

Stability analysis of periodic traveling waves in the diffusive CDIMA model

A. K. M. Nazimuddin *

Department of Mathematical and Physical Sciences

East West University

Dhaka 1212

Bangladesh

Md. Munna Hasan [†]

M. Showkat Akber [§]

Department of Mathematics

Jahangirnagar University

Savar

Dhaka 1342

Bangladesh

M. Osman Gani [‡]

Department of Mathematics

Jahangirnagar University

Savar

Dhaka 1342

Bangladesh

and

Meiji Institute for Advanced Study of Mathematical Sciences

Meiji University

4-21-1 Nakano

Nakano-ku

Tokyo 164-8525

Japan

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

* E-mail: nazimuddin@ewubd.edu (Corresponding Author)

† E-mail: mh.ju.math@gmail.com

§ E-mail: showkatakbers@gmail.com

‡ E-mail: osmangani2@juniv.edu

Abstract

Chemical interactions, heart arrhythmias, and morphogenesis processes are intimately associated with the development of spiral form and thereafter spiral instability. Periodic traveling waves are a possible feature of a dynamical system that can help explain spiral instability since from the core of regular spinning spirals, these kinds of waves can be released. This study deals with the persistence and stability analysis of periodic traveling waves in the diffusive chlorine dioxide-iodine-malonic acid model analytically and numerically. In the diffusive model, the emergence and uniqueness of the solution are examined. Additionally, we demonstrate that the diffusive model has a Hopf bifurcation, leading to periodic oscillating traveling waves. We calculate the stability threshold at which the periodic traveling waves become unstable due to the Eckhaus bifurcation. We investigate the spiral wave instability phenomena by using the numerical bifurcation analysis, where the spiral waves lead to far-field breakdown due to the Eckhaus instability of periodic traveling waves.

Subject Classification (2020): 35K57, 35Q92, 35C07, 92E20, 37M05.

Keywords: CDIMA model, Wave spiral, Periodic traveling waves, Bifurcation analysis, Hopf bifurcation.

1. Introduction

The periodic traveling wave (PTW) behavior was initially investigated with the oscillatory reaction-diffusion (R-D) system [1] and is a crucial one-dimensional result for many partial differential equations (PDEs). Over time, it emerged as a significant area of research for excitable systems. Additionally, PTWs have been documented in the fields of chemistry, biology, physical systems, and ecological systems [2–14]. Because periodic traveling waves may be produced from the core of regular spinning spirals, they are closely linked [15]. Biology [16–18] and chemistry [16, 19, 20] have regular rotating spiral patterns. The R-D model and chemical reactions are closely intertwined [21–23]. The chemical dynamics were studied using revolving spiral oscillations [16, 24]. In excitable systems, spinning spiral waves can fragment through core and far-field disintegration, producing a disorganized wave framework [25–27]. The stability analysis of a dynamical model may now be useful in practical settings, and a lot of dynamical models that are connected to stability analysis have been studied lately [28, 29]. An investigation of spiral disintegration events in the chemical Brusselator R-D model was recently conducted by Nazimuddin, Kabir, and Gani [30, 31]. Zhabotinsky in 1964 and Belousov in 1958 conducted the initial research on the oscillatory type BZ reactions [32, 33]. Chemical oscillations can have a very complicated behavior. There are over twenty reaction steps in the original BZ process. For additional information about BZ reaction, see [34, 35]. Similar to this, Lengyel, Rabai, and Epstein [36] looked at another very simple oscillatory chemical reaction involving malonic acid, iodine, and chlorine dioxide. An important component of oscillating reactions is chloride dioxide and chloride-related chemical oscillations. These studies focused on nonlinear characteristics, including processes, bio-stability,

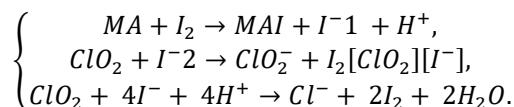
oscillations, and space phenomena [37]. The chlorite-iodide-malonic acid (CDIMA) oscillation system was studied in an open or batch environment [38–40]. Lengyel, Rabai, and Epstein [41, 42] investigated and produced a model of the reaction mechanism consisting of three functional reactions for the malonic acid, iodine, and chlorine dioxide oscillating reaction. Lengyel and Epstein [43] and Lengyel, Kadar, and Epstein [44] studied the CDIMA-starch reaction system using Turing structure simulation. Riaz and Ray [45] researched the CDIMA R-D model using a multiple scale perturbative analysis and shown that spiral solutions are admissible. Spiral wave breathing in the CDIMA R-D system under global periodic forcing has been demonstrated by Berenstein et al. [46].

Using the continuation technique and WAVETRAIN, the study investigates the role of PTW destabilization in spiral destabilization in the diffusive CDIMA system, exposing a unique solution and a Hopf bifurcation in the model [47]. In accordance to the model, waves in PTWs break into an oscillating wave structure when they experience an Eckhaus-type stability modification, which results in the loss of stability. Direct numerical simulations are performed in one and two dimensions. The degree of agreement between these studies and the continuation result is rather favorable.

This article uses diffusive CDIMA system, which is illustrated in 2. Section 3 discusses the diffusive model's development and uniqueness of solution. Section 4 investigates Hopf bifurcation persistence in the diffusive model. The stability and persistence results for PTWs are illustrated in Subsection 5.1. The findings of direct PDE simulation outcomes are presented in Subsection 5.2. To wrap up our investigation, we go over the explanation of the results in section 6.

2. Diffusive Chlorine Dioxide-Iodine-Malonic Acid System

An exquisite model of an oscillating CDIMA reaction was characterized and investigated by Lengyel, Rabai, and Epstein [36, 41, 42]. This experiment demonstrates that the following reactions and experimental rate equations are suitable for the model:



The model produces oscillations corresponding to those observed in the experiments, as illustrated by the above numerical integration. Nevertheless, the analytical handling of this model is still too challenging owing to its high level of complexity. The following form is assumed by the collection of ordinary differential equations (ODEs) when appropriate non-dimensionalities is used:

$$\begin{cases} \frac{du}{dt} = a - u - \frac{4uv}{1+u^2}, \\ \frac{dv}{dt} = bu \left(1 - \frac{v}{1+u^2}\right), \end{cases} \quad (1)$$

where I^- and ClO_2^- are the dimensionless concentrations, denoted by u and v . The parameters $a, b > 0$ are obtained by combining the experimental rate constants with the intentness given for the calm reactants [48-50]. The single equilibrium point of (1) is $(u^*, v^*) = \left(\frac{a}{5}, 1 + \left(\frac{a}{5}\right)^2\right)$. When $\frac{du}{dt} = 0$ and $\frac{dv}{dt} = 0$ are taken into consideration, fig. 1 shows the nullclines of (1).

The equation $b_{Hopf} = \left(\frac{3a}{5} - \frac{25}{a}\right)$ may be used to determine the Hopf bifurcation sites for (1). Fig. 2 denotes the stability curve at which the Hopf bifurcation point causes a change in stability.

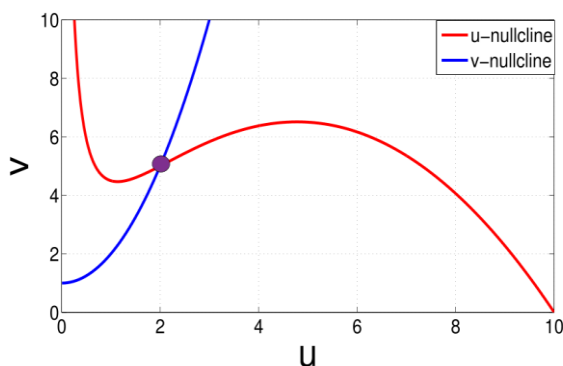


Figure 1
Nullclines for $a = 10.0$ and $b = 3.0$ in model (1).

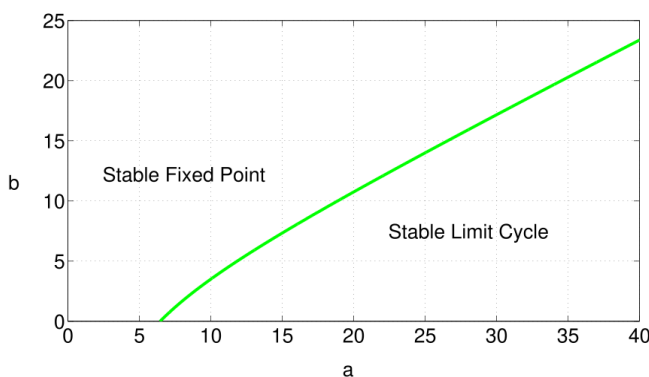


Figure 2
Hopf bifurcation curve of the model (1).

The system is unstable at $b = 2.0, 3.0$, and 4.0 , but stable at $b = 4.0$, as shown in fig. 3. The diffusive CDIMA system is obtained by combining the reaction system with diffusion terms as:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + a - u - \frac{4uv}{1+u^2}, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + bu \left(1 - \frac{v}{1+u^2}\right), \end{cases} \quad (2)$$

where $d_1, > 0$ and $d_2 > 0$, respectively, are constants that represent the activator and inhibitor diffusion coefficients [46, 51-53].

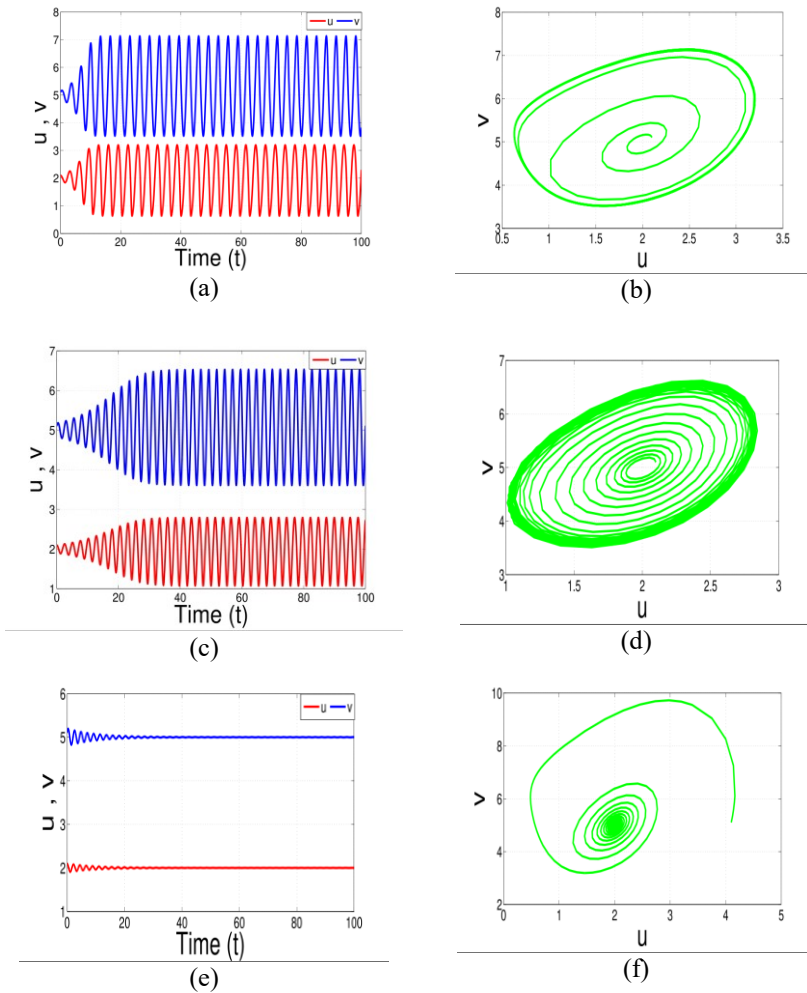


Figure 3

Limit cycle solution behavior of (1) by considering $a = 10.0$ and b is a free parameter. (a)-(b) $b = 2$, (c)-(d) $b = 3$, (e)-(f) $b = 4$.

Table 1
Typical values for the parameters of (2)

Parameters	d_1	d_2	a	b
Values	0.1	0.01	10	...

3. Solution's Presence and Uniqueness in the Diffusive Model

We will look at the diffusive CDIMA model presented in (2). Equation (2) initially produces when the Riemann-Liouville integral operator is implemented.

$$\begin{aligned} & u(x, t) - u(x, 0) \\ &= \int_0^t \left(d_1 u_{xx} + a - u - \frac{4uv}{1+u^2} \right) ds \end{aligned} \quad (3)$$

$$v(x, t) - v(x, 0) = \int_0^t \left(d_2 v_{xx} + bu \left(1 - \frac{v}{1+u^2} \right) \right) ds \quad (4)$$

Let,

$$R(t, u) = d_1 u_{xx} + a - u - \frac{4uv}{1+u^2} \quad (5)$$

and

$$S(t, v) = d_2 v_{xx} + bu \left(1 - \frac{v}{1+u^2} \right) \quad (6)$$

If the continuous functions u, u_1, v , and v_1 fall under $\Omega^2((0,1) \times (0,1))$, then $t_1 > 0$ and $t_2 > 0$ occur in such a way that

$$\begin{cases} \|(u_1)_{xx} - u_{xx}\| \leq \|u_1 - u\| t_1, \\ \|(v_1)_{xx} - v_{xx}\| \leq \|v_1 - v\| t_2. \end{cases} \quad (7)$$

Thus, the Lipschitz condition is satisfied by the sets $R(t, u)$ and $S(t, v)$.

Again,

$$\begin{aligned} \|R(t, u_1) - R(t, u)\| &= \left\| d_1 u_{xx} + a - u - \frac{4uv}{1+u^2} - d_1 (u_1)_{xx} - a + u_1 + \frac{4u_1 v_1}{1+u_1^2} \right\| \\ &\leq (4k_2 k_3 k_4 + 1 + k_1 t_1) \|u_1 - u\| \end{aligned}$$

Considering, $N_1 = k_1 t_1 + 1 + 4k_2 k_3 k_4$, we write

$$\|R(t, u_1) - R(t, u)\| \leq \|u_1 - u\| N_1, \quad (8)$$

Likewise, we may write

$$\|S(t, v_1) - S(t, v)\| \leq N_2 \|v_1 - v\|, \quad (9)$$

when $N_2 = l_1 t_2 + l_2 l_3$.

Thus, $R(t, u)$ and $S(t, v)$ are kernels that satisfy the Lipschitz criterion.

4. Hopf Bifurcation's Emergence in the Diffusive CDIMA Model

The traveling wave coordinate is defined by the equation $\zeta = \chi - ct$, based on our assumption. By adjusting $M(\zeta) = u(\chi, \tau)$ and $N(\zeta) = v(\chi, \tau)$ in equation (2), we obtain

$$\begin{cases} d_1 u_{\zeta\zeta} + cu_{\zeta} + a - u - \frac{4uv}{1+u^2}, \\ d_2 u_{\zeta\zeta} + cv_{\zeta} + bu \left(1 - \frac{v}{1+u^2}\right). \end{cases} \quad (10)$$

One way to rewrite system (10) is as the following set of ODEs:

$$\begin{cases} \frac{du}{d\zeta} = p, \\ \frac{dp}{d\zeta} = \frac{1}{d_1} \left(-cp - a + u + \frac{4uv}{1+u^2} \right), \\ \frac{dv}{d\zeta} = q, \\ \frac{dq}{d\zeta} = \frac{1}{d_2} \left(-cq - bu \left(1 - \frac{v}{1+u^2}\right) \right). \end{cases} \quad (11)$$

Considering $u^* = \frac{a}{5}$ and $v^* = 1 + \left(\frac{a}{5}\right)^2$, we obtain only a fixed point, namely $E^* = (u^*, 0, v^*, 0)$ in the system (11).

The following characteristic matrix occurs in the linearized version of (11) at equilibrium point E^* as:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1+u^{*4}+u^{*2}(2-4v^*)+4v^*}{d_1(1+u^{*2})^2} & -\frac{c}{d_1} & \frac{4u^*}{d_1+d_1u^{*2}} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b(1+u^{*4}-v^*+u^{*2}(2+v^*))}{d_2(1+u^{*2})^2} & 0 & \frac{bu^*}{d_1+d_2u^{*2}} & -\frac{c}{d_2} \end{pmatrix} \quad (12)$$

The related characteristic matrix can be expressed in the explicit manner shown below.

$$A(\omega, b) = \omega^4 + \sigma_3 \omega^3 + \sigma_2 \omega^2 + \sigma_1 \omega + \sigma_0, \quad (13)$$

where

$$\begin{aligned} \sigma_3 &= \frac{c(d_1+d_2)}{d_1 d_2}, \\ \sigma_2 &= \frac{c^2(25+a^2)-5abd_1-125d_2+3a^2d_2}{(25+a^2)d_1 d_2}, \\ \sigma_1 &= \frac{c(3a^2-5ab-125)}{(25+a^2)d_1 d_2}, \\ \sigma_0 &= \frac{25ab}{(25+a^2)d_1 d_2}. \end{aligned}$$

Utilizing the Routh-Hurwitz criteria, the stability of (11) may be stated as follows:

$$\Re(\omega) < 0 \Leftrightarrow \begin{cases} \sigma_0 > 0, \\ \sigma_1 > 0, \\ \sigma_3 > 0, \\ \sigma_1 \sigma_2 \sigma_3 > \sigma_1^2 + \sigma_0 \sigma_3^2. \end{cases} \quad (14)$$

The following simplified form is implied by system (14):

$$\Re(\omega) < 0 \Leftrightarrow$$

$$\left\{ \begin{array}{l} \frac{25ab}{(25+a^2)d_1d_2} > 0, \\ \frac{c(3a^2-5ab-125)}{(25+a^2)d_1d_2} > 0, \\ \frac{c(d_1+d_2)}{d_1d_2} > 0, \\ \frac{c^2}{(25+a^2)^2d_1^3d_2^3} (5a^2b(-8a+5b)d_1^2 - \\ 50a(25+a^2)bd_1d_2 + ((125-3a^2)^2 - \\ 40a^3b)d_2^2 + (25+a^2)(-125+3a^2-5ab) \\ c^2(d_1+d_2) > 0. \end{array} \right. \quad (15)$$

It is note that Hopf instability can be found by setting $\Re(\omega) = 0$, The set of all Hopf bifurcation points are indicated by the equations:

$$\Gamma_2 b^2 - \Gamma_1 b - \Gamma_0 = 0,$$

where

$$\begin{aligned} \Gamma_2 &= \frac{25a^2c^2}{(25+a^2)^2d_1d_2^3}, \\ \Gamma_1 &= -\frac{5ac^2(250d_1d_2+(25+a^2)c^2(d_1+d_2)+2a^2(4d_1^2+5d_1d_2+4d_2^2))}{(25+a^2)^2d_1^3d_2^3}, \\ \Gamma_0 &= \frac{c^2(-125+3a^2)((-125+3a^2)d_2^2+(25+a^2)c^2(d_1+d_2))}{(25+a^2)^2d_1^3d_2^3}. \end{aligned}$$

Theorem 4.1: If $d_1 + d_2 > 0$ & $k \neq 0$, the following requirement must be met by the equilibrium point of (11).

$$b < \frac{3a^2-125}{5a}$$

Proof: Let $\omega = ik$ be the form of the zeros in equation (13) as follows. Thus, we obtain (13),

$$A(ik, b) = \sigma_0 + \sigma_1 ik + \sigma_2 (ik)^2 + \sigma_3 (ik)^3 + (ik)^4 = 0.$$

After dividing the preceding equation's real and imaginary components, we obtain

$$k^4 - \sigma_2 k^2 + \sigma_0 = 0 \quad (16)$$

and

$$-\sigma_3 k^3 + \sigma_1 k = 0 \quad (17)$$

Now, from equation (17), we get

$$k^2 = \frac{3a^2-5ab-125}{(25+a^2)(d_1+d_2)} > 0$$

The inequality must thus be satisfied by the roots of (13) $b < \frac{3a^2-125}{5a}$ for $d_1 + d_2 > 0$.

Theorem 4.2: *There exists a constant $\rho_0 > 0$ when $c > 0$. Now, for each parameter value $b = b(\rho)$, The set of periodic solutions $l_n(\rho)$ for equation (11) with period $T_n = T_n(\rho)$.*

Proof: Verification of the transversality criterion is required. Thus, using equation (13), we obtain

$$A(ik, b_0) = 0 \ \& \ \frac{\partial}{\partial \omega} A(ik, b_0) \neq 0$$

For b in the region of b_0 , $A(\omega(b), b) = 0$, because the function A is continuous according to the implicit function theorem.

$$\omega'(b)|_{b=b_0} = \frac{5a(-5 + ick - d_1 k^2)p_0}{-5ab(c + 2id_1 k)p_0 + cp_0 p_1 + ik(-4k^2 + 2p_0(d_2 p_1 + p_2) + 3ikp_3)}$$

where

$$\begin{aligned} p_0 &= \frac{1}{(25 + a^2)d_1 d_2}, \\ p_1 &= -125 + 3a^2, \\ p_2 &= (25 + a^2)c^2, \\ p_3 &= \frac{c(d_1 + d_2)}{d_1 d_2}. \end{aligned}$$

Therefore,

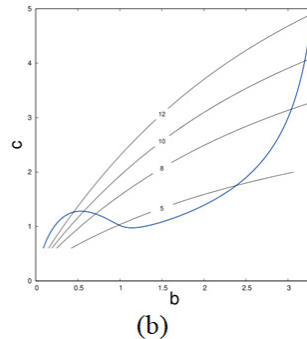
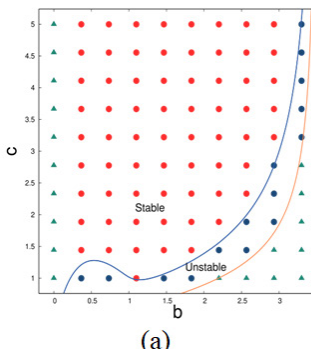
$$\begin{aligned} \Re(\omega'(b)|_{b=b_0}) &= \frac{1}{\mu} (-20ack^4 p_0 - 25a^2 b c d_1 k^2 p_0^2 - 25ac p_0^2 p_1 \\ &\quad - 5acd_1 d_2 k^2 p_0^2 p_1 + 125a^2 b c p_0^2 + 10ac p_0^2 p_1 d_2 k^2 \\ &\quad + 10ac p_0^2 k^2 p_2 + 75ap_0 k^2 p_3 + 15ap_0 p_3 d_1 d_2 k^4), \end{aligned}$$

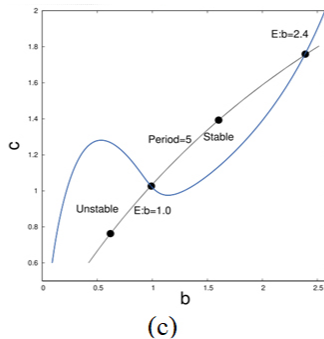
where

$$\begin{aligned} \mu &= \left(-10abd_1 d_2 k p_0 + k(-4k^2 + 2p_0(d_2 p_1 + p_2)) \right)^2 \\ &\quad + (-5abc p_0 + cp_0 p_1 3k^2 p_3)^2 > 0. \end{aligned}$$

We obtain, $\Re(\omega'(b)|_{b=b_0}) > 0$.

To get the result, ordinary differential equations are subjected to the Hopf bifurcation theorem [54, 55].



**Figure 4**

(a) Stability outcome of (2), (b) some PTW branches,
(c) single PTW branch in close-up view

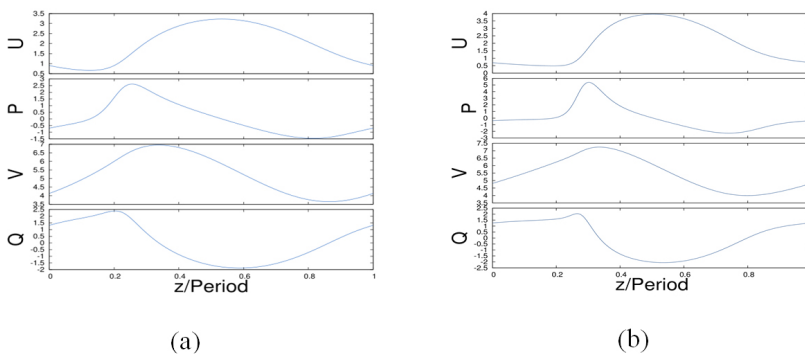
5. Numerical Investigation of the Diffusive CDIMA Model

We use several numerical techniques for diffusive CDIMA system analysis discussed in this part to validate these theoretical estimates. Subsection 5.2 provides the numerical strategies of (2), whereas subsection 5.1 shows the consistency and persistence of PTWs in (2).

5.1 Numerical Investigation through the Method of Continuation

After determining if PTWs exist, we apply WAVETRAIN [47], which makes use of AUTO [56], to analyze the stability of (2). The computations are predicated on research by [57,58]. Considering the appropriate parameter values stated in Table 1, the stability outcome of (2) are shown in fig. 4(a), which uses 10×10 grid elements.

With symbols denoting stable, unstable, or no PTWs, the Eckhaus-class stability change is determined. Some PTW branches are depicted in fig. 4(b). Also, fig. 4(c) shows the stability variations for $b = 1.0$ and $b = 2.4$.

**Figure 5**

PTWs of (2); (a) a durable graph of PTWs, and (b) an inconsistent graph of PTWs.

Fig. 5(a) displays a stable profile of PTWs with $b = 1.603$, $c = 1.393$, and a period of 5, whereas A single branch of PTWs verification parts is depicted in fig. 5(b), which displays an unstable wave solution profile with $b = 0.620$, $c = 0.763$, and a period of 5.

At $b = 3.0$, the dispersion graph of the system (2) is depicted in fig. 6(a). We discover that consistency evolve of the Eckhaus class occurs at $c = 3.0$ when $b = 3.0$. Additionally, we determine that for $b = 3.0$ and $c = 3.0$, the minimum stable period is 7.65 (roughly). We validate the outcome shown in fig. 6(a) by utilizing the relevant spectra displayed in figs. 6(b) and 6(c).

The dispersion graph of system (2) is shown in fig. 7(a) at $c = 1.25$, where we take into account the period and b . The remaining values of the parameters match those in fig. 4(a). We discover that the consistency evolve of the Eckhaus class occurs at $b = 0.56$ for $c = 1.25$. The consistent and inconsistent nature of the eigenvalue spectrum is depicted in fig. 7(b) and fig. 7(c).

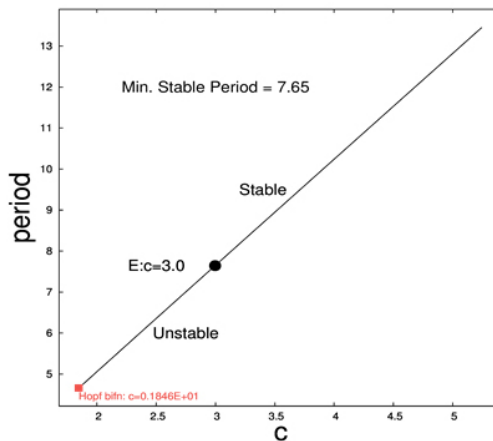
5.2 Results of Direct Numerical Simulation

The diffusive model is analyzed in the paper using certain numerical techniques, such as an implicit method for one-dimensional analysis and the ADI approach for two-dimensional analysis [59].

5.2.1 One-Space Dimensional Analysis

In order to validate a dispersal representation in fig. 6(a), we perform one-dimensional numerical simulations on (2) by utilizing periodic boundary conditions. The following may be used to express the model (2):

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + a - u - \frac{4uv}{1+u^2}, t > 0, x \in \Lambda, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + bu \left(1 - \frac{v}{1+u^2}\right), t > 0, x \in \Lambda. \end{cases}$$



(a)

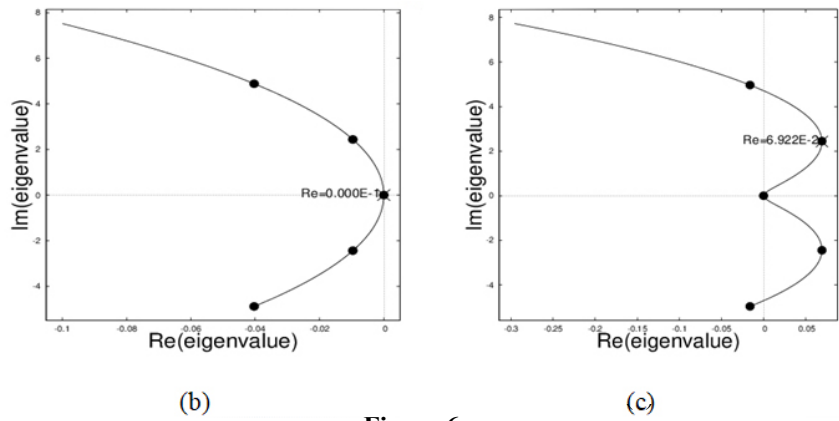
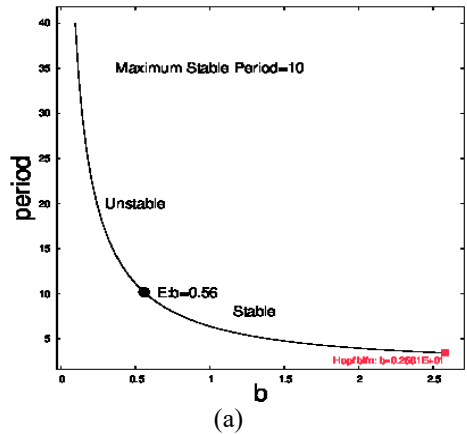


Figure 6

(a) Dispersion curve at $b = 3.0$, (b) Nature of eigenvalues of a durable structure, (c) Nature of eigenvalues of an inconsistent structure.



(a)

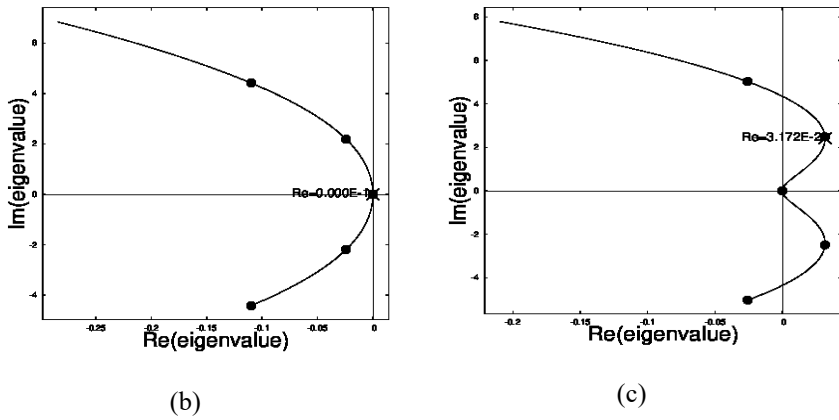
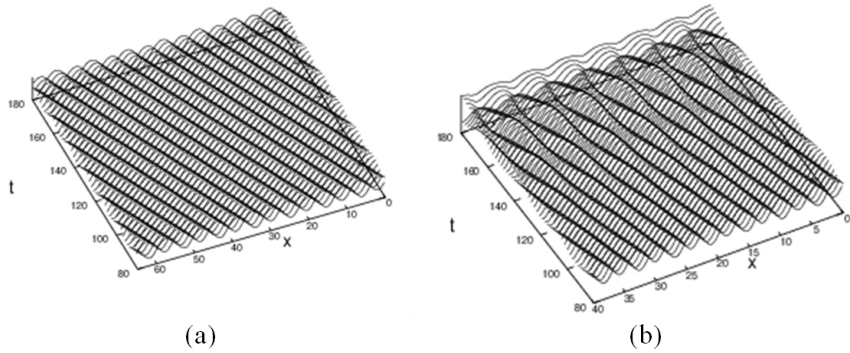


Figure 7

(a) Dispersion curve at $c = 1.25$, (b) Nature of eigenvalues of a durable structure, (c) Nature of eigenvalues of an inconsistent structure.

**Figure 8**

Direct PDE simulation in the single dimension; (a) a durable PTWs structure; (b) an inconsistent PTWs structure.

Using the formula $\Phi_x = n \times p$, we examine a space domain $[0, \Phi_x]$, with p and n denoting the duration and number of periodic pulses, correspondingly. 1000×1000 grid components, 0.01 time step, and 0.065 space step were employed in the simulation.

At the bifurcation point, we find a minimum stable period $p = 7.65$ roughly in fig. 6(a). For a system size of 65 with seven periodic pulses, we have selected $p_1 = 9.29$ as the spatial period. Throughout the time span $80 < t < 180$, the numerical simulation of (2) is conducted using the previously given parameters. The result is a stable PTWs for (2), as shown in fig. 8(a). With seven periodic pulses and a spatial period of 5.71, the system has a size of 40. 80–180 time periods are used for simulation. The conversion of pulses into oscillating pulses is depicted in fig. 8(b), which shows that numerical and continuation results agree well.

5.2.2 Two-Space Dimensional Analysis

We conducted two-dimensional simulations in the square region of $\Gamma = (0,100) \times (0,100)$. Presently we investigate the following:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + a - u - \frac{4uv}{1+u^2}, t > 0, (x,y) \in \Gamma, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + bu \left(1 - \frac{v}{1+u^2}\right), t > 0, (x,y) \in \Gamma. \end{cases}$$

Using Neumann boundary conditions, we build an IBVP for model (2). Moreover, PBCs may also be applied here as we have demonstrated the development of PTWs in the 1D event. The grid elements are 1000×1000 , the time interval is 0.01 and the step size is set to 0.1.

The simulation indicates that stability evolves at $b = 0.56$. For $b = 1.0$ and 0.8, a stable spiral pattern is produced, as shown in figs. 9(a) and 9(b); nevertheless, the excited state deepens as parameter b lowers. Instability of

spirals are shown in fig. 9(c) when $b = 0.5$ is selected close to the stability barrier, and more instability of spirals are shown in fig. 9(d) when $b = 0.4$. According to our observations, PTW oscillations cause the stable spiral pattern to breakup. The self-sustaining oscillation type known as the CDIMA model is dependent on the system's characteristics but independent of the starting conditions. Numerical simulation confirms that the spiral development and breakup are independent of the initial condition. As a result, we are able to derive a positive conclusion from both the direct simulation and the outcome obtained in section 5.1 for the system (2).

At $b = 0.4$, Figure 10 shows the spiral breakdown of model (2), using all parameter values taken from Table 1. Time $t = 0$ is the initial condition, as shown in fig. 10(a). Fig. 10(b-d) depicts the spiral pattern breakdown scenario.

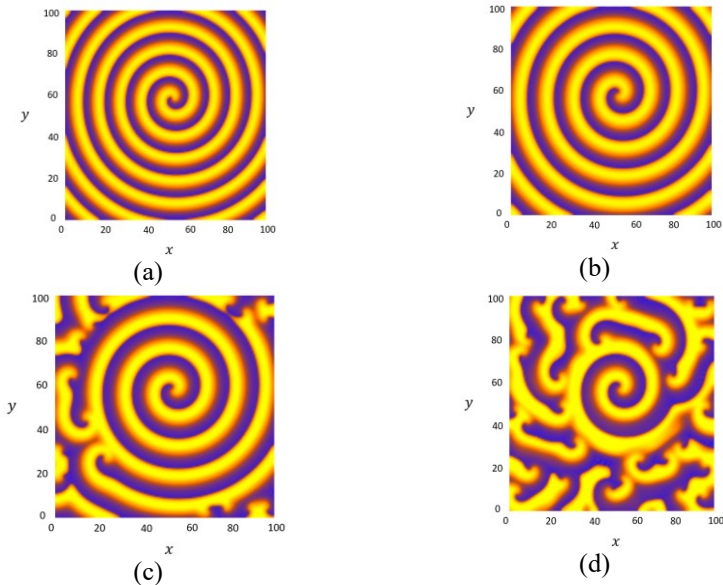
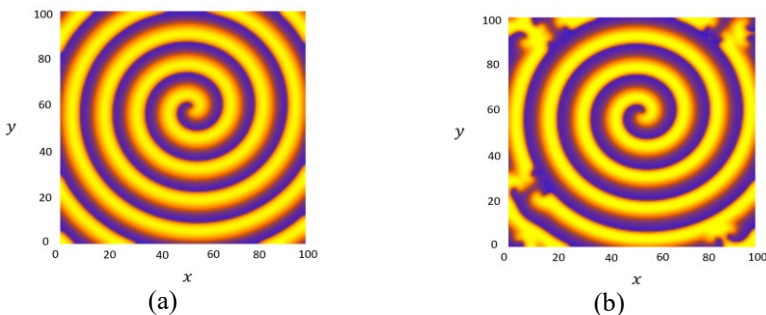
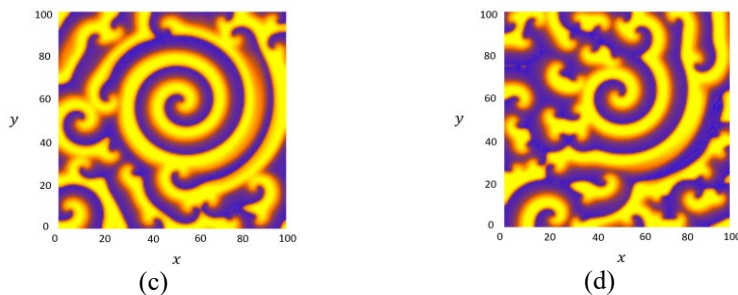


Figure 9

A spiral breakdown issues of (2), where (a) $b = 1.0$, (b) $b = 0.8$, (c) $b = 0.5$, and (d) $b = 0.4$.



**Figure 10**

A time sequence at $b = 4.0$ illustrates the spiral breakdown issues of (2); (a) $t = 0$, (b) $t = 78$, (c) $t = 126$, and (d) $t = 234$.

6. Conclusion

The spiral wave instability of a non-linear diffusive CDIMA system was examined in this research. PTWs were shown to be present and stable in the diffusive CDIMA model both analytically and numerically. We demonstrated that in the diffusive CDIMA model that gives rise to the PTWs, Hopf bifurcation emerges. The study shows that Eckhaus instability causes PTWs to lose stability. A revolving spiral progressively disintegrates, revealing the far-field disintegration property of a spiral through two-space dimensional analysis. The numerical simulation findings of the diffusive CDIMA model accurately describe the stability of PTWs, showing a slow breakdown of the spiral.

Core breakage of a stable rotating spiral occurs in chemical reaction models and excitable models [60–63]. In oscillating biochemical frameworks, far-field disintegration of the spiral has been observed [64, 65]. Researchers discovered that the Eckhaus-type instability of PTWs results in a far-field disintegration phenomenon [30,31,66].

References

- [1] N. Kopell and L. N. Howard, "Plane wave solutions to reaction-diffusion equations," *Stud. Appl. Math.*, vol. 52, no. 4, pp. 291–328 (1973).
- [2] R. L. DeVille and E. Vanden-Eijnden, "Wavetrain response of an excitable medium to local stochastic forcing," *Nonlinearity*, vol. 20, no. 1, p. 51 (2006).
- [3] J. A. Sherratt and G. J. Lord, "Nonlinear dynamics and pattern bifurcations in a model for vegetation stripes in semi-arid environments," *Theor. Popul. Biol.*, vol. 71, no. 1, pp. 1–11 (2007).
- [4] M. O. Gani and T. Ogawa, "Alternans and spiral breakup in an excitable reaction-diffusion system: A simulation study," *Int. Sch. Res. Notices*, vol. 2014, p. 459675 (2014).

- [5] V. Steinberg, J. Fineberg, E. Moses, and I. Rehberg, "Pattern selection and transition to turbulence in propagating waves," *Physica D*, vol. 37, no. 1-3, pp. 359–383 (1989).
- [6] M. Hecke, "Coherent and incoherent structures in systems described by the 1D CGL equation: Experiments and identification," *Physica D*, vol. 174, no. 1-4, pp. 134–151 (2003).
- [7] W. Van Saarloos, "Front propagation into unstable states," *Phys. Rep.*, vol. 386, no. 2-6, pp. 29–222 (2003).
- [8] I. R. Epstein and K. Showalter, "Nonlinear chemical dynamics: Oscillations, patterns, and chaos," *J. Phys. Chem.*, vol. 100, no. 31, pp. 13132–13147 (1996).
- [9] V. K. Vanag and I. R. Epstein, "Design and control of patterns in reaction-diffusion systems," *Chaos*, vol. 18, no. 2 (2008).
- [10] G. Bordyugov, N. Fischer, H. Engel, N. Manz, and O. Steinbock, "Anomalous dispersion in the Belousov-Zhabotinsky reaction: Experiments and modeling," *Physica D*, vol. 239, no. 11, pp. 766–775 (2010).
- [11] E. Ranta and V. Kaitala, "Travelling waves in vole population dynamics," *Nature*, vol. 390, no. 6659, p. 456 (1997).
- [12] S. M. Bierman, J. P. Fairbairn, S. J. Petty, D. A. Elston, D. Tidhar, and X. Lambin, "Changes over time in the spatiotemporal dynamics of cyclic populations of field voles (*Microtus agrestis* L.)," *Amer. Nat.*, vol. 167, no. 4, pp. 583–590 (2006).
- [13] J. A. Sherratt and M. J. Smith, "Periodic travelling waves in cyclic populations: Field studies and reaction-diffusion models," *J. R. Soc. Interface*, vol. 5, no. 22, pp. 483–505 (2008).
- [14] A. Panlov, R. Keldermann, and M. Nash, "Drift and breakup of spiral waves in reaction-diffusion-mechanics systems," *Proc. Natl. Acad. Sci. USA*, vol. 104, no. 19, pp. 7922–7926 (2007).
- [15] M. Bar and L. Brusch, "Breakup of spiral waves caused by radial dynamics: Eckhaus and finite wave number instabilities," *New J. Phys.*, vol. 6, no. 1, p. 5 (2004).
- [16] A. T. Winfree and S. H. Strogatz, "Organizing centres for three-dimensional chemical waves," *Nature*, vol. 311, no. 5987, pp. 611–615 (1984).
- [17] C. J. Weijer, "Dictyostelium morphogenesis," *Curr. Opin. Genet. Dev.*, vol. 14, no. 4, pp. 392–398 (2004).

- [18] J. Lechleiter, S. Girard, E. Peralta, and D. Clapham, "Spiral calcium wave propagation and annihilation in *Xenopus laevis* oocytes," *Science*, vol. 252, no. 5002, pp. 123–126 (1991).
- [19] A. Zaikin and A. Zhabotinsky, "Concentration wave propagation in two-dimensional liquid-phase self-oscillating system," *Nature*, vol. 225, no. 5232, pp. 535–537 (1970).
- [20] R. Imbihl and G. Ertl, "Oscillatory kinetics in heterogeneous catalysis," *Chem. Rev.*, vol. 95, no. 3, pp. 697–733 (1995).
- [21] A. Karma, "Physics of cardiac arrhythmogenesis," *Annu. Rev. Condens. Matter Phys.*, vol. 4, no. 1, pp. 313–337 (2013).
- [22] Z. Qu, G. Hu, A. Garfinkel, and J. N. Weiss, "Nonlinear and stochastic dynamics in the heart," *Phys. Rep.*, vol. 543, no. 2, pp. 61–162 (2014).
- [23] S. Alonso, M. Bar, and B. Echebarria, "Nonlinear physics of electrical wave propagation in the heart: A review," *Rep. Prog. Phys.*, vol. 79, no. 9, p. 096601 (2016).
- [24] J. M. Davidenko, A. V. Pertsov, R. Salomonsz, W. Baxter, and J. Jalife, "Stationary and drifting spiral waves of excitation in isolated cardiac muscle," *Nature*, vol. 355, no. 6358, pp. 349–351 (1992).
- [25] J. Nagumo, S. Arimoto, and S. Yoshizawa, "An active pulse transmission line simulating nerve axon," *Proc. IRE*, vol. 50, no. 10, pp. 2061–2070 (1962).
- [26] M. Gerhardt, H. Schuster, and J. J. Tyson, "A cellular automaton model of excitable media including curvature and dispersion," *Science*, vol. 247, no. 4950, pp. 1563–1566 (1990).
- [27] H. Ito and L. Glass, "Spiral breakup in a new model of discrete excitable media," *Phys. Rev. Lett.*, vol. 66, no. 5, p. 671 (1991).
- [28] F. Bozkurt and A. Yousef, "Flip bifurcation and stability analysis of a fractional-order population dynamics with Allee effect," *J. Interdiscip. Math.*, vol. 22, no. 6, pp. 1009–1029 (2019).
- [29] A. Khudhur and S. L. Jasim, "Stability analysis of robust control problems in linear descriptor systems with matching conditions," *J. Interdiscip. Math.*, vol. 26, no. 7, pp. 1541–1547 (2023).
- [30] A. Nazimuddin, M. H. Kabir, and M. O. Gani, "Oscillatory wave patterns and spiral breakup in the Brusselator model using numerical bifurcation analysis," *J. Comput. Sci.*, 101720 (2022).
- [31] A. Nazimuddin, M. H. Kabir, and M. O. Gani, "Spiral patterns and numerical bifurcation analysis in a three-component Brusselator

- model for chemical reactions," *Math. Comput. Simul.*, vol. 203, pp. 577–591 (2023).
- [32] B. Belousov, "An oscillating reaction and its mechanism," *Sborn. Referat. Radiat. Med.*, vol. 145 (1959).
- [33] Q. Din, T. Donchev, and D. Kolev, "Stability, bifurcation analysis and chaos control in chlorine dioxide-iodine-malonic acid reaction," *MATCH Commun. Math. Comput. Chem.*, vol. 79, no. 3, pp. 577–606 (2018).
- [34] J. D. Murray, "Biological waves: multi-species reaction diffusion models," *Mathematical Biology*, 311–359 (1993)
- [35] D. S. Jones, M. Plank, and B. D. Sleeman, *Differential Equations and Mathematical Biology*. Boca Raton, FL: Chapman & Hall/CRC (2009).
- [36] I. Lengyel, G. Rabai, and I. R. Epstein, "Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction," *J. Amer. Chem. Soc.*, vol. 112, no. 25, pp. 9104–9110 (1990).
- [37] P. De Kepper, J. Boissonade, and I. R. Epstein, "Chlorite-iodide reaction: A versatile system for the study of nonlinear dynamic behavior," *J. Phys. Chem.*, vol. 94, no. 17, pp. 6525–6536 (1990).
- [38] P. De Kepper, I. R. Epstein, K. Kustin, and M. Orbán, "Batch oscillations and spatial wave patterns in chlorite oscillating systems," *J. Phys. Chem.*, vol. 86, pp. 170–171 (1982).
- [39] P. De Kepper and I. R. Epstein, "Mechanistic study of oscillations and bistability in the Briggs-Rauscher reaction," *J. Amer. Chem. Soc.*, vol. 104, no. 1, pp. 49–55 (1982).
- [40] P. De Kepper, J. Boissonade, and I. R. Epstein, "Chlorite-iodide reaction: A versatile system for the study of nonlinear dynamic behavior," *J. Phys. Chem.*, vol. 94, no. 17, pp. 6525–6536 (1990).
- [41] I. Lengyel, G. Rabai, and I. R. Epstein, "Batch oscillations in the reaction of chlorine dioxide with iodine and malonic acid," *J. Amer. Chem. Soc.*, vol. 112, no. 11, pp. 4606–4607 (1990).
- [42] I. Lengyel, G. Rabai, and I. R. Epstein, "Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction," *J. Amer. Chem. Soc.*, vol. 112, no. 25, pp. 9104–9110 (1990).
- [43] I. Lengyel and I. R. Epstein, "Modeling of Turing structures in the chlorite-iodide-malonic acid-starch reaction system," *Science*, vol. 251, no. 4994, pp. 650–652 (1991).

- [44] I. Lengyel, S. Kadar, and I. R. Epstein, "Transient Turing structures in a gradient-free closed system," *Science*, vol. 259, no. 5094, pp. 493–495 (1993).
- [45] S. S. Riaz and D. S. Ray, "Spiral pattern in chlorite-iodide-malonic acid reaction: A theoretical and numerical study," *J. Chem. Phys.*, vol. 123, no. 17 (2005).
- [46] I. Berenstein, L. Yang, M. Dolnik, A. M. Zhabotinsky, and I. R. Epstein, "Dynamic mechanism of photochemical induction of Turing superlattices in the chlorine dioxide-iodine-malonic acid reaction-diffusion system," *J. Phys. Chem. A*, vol. 109, no. 24, pp. 5382–5387 (2005).
- [47] J. A. Sherratt, "Numerical continuation methods for studying periodic travelling wave (wavetrain) solutions of partial differential equations," *Appl. Math. Comput.*, vol. 218, no. 9, pp. 4684–4694 (2012).
- [48] D. Mu, C. Xu, Z. Liu, and Y. Pang, "Further insight into bifurcation and hybrid control tactics of a chlorine dioxide-iodine-malonic acid chemical reaction model incorporating delays," *MATCH Commun. Math. Comput. Chem.*, vol. 89, no. 3, pp. 529–566 (2023).
- [49] L. Shi, W. Li, and F. Wang, "Experimental study of a closed system in the chlorine dioxide-iodine-malonic acid-sulfuric acid oscillation reaction by UV-Vis spectrophotometric method," *J. Solution Chem.*, vol. 38, pp. 571–588 (2009).
- [50] Q. Din, T. Donchev, and D. Kolev, "Stability, bifurcation analysis and chaos control in chlorine dioxide-iodine-malonic acid reaction," *MATCH Commun. Math. Comput. Chem.*, vol. 79, no. 3, pp. 577–606 (2018).
- [51] P. Ghosh, S. Sen, S. S. Riaz, and D. S. Ray, "Galerkin analysis of light-induced patterns in the chlorine dioxide-iodine-malonic acid reaction-diffusion system," *Phys. Rev. E*, vol. 79, no. 5, p. 056216 (2009).
- [52] A. Preska Steinberg, I. R. Epstein, and M. Dolnik, "Target Turing patterns and growth dynamics in the chlorine dioxide-iodine-malonic acid reaction," *J. Phys. Chem. A*, vol. 118, no. 13, pp. 2393–2400 (2014).
- [53] S. Setayeshgar, Turing Pattern Formation in the Chlorine Dioxide-Iodine-Malonic Acid Reaction-Diffusion System. Pasadena, CA: California Institute of Technology (1998).

- [54] B. D. Hassard, N. D. Kazarinoff, and Y.-H. Wan, *Theory and Applications of Hopf Bifurcation*. Cambridge, U.K.: Cambridge Univ. Press (1981).
- [55] J. E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*. New York, NY: Springer (2012).
- [56] E. Doedel, *Software for Continuation and Bifurcation Problems in Ordinary Differential Equations*. Pasadena, CA: California Institute of Technology (1986).
- [57] J. A. Sherratt, "Numerical continuation of boundaries in parameter space between stable and unstable periodic travelling wave (wave-train) solutions of partial differential equations," *Adv. Comput. Math.*, vol. 39, pp. 175–192 (2013).
- [58] J. D. Rademacher, B. Sandstede, and A. Scheel, "Computing absolute and essential spectra using continuation," *Physica D*, vol. 229, no. 2, pp. 166–183 (2007).
- [59] K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge, U.K.: Cambridge Univ. Press (2005).
- [60] A. Karma, "Spiral breakup in model equations of action potential propagation in cardiac tissue," *Phys. Rev. Lett.*, vol. 71, no. 7, p. 1103 (1993).
- [61] M. Bar and M. Eiswirth, "Turbulence due to spiral breakup in a continuous excitable medium," *Phys. Rev. E*, vol. 48, no. 3, p. 1635 (1993).
- [62] A. Panlov and P. Hogeweg, "Spiral breakup in a modified FitzHugh-Nagumo model," *Phys. Lett. A*, vol. 176, no. 5, pp. 295–299 (1993).
- [63] Q. Ouyang, H. L. Swinney, and G. Li, "Transition from spirals to defect-mediated turbulence driven by a Doppler instability," *Phys. Rev. Lett.*, vol. 84, no. 5, p. 1047 (2000).
- [64] Q. Ouyang and J.-M. Flesselles, "Transition from spirals to defect turbulence driven by a convective instability," *Nature*, vol. 379, no. 6561, pp. 143–146 (1996).
- [65] L. Q. Zhou and Q. Ouyang, "Experimental studies on long-wavelength instability and spiral breakup in a reaction-diffusion system," *Phys. Rev. Lett.*, vol. 85, no. 8, p. 1650 (2000).
- [66] M. O. Gani and T. Ogawa, "Stability of periodic traveling waves in the Aliev-Panfilov reaction-diffusion system," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 33, pp. 30–42 (2016).

Received November, 2024