

Laplace provides for movement and momentum conservation

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Abstract

The Laplace equation can be generally solved for dimensions up to eight. The solution functions have rotational characteristics, and there are transitory as well as stationary solutions. The Laplace equation provides for time-dependent dynamics and for nonlinear interactions, as well as for an individual movement of parts of a system, under the condition that the individually moving parts conserve the total momentum of the system. A fundamental indeterminism about the outcome of interactions between such moving parts is also present.

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1. Introduction

The Laplace equation, $\Delta A = \left(\sum_{i=0}^{n-1} \frac{\partial^2}{\partial x_i^2} \right) A = 0$, in shorthand $\Delta A = \left(\sum_{i=0}^{n-1} \partial_i^2 \right) A = 0$, arises in many fields of physics, such as in hydrodynamics and in electrodynamics. It describes the flow of an incompressible¹, irrotational² and lossless³ fluid, which does not appear, nor disappear, nor change its density at any point of the EUCLIDEAN space⁴ [1] spanned by the dimensions

¹ An incompressible fluid is one whose density remains constant, i.e. whose volume does not change under pressure.

² Irrotational means that the fluid particles are not rotating around their own centers of mass.

³ A lossless fluid is one wherein no energy is dissipated or lost during its movement.

⁴ Only EUCLIDEAN spaces have additive, positive metrics; physical quantities obeying conservation laws must necessarily be described in a EUCLIDEAN space.

$(x_0, x_1, \dots, x_{n-1})$. The Laplace equation, which follows out of the Continuity equation⁵, appears wherever a conservation law is present. The 4-dimensional Laplace equation is also related to the 4-dimensional wave equation of relativistic physics⁶ [2].

As shown hereafter, the characteristics of the solution functions of the Laplace equation, $r \cdot e^{\underline{u} \cdot \theta}$ and $r \cdot e^{-\underline{u} \cdot \theta}$, are *scaling-rotators* and correspond to the kernel $e^{-st} = e^{-(\sigma+i\omega)t} = r(t) \cdot e^{-i\omega t}$ of the Laplace transform. The Laplace transform has been successfully used in the numerical solution of differential and partial differential equations by the ADOMIAN decomposition method [3], [4].

Despite the importance of the Laplace equation and of EUCLIDEAN metrics, contemporary physics is considerably fixed on curved space-time, which lacks additivity and conservation laws. The aim of the present article is therefore to give *food for thought* for exploring as well other fields of mathematical physics.

2. Solving the Laplace equation

For one single dimension of Euclidean space, the Laplace equation's solution is $A = \text{constant}$. Solutions of the type $\partial A = \text{constant}$ are excluded by the condition that A should neither appear, nor disappear, nor change its density. The one-dimensional Laplace equation $\Delta A = \partial(\partial A) = 0$ appears as a nested couple of two first order *transport equations* along a single dimension, where no way exists for transporting something without sources nor sinks nor change of density.

In EUCLIDEAN spaces of dimensions 2, 4, and 8, the Laplace equation can also be written as a nested couple of two first-order transport equations. The Laplace operator, $\Delta = (\sum_{i=0}^{n-1} \partial_i^2)$, a sum of squares, can noteworthy be factored in 2, 4, and 8 dimensions, yielding products of two first-order partial differential operators in the *Complex*, the *Quaternion*, and the *Octonion* number space, respectively.⁷ The resulting nested *transport equations* can be handled, and the Laplace equation can thus be generally solved for dimensions up to 8, because dimension 3 can be included in dimension 4; and dimensions 5, 6, and 7 can be included in dimension 8 [5]. For dimensions higher than 8, no such factoring is any more possible⁸.

Complex numbers, Quaternions and Octonions are rotation-scaling operators around a point in space. We therefore have advantage in

⁵ See **Annex 1**.

⁶ The 4D Laplace equation, $\Delta A = (\partial_0^2 + \sum_{i=1}^3 \partial_i^2)A = 0$, can be conformally mapped in *quaternion* space onto the 4D wave equation, $\Delta A = (\partial_0^2 - \sum_{i=1}^3 \partial_i^2)A = 0$, by the operation $(x_0 \rightarrow x_0; x_1 \rightarrow ix_1; x_2 \rightarrow jx_2; x_3 \rightarrow kx_3)$.

⁷ The possibility of factoring a sum of squares relies on the algebraic “*n-squares identities*”, which exist for $n = 2, 4$, and 8 , and according to which the product of two sums of n squares is again a sum of n squares. The resulting *Complex Algebras* are shortly resumed in **Annex 2**.

⁸ Hurwitz [14] proved that the only possible division algebras having a multiplicative norm $(\|xy\| = \|x\| \|y\|)$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

reformulating Laplace's equation in **polar coordinates, in terms of a radius and a principal rotation angle**⁹:

The Laplace operator in *complex, quaternion* or *octonion* polar coordinates (r, θ) is given by the product of the *Nabla* operator with its *complex conjugate* $\Delta = \nabla \nabla^*$ ¹⁰. The Laplace equation is then:

$$\begin{aligned}\Delta A &= \left((\partial_\theta + \underline{u} \cdot r \cdot \partial_r) (\partial_\theta - \underline{u} \cdot r \cdot \partial_r) \right) A \\ &= \left((\partial_\theta - \underline{u} \cdot r \cdot \partial_r) (\partial_\theta + \underline{u} \cdot r \cdot \partial_r) \right) A \\ &= \left(\partial_\theta^2 + r \partial_r (r \partial_r) \right) A = (\partial_\theta^2 + r \partial_r + r^2 \partial_r^2) A = 0.\end{aligned}$$

Therein, for the 2-dimensional case, $\underline{u} = i$ (with $i^2 = -1$); for the 4-dimensional case, $\underline{u} = (iu_1 + ju_2 + ku_3)$ (with $i^2 = j^2 = k^2 = ijk = -1$); and for the 8-dimensional case,

$$\underline{u} = (iu_1 + ju_2 + ku_3 + lu_4 + mu_5 + nu_6 + ou_7)$$

(with $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = ijklmno = -1$);

$\underline{u}$ is always a unit vector with $\underline{u} \cdot \underline{u} = -1$.

We then obtain, applying D'ALEMBERT'S [6] *method of characteristics*, two nested first-order equations:

$$\begin{aligned}\Delta A &= \left((\partial_\theta + \underline{u} \cdot r \cdot \partial_r) (\partial_\theta - \underline{u} \cdot r \cdot \partial_r) \right) A \\ &= (\partial_\theta + \underline{u} \cdot r \cdot \partial_r) \left((\partial_\theta - \underline{u} \cdot r \cdot \partial_r) A \right) = 0,\end{aligned}$$

or, the other way around:

$$\begin{aligned}\Delta A &= \left((\partial_\theta - \underline{u} \cdot r \cdot \partial_r) (\partial_\theta + \underline{u} \cdot r \cdot \partial_r) \right) A \\ &= (\partial_\theta - \underline{u} \cdot r \cdot \partial_r) \left((\partial_\theta + \underline{u} \cdot r \cdot \partial_r) A \right) = 0.\end{aligned}$$

The associative shift of parentheses is allowed, according to ARTIN's theorem, even in the case of the non-associative *octonions*, because there are only two *octonions* involved in the differential equation, noteworthy ∇ and A .¹¹ [7]

The Laplace equation is therefore satisfied if either

$$(\partial_\theta - \underline{u} \cdot r \cdot \partial_r) A = 0 \text{ or } (\partial_\theta + \underline{u} \cdot r \cdot \partial_r) A = 0.¹²$$

⁹ Hereafter we limit the factor decomposition to the radial part in r and the principal rotation in θ ; in 4D there are two rotations more, i.e. totally three rotations implied in the vector $\underline{u} = (iu_1 + ju_2 + ku_3)$, and in 8D there are six rotations more, i.e. totally seven rotations implied in the vector $\underline{u} = (iu_1 + ju_2 + ku_3 + lu_4 + mu_5 + nu_6 + ou_7)$.

¹⁰ The complex *Nabla* operator in polar form (r, θ) is $\nabla = \left(\frac{1}{r} \partial_\theta + \underline{u} \partial_r \right)$, with $\underline{u}$ as defined in the text. Given that the Laplace equation equals zero, we can multiply the *Nabla* operator by r , to obtain $\nabla = (\partial_\theta + \underline{u} \cdot r \cdot \partial_r)$.

¹¹ EMIL ARTIN's argument goes as follows: Each *octonion* has a real component, as well as an imaginary component pointing somewhere into the 7-dimensional imaginary space. Two octonions can therefore at most span 3 dimensions in the 7-dimensional imaginary space: i, j , and $ij = k$. This corresponds to a *quaternion* sub-space, wherein the associative law still holds.

¹² The cases that $(\partial_\theta - \underline{u} \cdot r \cdot \partial_r) A = \text{const}$, or $(\partial_\theta + \underline{u} \cdot r \cdot \partial_r) A = \text{const}$, would also satisfy the Laplace equation. But these options are excluded by the condition that A should not appear, nor disappear, nor change its density anywhere in space.

This is the case for any twice differentiable function

$A_f(r \cdot e^{\underline{u}\theta} + \text{const})$ rotating in the forward sense, and for any twice differentiable function $A_b(r \cdot e^{-\underline{u}\theta} + \text{const})$ rotating in the backward sense, respectively, due to the zero inner derivatives¹³. The constants of integration (*const*) provide freedom for independently choosing the origin (*orig*) of each rotating function anywhere in the *complex*, the *quaternion* or the *octonion* number space, respectively. The solutions of the Laplace equation are thus **functions of the forward- or backward-rotating characteristics**, $(r \cdot e^{\underline{u}\theta} + \text{orig})$ or $(r \cdot e^{-\underline{u}\theta} + \text{orig})$, respectively, in the space of the *Complex numbers*, the *Quaternions*, or the *Octonions*, respectively, as well as any sums of such functions.

A general solution of Laplace's equation in 2- to 8-dimensional EUCLIDEAN spaces can therefore be written as:

$$A = \sum_n A_{fn}(r \cdot e^{\underline{u}\theta} + \text{orig}_n) + \sum_m A_{bm}(r \cdot e^{-\underline{u}\theta} + \text{orig}_m). \quad (1)$$

The 2-dimensional case must be handled with Complex numbers; the 3- and 4-dimensional cases must be handled with Quaternions, and the 5-, 6-, 7-, and 8-dimensional cases must be handled with Octonions.

3. Time and Dynamics

Equation (1) is a static approach to the Laplace equation. We can ask now about whether it is possible to introduce a time-dependence into the general solution function (1), whilst still satisfying the Laplace equation.

A first observation in this line is, that any time-dependence can only affect the angular parts, $e^{-\underline{u}\theta}$ and $e^{\underline{u}\theta}$, of the solution function A, because a time-dependent variation of the radial part *r* would necessarily increase or decrease A, or change its density. The argument holding for one single dimension holds as well for the radial parts *r* of the solution function (1).

A time-dependence in the angular parts, on the other hand, results in a mere cyclic permutation of elements within A, and is thus possible without A appearing, or disappearing, or changing density anywhere in space.

A second observation is that a general substitution of the rotation angle θ by time *t* does not allow for an individual time dependence of parts within the solution function A.

Substituting, in equation (1), the angle θ by time *t* yields:

$$A = \sum_n A_{fn}(r \cdot e^{\underline{u}t} + \text{orig}_n) + \sum_m A_{bm}(r \cdot e^{-\underline{u}t} + \text{orig}_m),$$

which obviously satisfies the time-dependent Laplace equation, where ∂_θ is replaced by ∂_t , but this adds nothing new and results

¹³ Recall that the derivative of a nested function $f(g(x))$ is $f'(g(x)) \cdot g'(x)$. If $g'(x) = 0$, the derivative vanishes.

in a strict synchronicity of all moving parts of the solution function A, excluding all relative movement of parts against each other.

The individual time dependence of parts within the solution function A requires the introduction of individual circular frequencies ω_i for each rotating part:

$$A_f(r \cdot e^{\underline{u} \cdot \omega_i t} + orig_i) \text{ and } A_b(r \cdot e^{-\underline{u} \cdot \omega_i t} + orig_i).$$

The function A takes then the form of a sum of n individually forward-rotating and m individually backward rotating parts, with their respective circular frequencies ω_n and ω_m , and about their respective individual origins $orig_n$ and $orig_m$:

$$A = \sum_n A_{fn}(r \cdot e^{\underline{u} \cdot \omega_n t} + orig_n) + \sum_m A_{bm}(r \cdot e^{-\underline{u} \cdot \omega_m t} + orig_m). \quad (2)$$

For this new function (2) to solve the Laplace equation, however, additional constraints must be fulfilled.

4. Conservation of momentum

The Laplace operator, $\Delta A = (\sum_{i=0}^{n-1} \partial_i^2)A$, is a linear operator, satisfying the properties of *additivity* and *homogeneity*. If its general solution A is a sum of individually rotating parts, we can apply the Laplace operator separately to each of these parts or to any sub-sum of such parts of the solution function A.

Without loss of generality, we can thus introduce the solution (2) into the time-dependent Laplace equation as follows:

$$\begin{aligned} \Delta A &= (\partial_t + \underline{u} \cdot r \cdot \partial_r) \left((\partial_t - \underline{u} \cdot r \cdot \partial_r) \sum_n A_{fn}(r \cdot e^{\underline{u} \cdot \omega_n t} + orig_n) \right) \\ &+ (\partial_t - \underline{u} \cdot r \cdot \partial_r) \left((\partial_t + \underline{u} \cdot r \cdot \partial_r) \sum_m A_{bm}(r \cdot e^{-\underline{u} \cdot \omega_m t} + orig_m) \right) \\ &= 0. \end{aligned} \quad (3)$$

For $\omega_n = \omega_m = 1$, i.e. the already contemplated case of synchronicity, all forward-rotating derivatives cancel individually under $(\partial_t - \underline{u} \cdot r \cdot \partial_r)$:

$$\sum_n (A'_{fn}) \cdot (r \cdot \underline{u} \cdot \omega_n \cdot e^{\underline{u} \cdot \omega_n t} - r \cdot \underline{u} \cdot e^{\underline{u} \cdot \omega_n t}) = 0,$$

with $A'_{fn} = \partial_t A_{fn} - \underline{u} \cdot r \cdot \partial_r A_{fn}$, and all backward-rotating derivatives cancel individually under $(\partial_t + \underline{u} \cdot r \cdot \partial_r)$:

$$\sum_m (A'_{bm}) \cdot (-r \cdot \underline{u} \cdot \omega_m \cdot e^{-\underline{u} \cdot \omega_m t} + r \cdot \underline{u} \cdot e^{-\underline{u} \cdot \omega_m t}) = 0,$$

with $A'_{fn} = \partial_t A_{fn} + \underline{u} \cdot r \cdot \partial_r A_{fn}$.

Otherwise, if at least two $\omega_n \neq 1$, the sum of the forward-rotating derivatives cancels under the additional condition of:

$$\begin{aligned} &\sum_n \left((A'_{fn}) \cdot (r \cdot \underline{u} \cdot \omega_n \cdot e^{\underline{u} \cdot \omega_n t} - r \cdot \underline{u} \cdot e^{\underline{u} \cdot \omega_n t}) \right) \\ &= \sum_n \left((\omega_n - 1) \cdot (A'_{fn}) \cdot (r \cdot \underline{u} \cdot e^{\underline{u} \cdot \omega_n t}) \right) = 0, \end{aligned} \quad (4)$$

with $A'_{fn} = \partial_t A_{fn} - \underline{u} \cdot r \cdot \partial_r A_{fn}$, and if at least two $\omega_m \neq 1$, the sum of all backward-rotating derivatives cancels under the additional condition of:

$$\sum_m \left((A'_{bm}) \cdot (-r \cdot \underline{u} \cdot \omega_m \cdot e^{-\underline{u} \cdot \omega_m t} + r \cdot \underline{u} \cdot e^{-\underline{u} \cdot \omega_m t}) \right)$$

$$= \sum_m \left((1 - \omega_m) \cdot (A'_{bm}) \cdot (r \cdot \underline{u} \cdot e^{-\underline{u} \cdot \omega_m t}) \right) = 0, \quad (5)$$

with $A'_{fn} = \partial_t A_{fn} + u \cdot r \cdot \partial_r A_{fn}$.

But this is still not the most general case, which is given by the conditions imposed by equation (3), wherein the sum of all second partial derivatives over all forward- and backward-rotating parts of the solution function A must be zero.

These additional conditions represent the momentum-conservation law, which is imposed by the Laplace equation onto the individually moving parts in an incompressible, irrotational and lossless fluid, which does not appear, nor disappear, nor change its density at any point of the EUCLIDEAN space spanned by the equation.

The Laplace equation provides thus for an individual movement of parts of a system, under the condition that the total momentum (i.e. the quantity of movement) of the different moving parts is conserved. In other words: within a system obeying Laplace's equation, momentum cannot be created, nor lost; it can only be transferred, cumulated, and shared.

5. Periodicity and stationary solutions

A further observation concerns the rotational parts of the solution function,

$$A_{fn}(r \cdot e^{\underline{u} \cdot \omega_n t} + \text{orig}_n), \text{ and}$$

$$A_{bm}(r \cdot e^{-\underline{u} \cdot \omega_m t} + \text{orig}_m).$$

These can noteworthy give rise to **stationary solutions**, if they are periodic in $\omega t = 2k\pi$ (with $k = 0, 1, 2, \dots$; a non-negative integer). In such case, the functions always confirm what they had written in the cycles before; in all other cases, they overwrite it and blur it out.

These **stationary, time-stable solutions** constitute discrete points in the continuum of all possible time-dependent solutions, and, applied to electrodynamics, they can offer an explanation for the appearance of *massive particles of discrete rest masses m_0* in the electromagnetic *continuum*. [8]

6. Non-linearity and information processing potential

A further consideration concerns the intrinsic information processing potential contained in the solutions (3) of Laplace's equation.

Information processing noteworthy requires a time-stable support for storing the data, **and non-linear interactions**, for establishing interdependencies between the data, and not just superpositions of data.

The stationary solutions of Laplace's equations offer the required time-stable support for storing data, and the **nonlinear cyclic nature** of its transitory solutions provides for the nonlinear interactions required for processing the data.

Living nature relies on data processing and storage, an obvious fact in biology (genetic code, biochemical machinery of the living cell, nerve system of animals and human beings), but such data processing and storage is anchored much deeper in nature; its basis is ultimately found in the fundamental non-linearity of the solutions of Laplace's equation.

7. Fundamental indeterminism

An ultimate question we raise is about whether the outcome of a physical interaction is predetermined by the physical constellation, i.e. the measurable parameters of the physical system before the interaction takes place, or not?

If the physical system obeys the time-dependent Laplace equation, which has solutions as given by equation (2), under the supplementary conditions imposed by equation (3), the following applies:

Equation (3) is a composite equation in the space of the *Complex numbers*, the *Quaternions* or the *Octonions*, thus comprising 2, 4, or 8 component equations, whereby each of them must be individually satisfied. The momentum-conservation laws of a physical system span therefore at most eight dimensions in space, i.e. they give rise to at most eight equations which must be satisfied during a physical interaction in the contemplated system.

On the other hand, there is no upper limit to the number of components n and m , i.e. the number of individual parts of the solution function A implied in a physical interaction. There are thus, generally, not sufficient equations for unambiguously determining the outcome of an interaction between moving parts in a determined system of measurable constellation. This points towards a fundamental indeterminism of interactions in such systems.

Conclusion

Systems obeying Laplace's equation can be described by sums of functions of rotational characteristics, turning in a forward and/or in a backward sense. Said functions give rise to transitory, as well as to stationary solutions, and also provide for the individual movement of parts of the system, as well as for nonlinear interactions between those parts, under the condition that the individually moving parts conserve the total momentum of the system. The momentum conservation laws do not generally provide a sufficient number of equations for determining, in a given constellation, the outcome of an interaction between the moving parts of the system; pointing thus towards a fundamental indeterminism in such systems.

Annex 1

Deriving the Laplace equation

THE Laplace equation can be derived from the *Continuity Equation* (also called the general STOKES' law), which mathematically expresses the **accountability principle**:

$$\int_V d\omega = \int_{dV} \omega$$

Therein, V represents a volume (or a hyper-volume), and ω its contents. The quantity dV is the boundary surface of the volume or hyper-volume, and $d\omega$ is the change of contents in the volume or hyper-volume V .¹⁴ The integrals mean

¹⁴ If V is a cash, the integrated change of its contents $d\omega$ corresponds to the total cash flows ω (input – output) through the cash boundary dV .

that we have to sum over the (hyper)volume and the (hyper)surface, respectively.

The differential form of the general STOKES' law is GAUSS' law: $\text{div}(\phi(x)) = \rho(x)$.

The quantity ϕ represents a flux, and ρ is a source or a sink density.

If the flux ϕ is expressed as the gradient of a potential φ ¹⁵, $\phi(x) = -\text{grad}(\varphi(x))$, we obtain POISSON'S equation: $\text{div}(\text{grad}(\varphi(x))) = \Delta(\varphi(x)) = -\rho(x)$.

In the absence of sources and sinks ρ , POISSON'S equation becomes Laplace's equation: $\Delta(\varphi(x)) = 0$.

Annex 2

Complex algebras

The *Complex* $\mathbb{C}$, *Quaternion* $\mathbb{Q}$, and *Octonion* $\mathbb{O}$ number systems are *division algebras* analogous to the real numbers $\mathbb{R}$, supporting the basic algebraic operations, as well as the usual functions and the differential calculus.

Arithmetic bases of the complex number systems

The three complex algebras are derived from the three algebraic n-squares identities¹⁶ [9], which can be verified by simple algebraic evaluation, noteworthy:

The 2-squares identity:¹⁷ [10]

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}^2 \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}^2 = \begin{pmatrix} (a_0 b_0 - a_1 b_1) \\ (a_0 b_1 + a_1 b_0) \end{pmatrix}^2,$$

defines the multiplication rule of the *complex* number system.

The 4-squares identity: [11] [12]

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}^2 \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}^2 = \begin{pmatrix} (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \\ (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) \\ (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) \\ (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0) \end{pmatrix}^2,$$

defines the multiplication rule of the *quaternion* number system.

Quaternions are not commutative.

¹⁵ By simplifying with the HELMHOLTZ-HODGE theorem, which states that $\omega = \text{rot}(A) - \text{grad}(\varphi)$; we are going to take the divergence of the quantity ω , and $\text{div rot} = 0$ in every case.

¹⁶ For didactical reasons, the n-squares identities are hereafter written in vectorial form,

¹⁷ This identity is also known as the BRAMAGUPTA or the FIBONACCI identity.

The 8-squares identity: [13]

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}^2 \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}^2 = \begin{pmatrix} (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7) \\ (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4 - a_6b_7 + a_7b_6) \\ (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 + a_4b_6 + a_5b_7 - a_6b_4 - a_7b_5) \\ (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 + a_4b_7 - a_5b_6 + a_6b_5 - a_7b_4) \\ (a_0b_4 - a_1b_5 - a_2b_6 - a_3b_7 + a_4b_0 + a_5b_1 + a_6b_2 + a_7b_3) \\ (a_0b_5 + a_1b_4 - a_2b_7 + a_3b_6 - a_4b_1 + a_5b_0 - a_6b_3 + a_7b_2) \\ (a_0b_6 + a_1b_7 + a_2b_4 - a_3b_5 - a_4b_2 + a_5b_3 + a_6b_0 - a_7b_1) \\ (a_0b_7 - a_1b_6 + a_2b_5 + a_3b_4 - a_4b_3 - a_5b_2 + a_6b_1 + a_7b_0) \end{pmatrix}^2,$$

defines the multiplication rule of the *octonion* number system.

Octonions are neither commutative, nor associative. However, any sub-algebra spanned by only two octonions is still associative (ARTIN'S theorem) [7].

Algebraic properties of the complex number systems

Complex numbers can be written in **algebraic form**, as a sum of components, e.g.

$$C = (x_0 + ix_1) ;$$

$$Q = (x_0 + ix_1 + jx_2 + kx_3) ;$$

$$O = (x_0 + ix_1 + jx_2 + kx_3 + lx_4 + mx_5 + nx_6 + ox_7) .$$

In programming applications, *complex numbers* are generally written as **vectors**, i.e. as doublets, quartets, or octets of real numbers.

Complex numbers have, further to their **algebraic form**, also a **polar form** (r, φ) and an **exponential form**:

$$C = r \cdot (\cos \varphi + i \sin \varphi) = r \cdot e^{i\varphi} ;$$

$$Q = r \cdot (\cos \varphi + (iu_1 + ju_2 + ku_3) \sin \varphi) = r \cdot e^{(iu_1 + ju_2 + ku_3)\varphi}, \text{ with } u_1^2 + u_2^2 + u_3^2 = 1 ;$$

$$O = r \cdot (\cos \varphi + (iu_1 + ju_2 + ku_3 + lu_4 + mu_5 + nu_6 + ou_7) \sin \varphi) = r \cdot e^{(iu_1 + ju_2 + ku_3 + lu_4 + mu_5 + nu_6 + ou_7)\varphi},$$

$$\text{with } u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 = 1.$$

All these forms are equivalent, which provides interesting mathematical properties, as well as many calculation shortcuts.

To multiply complex numbers, we can e.g. multiply their radii and add their angles.

The norm (i.e. the **length**) of a *complex vector* (*complex number*) u is the square root of the product of the *complex number* with its **complex conjugate** u^* : $|u| = \sqrt{u \cdot u^*} = \sqrt{u^* \cdot u}$. The *complex conjugate* of a *Complex number*, *Quaternion* or *Octonion* is obtained by reversing the sign of its imaginary part, or reversing the sign of its exponent:

$$C \cdot C^* = (x_0 + ix_1) \cdot (x_0 - ix_1) ;$$

$$Q \cdot Q^* = (x_0 + ix_1 + jx_2 + kx_3) \cdot (x_0 - ix_1 - jx_2 - kx_3) ;$$

$$O \cdot O^* = (x_0 + ix_1 + jx_2 + kx_3 + lx_4 + mx_5 + nx_6 + ox_7) \cdot (x_0 - ix_1 - jx_2 - kx_3 - lx_4 - mx_5 - nx_6 - ox_7).$$

Complex numbers, Quaternions and Octonions act as rotation-scaling operators, which rotate, upon multiplication, a vector around a point in 2-, 4-, or 8-dimensional space, [9] and scale it by their length.

Complex numbers, Quaternions and Octonions also support **differential calculus**, just as if they were real numbers (with certain exceptions, however), and they support all kind of **mathematical functions**, insofar these functions can be expressed by a **Taylor series**, i.e. as a weighted sum of powers of the function's argument.

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