ddot{\alpha} - \mathfrak{F}$ contractions has been investigated in the framework of complete 2-m.s. Under suitable admissibility conditions, a f.p existence and uniqueness theorem has been established. The proof is based on the construction of an iterative sequence and the verification of its convergence in the complete 2-m.s.

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