

Global dynamics of a discrete mosquito model with Allee effect

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Abstract

In this paper, we investigate the dynamical behavior of a two-dimensional discrete mosquito model incorporating an Allee effect. We analyze the system's trajectory comprehensively, examining both local and global dynamics. Our analysis begins by establishing the invariance of the positive trajectory within the first quadrant. We then investigate the existence of fixed points within this region. Following this, we perform a stability analysis to determine both local and global asymptotic stability of the identified fixed points. Finally, we validate the effectiveness and applicability of our theoretical results through numerical examples.

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1. Introduction

Analyzing the global dynamics and stability of complex systems remains a significant challenge in discrete dynamical systems. This has spurred recent interest in the global stability of discrete-time systems, a key area within dynamic system theory. In this context, we investigate the global dynamics of a discrete mosquito population model.

Mosquitoes transmit a variety of diseases, including malaria and several viruses, making their population control a crucial public health practice, especially in tropical regions.

Mathematical models play a crucial role in understanding mosquito population dynamics. These models can be continuous-time or discrete-time, as exemplified by the works in [3-8, 16-18, 20, 22].

This work focuses on a discrete-time dynamical system for a simplified mosquito population model presented in [17]. This model incorporates Allee effects, which are situations where population growth is limited at low densities. Mosquitoes undergo complete metamorphosis, with four distinct development stages: egg, larva, pupa, and adult. To simplify the model, we combine the three aquatic stages (egg, larva, pupa) into a single larvae class denoted by x . The adult mosquito population is denoted by y [17]. Moreover, we assume that density dependence exists only in the larvae stage.

Denote the birth rate, i.e., the oviposition rate of adults by $\beta(t)$; the rate of emergence from larvae to adults by a function of the larvae with the form of $\alpha(1-k(x))$, where $\alpha > 0$ is the maximum emergence rate, $0 \leq k(x) \leq 1$, with $k(0) = 0, k'(x) > 0$, and $\lim_{x \rightarrow \infty} k(x) = 1$, is the functional response due to the intraspecific competition [16]. Moreover, we assume the death rate of larvae be a linear function, denoted by $d_0 + d_1x$, and the death rate of adults be constant, denoted by μ . Then, in the absence of sterile mosquitoes, we get the following system of equations:

$$\begin{cases} \frac{dx}{dt} = \beta(t)y - \alpha(1-k(x))x - (d_0 + d_1x)x, \\ \frac{dy}{dt} = \alpha(1-k(x))x - \mu y. \end{cases} \quad (1)$$

We further assume a functional response for $k(x)$, as in [16], in the form

$$k(x) = \frac{x}{1+x}.$$

The dynamical system (1) was previously studied in [16] for the case where the birth rate β is constant (i.e., $\beta(t) = \beta$), in which mosquito adults have no difficulty finding their mates such that no Allee effects are concerned. The discrete-time version of this model was then investigated in [3-6].

The Allee effect plays an important role in population dynamics [2, 11, 16, 17, 24]. A component Allee effect describes a decrease in any fitness component (e.g., survival, reproduction) as population size or density declines. A prime example is the decrease in a female's probability of mating with fewer males around. This specific phenomenon is commonly referred to as a "mate-finding Allee effect".

When adult mosquitoes face difficulty finding mates, the model incorporates Allee effects. This is reflected in the adult birth rate, which is given by:

$$\beta(t) = \frac{\beta y(t)}{\gamma + y(t)},$$

where β is birth rate, γ is the Allee effect constant. Then stage-structured wild mosquito population model is given by:

$$\begin{cases} \frac{dx}{dt} = \frac{\beta y^2}{\gamma + y} - \frac{\alpha x}{1+x} - (d_0 + d_1 x)x, \\ \frac{dy}{dt} = \frac{\alpha x}{1+x} - \mu y. \end{cases} \quad (2)$$

This continuous system has been discussed in [17], but its discrete version has not been investigated as of yet. Discretization of the continuous-time model offers several advantages for studying mosquito dynamics. It allows simulations to be conducted on discrete time scales, which are more relevant to biological processes. This facilitates a closer alignment with actual mosquito population data and enables the effective integration of time-sensitive ecological components. Continuous systems can be approached by several discrete methods, including Euler forward/backward and semi-discretization. We utilize the Euler forward difference scheme here. In this paper, we study the discrete time dynamical systems associated with the system (2) by employing Euler's forward discretization method. Define the operator $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{cases} x' = \frac{\beta y^2}{\gamma + y} - \frac{\alpha x}{1+x} - (d_0 + d_1 x)x + x, \\ y' = \frac{\alpha x}{1+x} - \mu y + y, \end{cases} \quad (3)$$

where $\alpha > 0, \beta > 0, \gamma > 0, \mu > 0, d_0 \geq 0, d_1 \geq 0$. In the system (3), x', y' means the next state relative to the initial state x, y respectively.

If for each possible state $z = (x, y) \in \mathbb{R}^2$ describing the current generation, the state $z' = (x', y') \in \mathbb{R}^2$ is uniquely defined as $z' = W(z)$ then the map $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ called the evolution operator [15, 20].

The main problem for a given operator W and arbitrarily initial point $z^{(0)} = (x^{(0)}, y^{(0)}) \in \mathbb{R}^2$ is to describe the limit points of the trajectory $\{z^{(m)}\}_{m=0}^\infty$, where $z^{(m)} = W^m(z^{(0)})$.

In [21] the dynamics of the operator defined by (3) is studied under the condition $d_0 = d_1 = 0$. In this paper we consider the operator W (defined by (3)) for the case $d_0 > 0$.

2. Preliminaries and Known Results

Let an operator $W : \mathfrak{R} \rightarrow \mathfrak{R}$ be defined by

$$W(x, y) = (f_1(x, y), f_2(x, y)),$$

where f_1 and f_2 are given functions and $\mathfrak{R} \subset \mathbb{R}^2$ has nonempty interior. The system of difference equations generated by W is

$$\begin{cases} x_{n+1} = f_1(x_n, y_n), \\ y_{n+1} = f_2(x_n, y_n), \end{cases} \quad n = 0, 1, 2, \dots \quad (4)$$

with initial point (x_0, y_0) taken from the intersection of the domains of f_1 and f_2 .

If $f_1(x, y)$ is increasing in x and decreasing in y , while $f_2(x, y)$ is decreasing in x but increasing in y , then (4) is called a *competitive* system. If both $f_1(x, y)$ and $f_2(x, y)$ are increasing in each variable, then the system is said to be *cooperative*. These competitive and cooperative transformations are usually referred to as *monotone maps*. Moreover, the system (4) (or the corresponding map) is called *strongly cooperative* when both coordinate functions f_1 and f_2 are strictly monotone [1].

Let $\preceq$ be a partial order on $\mathbb{R}^n$ with nonnegative cone P . For $z_1, z_2 \in \mathbb{R}^n$ the order interval $\llbracket z_1, z_2 \rrbracket$ is the set of all z such that $z_1 \preceq z \preceq z_2$. We say $z_1 \prec z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$, and $z_1 \ll z_2$ if $z_2 - z_1 \in \text{int}(P)$. An

operator W on a subset of $\mathbb{R}^n$ is order-preserving if $W(z_1) \preceq W(z_2)$ whenever $z_1 \prec z_2$, strictly order-preserving $W(z_1) \prec W(z_2)$ whenever $z_1 \prec z_2$, and strongly order-preserving if $W(z_1) \ll W(z_2)$ whenever $z_1 \prec z_2$.

A point $(x^*, y^*) \in \mathbb{R}^2$ is called a fixed point of W if $W((x^*, y^*)) = (x^*, y^*)$. The basin of attraction of a fixed point (x^*, y^*) of an operator W , is defined as the set of all initial points $(x^{(0)}, y^{(0)})$ in the domain of W such that the sequence of iterates $W^n(x^{(0)}, y^{(0)})$ converges to (x^*, y^*) .

The following result is a direct consequence of the Trichotomy Theorem of Dancer and Hess, (see [9, 13, 14]) proves useful in determining the basins of attraction for the fixed points.

Proposition 1: If the non-negative cone of a partial ordering $\preceq$ is a generalized quadrant in $\mathbb{R}^n$, and if W has no fixed points in $\llbracket z_1, z_2 \rrbracket$ other than z_1 and z_2 , then the interior of $\llbracket z_1, z_2 \rrbracket$ is either a subset of the basin of attraction of z_1 or a subset of the basin of attraction of z_2 .

Consider the North-East (NE) partial order on $\mathbb{R}^2$, where the positive cone corresponds to the first quadrant. Explicitly, we write

$$(x_1, y_1) \preceq_{NE} (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Similarly, the South-East (SE) partial order on $\mathbb{R}^2$ is defined by $(x_1, y_1) \preceq_{SE} (x_2, y_2)$ whenever $x_1 \leq x_2$ and $y_1 \geq y_2$.

A mapping W defined on a nonempty subset $\mathfrak{A} \subset \mathbb{R}^2$ is said to be *cooperative* if it is monotone with respect to the NE order. Analogously, W is called *competitive* whenever it is monotone with respect to the SE order.

We define the non-negative quadrant of the plane as $\mathbb{R}_+^2 = \{(x, y) : x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$. The operator W is well-defined on $\mathbb{R}^2$ except for the lines $x = -1$ and $y = -\gamma$. However, to model a population dynamics system using a continuous operator, we require both population values (represented by x and y) to be non-negative. Therefore, we restrict our analysis to $\mathbb{R}_+^2$ and choose the parameters of the operator W such that it maps this set onto itself.

Lemma 1: *If*

$$\alpha > 0, \beta > 0, \gamma > 0, 0 < \mu \leq 1, d_0 > 0, \alpha + d_0 \leq 1, d_1 = 0 \quad (5)$$

then the operator (3) maps the set $\mathbb{R}_+^2$ to itself.

Proof: This can be proved by direct computation.

Lemma 1 provides sufficient conditions for non-negative values of x and y . In this case the system (3) becomes

$$W_0 : \begin{cases} x' = \frac{\beta y^2}{\gamma + y} + \left(1 - d_0 - \frac{\alpha}{1+x}\right)x, \\ y' = \frac{\alpha x}{1+x} + (1-\mu)y. \end{cases} \quad (6)$$

Definition 1: (see [10]) A fixed point z of an operator W_0 is called hyperbolic if its Jacobian J at z has no eigenvalues on the unit circle.

Definition 2: (see [10]) A hyperbolic fixed point z is called:

- attracting if all the eigenvalues of the Jacobian matrix $J(z)$ are less than 1 in absolute value;
- repelling if all the eigenvalues of the Jacobian matrix $J(z)$ are greater than 1 in absolute value;
- a saddle otherwise.

To investigate the local stability analysis of the fixed point of system (6), we present the following lemma [23].

Lemma 2: *Let*

$$F(\lambda) = \lambda^2 + B\lambda + C, \quad B, C \in \mathbb{R},$$

with roots λ_1, λ_2 . Then

1. *If $F(1) > 0$:*
 - *both roots inside unit circle $\Leftrightarrow F(-1) > 0, C < 1$;*
 - *one root $= -1$ and the other different $\Leftrightarrow F(-1) = 0, B \neq 2$;*
 - *one root inside, one outside $\Leftrightarrow F(-1) < 0$;*
 - *both roots outside $\Leftrightarrow F(-1) > 0, C > 1$;*
 - *complex conjugate roots on unit circle $\Leftrightarrow -2 < B < 2, C = 1$;*
 - *double root at $-1 \Leftrightarrow F(-1) = 0, B = 2$.*
2. *If $F(1) = 0$, meaning that 1 is a root, then the other root satisfies $|\lambda| (<, =, >) 1 \Leftrightarrow |C| (<, =, >) 1$.*
3. *If $F(1) < 0$, one root > 1 and the other is $\lambda \leq -1 \Leftrightarrow F(-1) \leq 0, \quad -1 < \lambda < 1 \Leftrightarrow F(-1) > 0$.*

3. Analyzing the Existence and Local Stability of Fixed Points

We begin by investigating the existence of fixed points.

The fixed points of the system (6) satisfy the following system of equations:

$$\begin{aligned}x &= \frac{\beta y^2}{\gamma + y} + \left(1 - d_0 - \frac{\alpha}{1+x}\right)x, \\y &= \frac{\alpha x}{1+x} + (1-\mu)y.\end{aligned}\tag{7}$$

It can be seen that the point $(0,0)$ is a solution to system (7). By solving the second equation of system (7) for y and substituting it into the first equation, we obtain the following equation:

$$\begin{aligned}d_0\mu(\alpha + \gamma\mu)x^2 + \left(\mu(d_0\gamma\mu + (\alpha + d_0)(\alpha + \gamma\mu)) - \alpha^2\beta\right)x \\+ \gamma\mu^2(\alpha + d_0) = 0.\end{aligned}\tag{8}$$

We denote

$$\begin{aligned}D &= \left(\mu(d_0\gamma\mu + (\alpha + d_0)(\alpha + \gamma\mu)) - \alpha^2\beta\right)^2 \\&\quad - 4d_0\gamma\mu^3(\alpha + d_0)(\alpha + \gamma\mu), \\v &= \frac{\mu}{\alpha^2} \left(\sqrt{d_0\gamma\mu} + \sqrt{(\alpha + d_0)(\alpha + \gamma\mu)}\right)^2.\end{aligned}\tag{9}$$

Conditions in (5) ensure $d_0\mu(\alpha + \gamma\mu) > 0$ and $\gamma\mu^2(\alpha + d_0) > 0$. For positive roots in equation (8), we additionally require $\mu(d_0\gamma\mu + (\alpha + d_0)(\alpha + \gamma\mu)) - \alpha^2\beta < 0$ and $D > 0$. From here we get $\beta > v$. The number of solutions to equation (8) depends on the relationship between β and v . If $\beta = v$, equation (8) has one solution. If $\beta < v$, equation (8) has no solution.

The following proposition holds for fixed points of the operator W_0 :

Proposition 2: The fixed points for (6) are as follows:

- if $\beta < v$ then the operator (6) has a unique fixed point $z_1 = (0,0)$.
- if $\beta = v$ then mapping (6) has two fixed points

$$z_{1,1} = (0,0), \quad z_{1,2} = (x_1^*, y_1^*).$$

- if $\beta > v$ then mapping (6) has three fixed points

$$z_{2,1} = (0,0), \quad z_{2,2} = (x_2^*, y_2^*), \quad z_{2,3} = (x_3^*, y_3^*),$$

where

$$\begin{aligned}x_1^* &= \sqrt{\frac{\gamma\mu(\alpha + d_0)}{d_0(\alpha + \gamma\mu)}}, \\x_2^* &= \frac{\alpha^2\beta - \mu(d_0\gamma\mu + (\alpha + d_0)(\alpha + \gamma\mu)) - \sqrt{D}}{2d_0\mu(\alpha + \gamma\mu)}, \\x_3^* &= \frac{\alpha^2\beta - \mu(d_0\gamma\mu + (\alpha + d_0)(\alpha + \gamma\mu)) + \sqrt{D}}{2d_0\mu(\alpha + \gamma\mu)}, \\y_i^* &= \frac{\alpha x_i^*}{\mu(1 + x_i^*)}, i = 1, 2, 3.\end{aligned}\tag{10}$$

Now we shall examine the type of the fixed point.

To determine the type of a fixed point $z = (x^*, y^*)$ of the operator defined by (6), we can analyze the Jacobian matrix:

$$J_{w_0} = \begin{pmatrix} 1 - d_0 - \frac{\alpha}{(1+x)^2} & \frac{\beta y(2\gamma + y)}{(\gamma + y)^2} \\ \frac{\alpha}{(1+x)^2} & 1 - \mu \end{pmatrix}.$$

The corresponding characteristic polynomial is given by

$$F(\lambda) = \lambda^2 + B\lambda + C,\tag{11}$$

with coefficients

$$B = \mu + d_0 + \alpha / (1 + x^*)^2 - 2,$$

$$\begin{aligned}C &= 1 - \mu - d_0 - \alpha / (1 + x^*)^2 + \mu d_0 \\ &\quad + \alpha / (1 + x^*)^2 \cdot (\mu - \beta y^*(2\gamma + y^*) / (\gamma + y^*)^2).\end{aligned}$$

From (11) we get

$$\begin{aligned}F(1) &= 1 + B + C = \mu(d_0 + \alpha / (1 + x^*)^2) \\ &\quad - \alpha / (1 + x^*)^2 \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right),\end{aligned}$$

$$\begin{aligned}F(-1) &= 1 - B + C = 4 - 2(\mu + d_0 + \alpha / (1 + x^*)^2) \\ &\quad + \mu(d_0 + \alpha / (1 + x^*)^2) - \alpha / (1 + x^*)^2 \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right).\end{aligned}$$

(1) Let $\beta < \nu$, i.e., $x^* = 0, y^* = 0$. Then

$$F(1) = \mu(\alpha + d_0) > 0, C = (1 - d_0 - \alpha)(1 - \mu) < 1,$$

$$F(-1) = 2(2 - \mu - d_0 - \alpha) + \mu(d_0 + \alpha) > 0.$$

According to item (i) of Lemma 2, $|\lambda_{1,2}| < 1$. Therefore, the fixed point $(0, 0)$ is attracting.

(2) Let $\beta = \nu$, i.e., $x^* = x_1^* = \sqrt{\frac{\mu(\alpha + d_0)}{d_0(\alpha + \mu)}}$. Then

$$\begin{aligned} F(1) &= \mu \left(d_0 + \alpha / (1 + x^*)^2 \right) \\ &\quad - \alpha / (1 + x^*)^2 \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right) \\ &= \mu \left(d_0 + \alpha / (1 + x^*)^2 \right) - \frac{\mu y^*}{(1 + x^*)x^*} \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right) \\ &= \frac{\mu}{(1 + x^*)x^*} \left(d_0 x^* (1 + x^*) + \alpha x^* / (1 + x^*) \right) \\ &\quad - \frac{\mu y^*}{(1 + x^*)x^*} \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right) \\ &= \frac{\mu}{(1 + x^*)x^*} \left(d_0 x^{*2} + \frac{\beta y^{*2}}{\gamma + y^*} \right) - \frac{\mu y^*}{(1 + x^*)x^*} \cdot \frac{\beta y^*}{\gamma + y^*} \left(1 + \frac{\gamma}{\gamma + y^*} \right) \\ &= \frac{\mu}{(1 + x^*)x^*} \left(d_0 x^{*2} - \frac{\beta \gamma y^{*2}}{(\gamma + y^*)^2} \right) \\ &= \frac{\mu}{(1 + x^*)x^*} \left(d_0 x^{*2} - \frac{\beta \gamma \alpha^2 x^{*2}}{(\mu + (\alpha + \mu)x^*)^2} \right) \\ &= \frac{\mu x^{*2}}{(1 + x^*)x^*} \left(d_0 - \frac{\beta \gamma \alpha^2}{(\mu + (\alpha + \mu)x^*)^2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\mu x^*}{1+x^*} \left(d_0 - \frac{\gamma \alpha^2 \mu \left(\frac{\sqrt{d_0 \gamma \mu} + \sqrt{(\alpha + d_0)(\alpha + \gamma \mu)}}{\alpha} \right)^2}{\left(\gamma \mu + (\alpha + \gamma \mu) \sqrt{\frac{\gamma \mu (\alpha + d_0)}{d_0 (\alpha + \gamma \mu)}} \right)^2} \right) \\
&= \frac{\mu x^*}{1+x^*} (d_0 - d_0) = 0,
\end{aligned}$$

$$-1 < C = 1 - \mu - d_0 - \frac{\alpha}{(1+x^*)^2} + F(1)$$

$$= 1 - \mu - d_0 - \frac{\alpha}{(1+x^*)^2} < 1.$$

According to item (ii) of Lemma 2, $\lambda_1 = 1$, $|\lambda_2| < 1$. So, the fixed point (x_1^*, y_1^*) is non-hyperbolic.

(3) Let $\beta > \nu$, i.e., $x^* = x_{2,3}^*$. Then

$$\begin{aligned}
F(1) &= \frac{\mu x^{*2}}{(1+x^*)x^*} \left(d_0 - \frac{\beta \gamma \alpha^2}{(\gamma \mu + (\alpha + \gamma \mu)x^*)^2} \right) \\
&= \frac{\mu x^*}{1+x^*} \cdot \left(d_0 - \beta \gamma \alpha^2 / (\gamma \mu + (\alpha + \gamma \mu) \cdot \right. \\
&\quad \left. \cdot \frac{\alpha^2 \beta - \mu(d_0 \gamma \mu + (\alpha + d_0)(\alpha + \gamma \mu)) \pm \sqrt{D}}{2d_0 \mu (\alpha + \gamma \mu)} \right)^2 \Bigg) \\
&= \frac{\mu d_0 x^*}{1+x^*} \cdot \frac{D \pm \sqrt{D^2 + 4\alpha^2 \beta \gamma \mu^2 d_0 D}}{D + 2\alpha^2 \beta \gamma \mu^2 d_0 \pm \sqrt{D^2 + 4\alpha^2 \beta \gamma \mu^2 d_0}}.
\end{aligned}$$

If $x^* = x_3^*$, then $F(1) > 0$, $F(-1) > 0$, $C < 1$. According to item (i) of Lemma 2, $|\lambda_{1,2}| < 1$. Therefore, the fixed point (x_3^*, y_3^*) is attracting. If $x^* = x_2^*$, then $F(1) < 0$, $F(-1) < 0$. According to item (iii) of Lemma 2, $\lambda_1 > 1$, $\lambda_2 < -1$. So, the fixed point (x_2^*, y_2^*) is repelling.

Thus for the type of fixed points the following theorem holds.

Theorem 1: *The type of the fixed points for (6) are as follows:*

- if $\beta < v$, then the operator (6) has unique fixed point $(0,0)$ which is attracting;
 - if $\beta = v$, then the operator (6) has two fixed points $(0,0)$, (x_1^*, y_1^*) , and the point $(0,0)$ is attracting, the point (x_1^*, y_1^*) is non-hyperbolic (but (x_1^*, y_1^*) is semi-attracting¹);
 - if $\beta > v$, then the operator (6) has three fixed points $(0,0)$, (x_2^*, y_2^*) , (x_3^*, y_3^*) , and the points $(0,0)$ and (x_3^*, y_3^*) are attracting, the point (x_2^*, y_2^*) is repelling;
- where (x_1^*, y_1^*) , (x_2^*, y_2^*) and (x_3^*, y_3^*) are defined by (10).

4. Global Dynamics of the Operator W_0

In this section for any initial point $(x^{(0)}, y^{(0)}) \in \mathbb{R}_+^2$ we investigate behavior of the trajectories $(x^{(n)}, y^{(n)}) = W_0^n(x^{(0)}, y^{(0)})$, $n \geq 1$.

Let

$$\Omega = [0, \omega_1] \times [0, \omega_2] \quad (12)$$

with

$$\omega_1 = \frac{\alpha^2 \beta}{\mu d_0 (\alpha + \gamma \mu)}, \quad \omega_2 = \frac{\alpha}{\mu}.$$

Definition 3. (see [19]) The set A is called the absorbing set for the operator W_0 if it satisfies the following properties:

- invariance: the set A is invariant with respect to W_0 , i.e., $W_0(A) \subset A$;
- eventual absorption: for any initial point $(x^{(0)}, y^{(0)}) \in \mathbb{R}_+^2$, there exists $k_0 \in \mathbb{N}$ such that $W_0^k(x^{(0)}, y^{(0)}) \in A$ for all $k \geq k_0$.

Lemma 3: (Absorbing set). The set Ω is an absorbing set for the operator W_0 given by (6).

Proof: We have

$$\begin{aligned} x^{(n)} &= \frac{\beta(y^{(n-1)})^2}{\gamma + y^{(n-1)}} + \left(1 - d_0 - \frac{\alpha}{1 + x^{(n-1)}}\right) x^{(n-1)}, \\ y^{(n)} &= \frac{\alpha x^{(n-1)}}{1 + x^{(n-1)}} + (1 - \mu) y^{(n-1)}. \end{aligned} \quad (13)$$

¹ meaning that the second eigenvalue is less than 1 in absolute value.

Assume that all values of $y^{(n)}$ are greater than ω_2 . Then

$$\begin{aligned} y^{(n+1)} - y^{(n)} &= \alpha x^{(n)} / (1 + x^{(n)}) - \mu y^{(n)} < \alpha - \mu y^{(n)} \\ &= \mu(\omega_2 - y^{(n)}) < 0. \end{aligned}$$

Thus, the sequence $\{y^{(n)}\}$ is monotone decreasing. Since it is bounded below, the limit exists and satisfies

$$\lim_{n \rightarrow \infty} y^{(n)} \geq \omega_2. \quad (14)$$

Moreover, we obtain the following estimate for $y^{(n)}$:

$$\begin{aligned} y^{(n)} &= \frac{\alpha x^{(n-1)}}{1 + x^{(n-1)}} + (1 - \mu)y^{(n-1)} < \alpha + (1 - \mu)y^{(n-1)} \\ &< \alpha + (1 - \mu)(\alpha + (1 - \mu)y^{(n-2)}) \\ &< \alpha + \alpha(1 - \mu) + (1 - \mu)^2(\alpha + (1 - \mu)y^{(n-3)}) < \dots \\ &< \alpha + \alpha(1 - \mu) + \alpha(1 - \mu)^2 + \dots + \alpha(1 - \mu)^{n-1} + (1 - \mu)^n y^{(0)} \\ &= \omega_2 + (1 - \mu)^n (y^{(0)} - \omega_2). \end{aligned}$$

Thus $y^{(n)} < \omega_2 + (1 - \mu)^n (y^{(0)} - \omega_2)$. Consequently

$$\lim_{n \rightarrow \infty} y^{(n)} \leq \omega_2. \quad (15)$$

From (14) together with (15), it follows that

$$\lim_{n \rightarrow \infty} y^{(n)} = \omega_2. \quad (16)$$

From (16) and $y^{(n)} = \frac{\alpha x^{(n-1)}}{1 + x^{(n-1)}} - \mu y^{(n-1)} + y^{(n-1)}$ it follows

$$\lim_{n \rightarrow \infty} x^{(n)} = +\infty. \quad (17)$$

Now we estimate the sequence $x^{(n)}$. The inequality $y^{(n)} \leq y^{(0)}$ holds for all n , as the sequence $y^{(n)}$ is decreasing. We denote $y_0^{(0)} = \frac{\beta(y^{(0)})^2}{\gamma + y^{(0)}}$.

Then

$$\begin{aligned} x^{(n)} &= \frac{\beta(y^{(n-1)})^2}{\gamma + y^{(n-1)}} + \left(1 - d_0 - \frac{\alpha}{1 + x^{(n-1)}}\right) x^{(n-1)} \\ &< y_0^{(0)} + (1 - d_0)x^{(n-1)} \end{aligned}$$

$$\begin{aligned}
&< y_0^{(0)} + y_0^{(0)}(1-d_0) + \dots + y_0^{(0)}(1-d_0)^{n-1} + (1-d_0)^n x^{(0)} \\
&= \frac{y_0^{(0)}}{d_0} + (1-d_0)^n \left(x^{(0)} - \frac{y_0^{(0)}}{d_0} \right).
\end{aligned}$$

From condition (5) it follows $d_0 \in (0, 1)$. Consequently

$$\lim_{n \rightarrow \infty} x^{(n)} \leq \frac{y_0^{(0)}}{d_0}. \quad (18)$$

Our initial assumption is invalid based on (17) and (18). Therefore, there exists n_0 such that $y^{(n_0)}$ is not greater than ω_2 . If $y^{(n_0)} \leq \omega_2$ then $y^{(n_0+1)} < \omega_2$. Indeed,

$$\begin{aligned}
y^{(n_0+1)} &= (1-\mu)y^{(n_0)} + \frac{\alpha x^{(n_0)}}{1+x^{(n_0)}} \leq (1-\mu)\omega_2 + \frac{\alpha x^{(n_0)}}{1+x^{(n_0)}} \\
&= \omega_2 - \alpha \left(1 - \frac{x^{(n_0)}}{1+x^{(n_0)}} \right) < \omega_2.
\end{aligned}$$

Assume that all values of $x^{(n)}$ are greater than ω_1 . Then

$$\begin{aligned}
x^{(n+1)} - x^{(n)} &= \beta(y^{(n)})^2 / (\gamma + y^{(n)}) - \alpha x^{(n)} / (1+x^{(n)}) - d_0 x^{(n)} \\
&< \beta\omega_2^2 / (\gamma + \omega_2) - d_0 x^{(n)} = d_0(\omega_1 - x^{(n)}) \leq 0.
\end{aligned}$$

As $x^{(n)}$ decreases and has a lower bound, one has

$$\lim_{n \rightarrow \infty} x^{(n)} \geq \omega_1. \quad (19)$$

We next provide an estimate for $x^{(n)}$

$$\begin{aligned}
x^{(n)} &= \beta(y^{(n-1)})^2 / (\gamma + y^{(n-1)}) + (1-d_0 - \alpha / (1+x^{(n-1)}))x^{(n-1)} \\
&< \beta\omega_2^2 / (\gamma + \omega_2) + (1-d_0)x^{(n-1)} \\
&< \beta\omega_2^2 / (\gamma + \omega_2) + (1-d_0)(\beta\omega_2^2 / (\gamma + \omega_2) + (1-d_0)x^{(n-2)}) < \dots \\
&< \beta\omega_2^2 / (\gamma + \omega_2) + \beta\omega_2^2 / (\gamma + \omega_2)(1-d_0) + \dots \\
&\quad + \beta\omega_2^2 / (\gamma + \omega_2)(1-d_0)^{n-1} + (1-d_0)^n x^{(0)} \\
&= \omega_1 + (1-d_0)^n (x^{(0)} - \omega_1).
\end{aligned}$$

From $d_0 \in (0, 1)$, consequently, the inequality holds.

$$\lim_{n \rightarrow \infty} x^{(n)} \leq \omega_1. \quad (20)$$

By (19) and (20) we have

$$\lim_{n \rightarrow \infty} x^{(n)} = \omega_1. \quad (21)$$

Since $x^{(n)}$ converges to ω_1 , it follows from the second equation of (13) that

$$\lim_{n \rightarrow \infty} (y^{(n)} - (1 - \mu)y^{(n-1)}) = \frac{\alpha\omega_1}{1 + \omega_1}.$$

Since $y^{(n)}$ is bounded, it possesses upper and lower limits, denoted by $\bar{y}$ and $\underline{y}$, respectively. To arrive at a contradiction, suppose $y^{(n)}$ has no limit (i.e., $\bar{y} \neq \underline{y}$). This would necessitate the existence of two subsequences: $y^{(n_i)}$ converging to $\bar{y}$ and $y^{(n_j)}$ converging to $\underline{y}$.

$$\begin{aligned} \lim_{i \rightarrow \infty} (y^{(n_i)} - (1 - \mu)y^{(n_i-1)}) &= \frac{\alpha\omega_1}{1 + \omega_1}, \Rightarrow \\ \lim_{i \rightarrow \infty} y^{(n_i-1)} &= \frac{1}{1 - \mu} \left(\bar{y} - \frac{\alpha\omega_1}{1 + \omega_1} \right) \leq \bar{y}, \Rightarrow \bar{y} \leq \frac{\alpha\omega_1}{\mu(1 + \omega_1)}; \end{aligned}$$

$$\begin{aligned} \lim_{i \rightarrow \infty} (y^{(n_i+1)} - (1 - \mu)y^{(n_i)}) &= \frac{\alpha\omega_1}{1 + \omega_1}, \Rightarrow \\ \lim_{i \rightarrow \infty} y^{(n_i+1)} &= (1 - \mu)\bar{y} + \frac{\alpha\omega_1}{1 + \omega_1} \leq \bar{y}, \Rightarrow \bar{y} \geq \frac{\alpha\omega_1}{\mu(1 + \omega_1)}; \end{aligned}$$

$$\begin{aligned} \lim_{j \rightarrow \infty} (y^{(n_j)} - (1 - \mu)y^{(n_j-1)}) &= \frac{\alpha\omega_1}{1 + \omega_1}, \Rightarrow \\ \lim_{j \rightarrow \infty} y^{(n_j-1)} &= \frac{1}{1 - \mu} \left(\underline{y} - \frac{\alpha\omega_1}{1 + \omega_1} \right) \geq \underline{y}, \Rightarrow \underline{y} \geq \frac{\alpha\omega_1}{\mu(1 + \omega_1)}; \end{aligned}$$

$$\begin{aligned} \lim_{j \rightarrow \infty} (y^{(n_j+1)} - (1 - \mu)y^{(n_j)}) &= \frac{\alpha\omega_1}{1 + \omega_1}, \Rightarrow \\ \lim_{j \rightarrow \infty} y^{(n_j+1)} &= (1 - \mu)\underline{y} + \frac{\alpha\omega_1}{1 + \omega_1} \geq \underline{y}, \Rightarrow \underline{y} \leq \frac{\alpha\omega_1}{\mu(1 + \omega_1)}. \end{aligned}$$

Therefore, we conclude that $\bar{y} = \underline{y} = \frac{\alpha\omega_1}{\mu(1 + \omega_1)}$ i.e.,

$$\lim_{n \rightarrow \infty} y^{(n)} = \frac{\omega_1\omega_2}{1 + \omega_1}.$$

Proposition 2 shows that $\left(\omega_1, \frac{\omega_1 \omega_2}{1 + \omega_1}\right)$ is not a fixed point of system (6), contradicting our hypothesis. Therefore, there exists m_0 such that $x^{(k_0)}$ is not greater than ω_1 . If $x^{(k_0)} \leq \omega_1$ then $x^{(k_0+1)} < \omega_1$. Indeed,

$$\begin{aligned} x^{(k_0+1)} &= \frac{\beta(y^{(k_0)})^2}{\gamma + y^{(k_0)}} + \left(1 - d_0 - \frac{\alpha}{1 + x^{(k_0)}}\right)x^{(k_0)} \\ &< \frac{\beta\omega_2^2}{\gamma + \omega_2} + (1 - d_0)x^{(k_0)} \leq \frac{\beta\omega_2^2}{\gamma + \omega_2} + (1 - d_0)\omega_1 = \omega_1. \end{aligned}$$

So for any initial point $(x^{(0)}, y^{(0)}) \in \mathbb{R}_+^2$, there exists $k_0 \in \mathbb{N}$ such that $W_0^k(x^{(0)}, y^{(0)}) \in \Omega$ for all $k \geq k_0$. Furthermore, W_0 maps the set Ω into itself.

Lemma 3 establishes that Ω is an absorbing set. Consequently, we can restrict our analysis to the dynamics of the operator W_0 within the set Ω .

Proposition 3. Let $[a, b]$ and $[c, d]$ be intervals of real numbers, and let $f_1 : [a, b] \times [c, d] \rightarrow [a, b]$, and $f_2 : [a, b] \times [c, d] \rightarrow [c, d]$ be continuous functions. Consider the system of difference equation

$$\begin{aligned} x_{n+1} &= f_1(x_n, y_n), \\ y_{n+1} &= f_2(x_n, y_n), \end{aligned} \quad n = 0, 1, 2, \dots \quad (22)$$

with initial condition $(x^{(0)}, y^{(0)}) \in [a, b] \times [c, d]$. Suppose the following conditions (i) and (ii) are satisfied:

- $f_1(x, y)$ and $f_2(x, y)$ are non-decreasing in both x and y .
- If $a_1, b_1 \in [a, b]$, $c_1, d_1 \in [c, d]$ is a solution of the system of equations

$$\begin{cases} a_1 = f_1(a_1, c_1), & b_1 = f_1(b_1, d_1) \\ c_1 = f_2(a_1, c_1), & d_1 = f_2(b_1, d_1) \end{cases} \quad (23)$$

then $a_1 = b_1$ and $c_1 = d_1$.

Then there exists exactly one fixed point (x^*, y^*) of system (22), and trajectory of any initial point $(x^{(0)}, y^{(0)})$ in $[a, b] \times [c, d]$ converges to (x^*, y^*) .

Proof: By the Brouwer fixed-point theorem, the operator

$$W : [a, b] \times [c, d] \rightarrow [a, b] \times [c, d]$$

defined by

$$W(x, y) = (f_1(x, y), f_2(x, y))$$

has a fixed point (x^*, y^*) (cf. with proof of Theorem 1.16 of [12]). Since (x^*, y^*) also satisfies system (22), we can conclude that system (22) has at least one fixed point.

Consider the sequence $(x^{(n)}, y^{(n)})$ generated by the operator W for any initial point $(x^{(0)}, y^{(0)})$ within the rectangle $[a, b] \times [c, d]$. It suffices to show that

$$\lim_{n \rightarrow \infty} (x^{(n)}, y^{(n)}) = (x^*, y^*).$$

We set the initial values as

$$a_1^{(0)} = a, \quad b_1^{(0)} = b, \quad c_1^{(0)} = c, \quad d_1^{(0)} = d,$$

and for each $i \geq 0$ we define the recursive relations

$$\begin{aligned} a_1^{(i+1)} &= f_1(a_1^{(i)}, c_1^{(i)}), & b_1^{(i+1)} &= f_1(b_1^{(i)}, d_1^{(i)}), \\ c_1^{(i+1)} &= f_2(a_1^{(i)}, c_1^{(i)}), & d_1^{(i+1)} &= f_2(b_1^{(i)}, d_1^{(i)}). \end{aligned} \quad (24)$$

Then

$$\begin{aligned} a_1^{(0)} = a &\leq f_1(a_1^{(0)}, c_1^{(0)}) \leq f_1(b_1^{(0)}, d_1^{(0)}) \leq b = b_1^{(0)}, \\ c_1^{(0)} = c &\leq f_2(a_1^{(0)}, c_1^{(0)}) \leq f_2(b_1^{(0)}, d_1^{(0)}) \leq d = d_1^{(0)}, \end{aligned}$$

and from this, we can see that

$$a_1^{(0)} \leq a_1^{(1)} \leq b_1^{(1)} \leq b_1^{(0)} \quad \text{and} \quad c_1^{(0)} \leq c_1^{(1)} \leq d_1^{(1)} \leq d_1^{(0)}.$$

Following the same approach, we have

$$\begin{aligned} a_1^{(1)} &= f_1(a_1^{(0)}, c_1^{(0)}) \leq f_1(a_1^{(1)}, c_1^{(1)}) \\ &\leq f_1(b_1^{(1)}, d_1^{(1)}) \leq f_1(b_1^{(0)}, d_1^{(0)}) = b_1^{(1)}, \\ c_1^{(1)} &= f_2(a_1^{(0)}, c_1^{(0)}) \leq f_2(a_1^{(1)}, c_1^{(1)}) \\ &\leq f_2(b_1^{(1)}, d_1^{(1)}) \leq f_2(b_1^{(0)}, d_1^{(0)}) = d_1^{(1)}, \end{aligned}$$

and from this, we can see that

$$a_1^{(0)} \leq a_1^{(1)} \leq a_1^{(2)} \leq b_1^{(2)} \leq b_1^{(1)} \leq b_1^{(0)} \quad \text{and} \quad c_1^{(0)} \leq c_1^{(1)} \leq c_1^{(2)} \leq d_1^{(2)} \leq d_1^{(1)} \leq d_1^{(0)}.$$

Note that

$$a_1^{(0)} = a \leq x^{(n)} \leq b = b_1^{(0)} \quad \text{and} \quad c_1^{(0)} = c \leq y^{(n)} \leq d = d_1^{(0)} \quad \text{for all } n \geq 0.$$

For all $n \geq 0$, we have

$$\begin{aligned} a_1^{(1)} &= f_1(a_1^{(0)}, c_1^{(0)}) \leq f_1(x^{(n)}, y^{(n)}) \leq f_1(b_1^{(0)}, d_1^{(0)}) = b_1^{(1)}, \\ c_1^{(1)} &= f_2(a_1^{(0)}, c_1^{(0)}) \leq f_2(x^{(n)}, y^{(n)}) \leq f_2(b_1^{(0)}, d_1^{(0)}) = d_1^{(1)}. \end{aligned}$$

Hence, it follows that

$$a_1^{(1)} \leq x^{(n)} \leq b_1^{(1)} \text{ and } c_1^{(1)} \leq y^{(n)} \leq d_1^{(1)} \text{ for all } n \geq 1.$$

For all $n \geq 1$, the following inequalities hold:

$$\begin{aligned} a_1^{(2)} &= f_1(a_1^{(1)}, c_1^{(1)}) \leq f_1(x^{(n)}, y^{(n)}) \leq f_1(b_1^{(1)}, d_1^{(1)}) = b_1^{(2)}, \\ c_1^{(2)} &= f_2(a_1^{(1)}, c_1^{(1)}) \leq f_2(x^{(n)}, y^{(n)}) \leq f_2(b_1^{(1)}, d_1^{(1)}) = d_1^{(2)}. \end{aligned} \quad (25)$$

This implies that

$$a_1^{(2)} \leq x^{(n)} \leq b_1^{(2)} \text{ and } c_1^{(2)} \leq y^{(n)} \leq d_1^{(2)} \text{ for all } n \geq 2.$$

It follows by induction that for $i \geq 0$, the following statements are true:

- (i) $a = a_1^{(0)} \leq a_1^{(1)} \leq \dots \leq a_1^{(i-1)} \leq a_1^{(i)}$
 $\leq b_1^{(i)} \leq b_1^{(i-1)} \leq \dots \leq b_1^{(1)} \leq b_1^{(0)} = b.$
- (ii) $c = c_1^{(0)} \leq c_1^{(1)} \leq \dots \leq c_1^{(i-1)} \leq c_1^{(i)}$
 $\leq d_1^{(i)} \leq d_1^{(i-1)} \leq \dots \leq d_1^{(1)} \leq d_1^{(0)} = d.$
- (iii) $a_1^{(i)} \leq x^{(n)} \leq b_1^{(i)}$ for all $n \geq i.$
- (iv) $c_1^{(i)} \leq y^{(n)} \leq d_1^{(i)}$ for all $n \geq i.$

Assertions (i) and (ii) guarantee that the sequences $a_1^{(i)}$, $b_1^{(i)}$, $c_1^{(i)}$ and $d_1^{(i)}$ converge. Let their limits be denoted by a_1 , b_1 , c_1 , and d_1 , respectively. Then

$$a \leq a_1 \leq b_1 \leq b \text{ and } c \leq c_1 \leq d_1 \leq d.$$

By the continuity of f_1 and f_2 , we have

$$\begin{cases} a_1 = f_1(a_1, c_1), & b_1 = f_1(b_1, d_1) \\ c_1 = f_2(a_1, c_1), & d_1 = f_2(b_1, d_1) \end{cases}$$

and hence $a_1 = b_1$ and $c_1 = d_1$ from which the proof follows.

The following theorem describes the trajectory of any initial point $(x^{(0)}, y^{(0)})$ in Ω .

Theorem 2: Assume that (5) holds and let $\beta < v$. Then the trajectory converges to the unique fixed point zero for any initial point $(x^{(0)}, y^{(0)}) \in \Omega$, i.e.,

$$\lim_{n \rightarrow \infty} W_0^n(x^{(0)}, y^{(0)}) = (0, 0).$$

Proof. Let

$$f_1(x, y) = \frac{\beta y^2}{\gamma + y} + x \left(1 - d_0 - \frac{\alpha}{1+x} \right),$$

$$f_2(x, y) = \frac{\alpha x}{1+x} + (1 - \mu)y.$$

Since functions $f_1(x, y)$ and $f_2(x, y)$ satisfy all the conditions of Proposition 3, the trajectory of any initial point within the set Ω converges to zero.

4.1 Dynamics on invariant sets.

A set A is called invariant with respect to W_0 if $W_0(A) \subset A$. We denote

$$\Omega_1(x^*, y^*) = \{(x, y) \in \Omega, 0 \leq x \leq x^*, 0 \leq y \leq y^* \setminus \{(x^*, y^*)\}\},$$

$$\Omega_2(x^*, y^*) = \{(x, y) \in \Omega, x^* \leq x \leq \omega_1, y^* \leq y \leq \omega_2\} \setminus \{(x^*, y^*)\},$$

where (x^*, y^*) is the positive fixed point defined by (10).

Lemma 4: The sets Ω_1 and Ω_2 are invariant with respect to W_0 .

Proof: 1) Let $0 \leq x \leq x^*, 0 \leq y \leq y^*$. Then

$$\begin{aligned} x' - x^* &= \frac{\beta y^2}{\gamma + y} + x \left(1 - d_0 - \frac{\alpha}{1+x} \right) - x^* \\ &\leq \frac{\beta y^{*2}}{\gamma + y^*} + x^* \left(1 - d_0 - \frac{\alpha}{1+x^*} \right) - x^* \\ &= \frac{\beta y^{*2}}{\gamma + y^*} - d_0 x^* - \frac{\alpha x^*}{1+x^*} = 0, \end{aligned}$$

$$\begin{aligned}
y^* - y' &= y^* - (1 - \mu)y - \frac{\alpha x}{1 + x} \\
&\geq y^* - (1 - \mu)y^* - \frac{\alpha x^*}{1 + x^*} = \mu y^* - \frac{\alpha x^*}{1 + x^*} = 0.
\end{aligned}$$

So, $(x', y') \in W_0(\Omega_1) \subset \Omega_1$

2) Let $x^* \leq x \leq \omega_1, y^* \leq y \leq \omega_2$. Then

$$\begin{aligned}
x^* - x' &= x^* - \frac{\beta y^2}{\gamma + y} - x(1 - d_0 - \frac{\alpha}{1 + x}) \\
&\leq x^* - \frac{\beta y^{*2}}{\gamma + y^*} - x^*(1 - d_0 - \frac{\alpha}{1 + x}) \\
&= -\frac{\beta y^{*2}}{\gamma + y^*} + d_0 x^* + \frac{\alpha x^*}{1 + x} = 0,
\end{aligned}$$

$$y' - y^* = \frac{\alpha x}{1 + x} + (1 - \mu)y - y^* \geq \frac{\alpha x^*}{1 + x^*} + (1 - \mu)y^* - y^* = 0.$$

Therefore, $(x', y') \in W_0(\Omega_2) \subset \Omega_2$.

Lemma 5: The operator W_0 defined in (6) is monotone with respect to the NE ordering, i.e., is a cooperative map.

Proof: By (6) we have

$$\begin{aligned}
x'_1 - x'_2 &= \frac{\beta(y_1 - y_2)(\gamma(y_1 + y_2) + y_1 y_2)}{(\gamma + y_1)(\gamma + y_2)} + (x_1 - x_2) \left(1 - d_0 - \frac{\alpha}{(1 + x_1)(1 + x_2)} \right), \\
y'_1 - y'_2 &= \frac{\alpha(x_1 - x_2)}{(1 + x_1)(1 + x_2)} + (1 - \mu)(y_1 - y_2). \tag{26}
\end{aligned}$$

Take $(x_1, y_1) \preceq_{NE} (x_2, y_2) \in \mathbb{R}_+^2$ then $x_1 \leq x_2$ and $y_1 \leq y_2$. By the last equalities from (26) one can see that $x'_1 \leq x'_2$ and $y'_1 \leq y'_2$, i.e.,

$$(x'_1, y'_1) = W_0(x_1, y_1) \preceq_{NE} (x'_2, y'_2) = W_0(x_2, y_2).$$

This completes the proof.

The following theorem describes the trajectory of any point $(x^{(0)}, y^{(0)})$ in invariant sets $\Omega_1(x_1^*, y_1^*)$ and $\Omega_2(x_1^*, y_1^*)$.

Theorem 3: Assume that (5) holds and let $\beta = v$. For the operator W_0 and any initial point $(x^{(0)}, y^{(0)})$, the following hold

$$\lim_{n \rightarrow \infty} x^{(n)} = \begin{cases} 0, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_1(x_1^*, y_1^*), \\ x_1^*, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_2(x_1^*, y_1^*), \end{cases}$$

$$\lim_{n \rightarrow \infty} y^{(n)} = \begin{cases} 0, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_1(x_1^*, y_1^*), \\ y_1^*, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_2(x_1^*, y_1^*), \end{cases}$$

where (x_1^*, y_1^*) is the fixed point defined by (10).

Proof: For $\beta = v$ (see Theorem 1), $z_{1,1} = (0, 0)$ is attracting, while $z_{1,2} = (x_1^*, y_1^*)$ is a non-hyperbolic. By applying Proposition 1 to the ordered interval $[[z_{1,1}, z_{1,2}]]_{NE}$, we can conclude that the trajectory of an arbitrary initial point in $\Omega_1(x_1^*, y_1^*)$ converges to zero. Proposition 3 implies that the trajectory of any initial point selected from set $\Omega_2(x_1^*, y_1^*)$ converges to a limit point (x_1^*, y_1^*) . See Fig. 1 (a) for numerical values and the visual illustration.

The following theorem describes the trajectory of any point $(x^{(0)}, y^{(0)})$ in invariant sets $\Omega_1(x_2^*, y_2^*)$ and $\Omega_2(x_2^*, y_2^*)$.

Theorem 4: Assume that (5) holds and let $\beta > v$. For the operator W_0 and any initial point $(x^{(0)}, y^{(0)})$, the following hold

$$\lim_{n \rightarrow \infty} x^{(n)} = \begin{cases} 0, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_1(x_2^*, y_2^*), \\ x_3^*, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_2(x_2^*, y_2^*), \end{cases}$$

$$\lim_{n \rightarrow \infty} y^{(n)} = \begin{cases} 0, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_1(x_2^*, y_2^*), \\ y_3^*, & \text{if } (x^{(0)}, y^{(0)}) \in \Omega_2(x_2^*, y_2^*), \end{cases}$$

where (x_2^*, y_2^*) , (x_3^*, y_3^*) are the fixed point defined by (10).

Proof: Under the condition $\beta > v$ (see Theorem 1), the fixed points $z_{2,1} = (0, 0)$ and $z_{2,3} = (x_3^*, y_3^*)$ are attracting, while $z_{2,2} = (x_2^*, y_2^*)$ is a repelling fixed point. Then, by applying Proposition 1 to the ordered interval $[[z_{2,1}, z_{2,2}]]_{NE}$, we can conclude that the trajectory of an arbitrary initial point in $\Omega_1(x_2^*, y_2^*)$ converges to zero.

Recall that ω_1 and ω_2 are defined by (12). We divide the set $\Omega_2(x_2^*, y_2^*)$ into four parts:

$$\Omega^1(x_3^*, y_3^*) = \{(x, y) \in \Omega_2(x_2^*, y_2^*) : x_3^* \leq x \leq \omega_1, y_3^* \leq y \leq \omega_2\},$$

$$\Omega^2(x_3^*, y_3^*) = \{(x, y) \in \Omega_2(x_2^*, y_2^*) : x_2^* \leq x \leq x_3^*, y_3^* \leq y \leq \omega_2\},$$

$$\Omega^3(x_3^*, y_3^*) = \{(x, y) \in \Omega_2(x_2^*, y_2^*) : x_2^* \leq x \leq x_3^*, y_2^* \leq y \leq y_3^*\},$$

$$\Omega^4(x_3^*, y_3^*) = \{(x, y) \in \Omega_2(x_2^*, y_2^*) : x_3^* \leq x \leq \omega_1, y_2^* \leq y \leq y_3^*\}.$$

By applying Proposition 1 to the ordered interval $[[z_{2,2}, z_{2,3}]]_{NE'}$ we can conclude that the trajectory of an arbitrary initial point in $\Omega^3(x_3^*, y_3^*)$ converges to (x_3^*, y_3^*) . Proposition 3 implies that the trajectory of any initial point in $\Omega^1(x_3^*, y_3^*)$ converges to (x_3^*, y_3^*) .

Assume that $(x^{(0)}, y^{(0)}) \in \text{int}(\Omega^2(x_3^*, y_3^*))$ or $(x^{(0)}, y^{(0)}) \in \text{int}(\Omega^4(x_3^*, y_3^*))$.

Then, there exists $(\overline{x^{(0)}}, \overline{y^{(0)}}) \in \text{int}(\Omega^3(x_3^*, y_3^*))$ such that

$$(\overline{x^{(0)}}, \overline{y^{(0)}}) \preceq_{NE} (x^{(0)}, y^{(0)})$$

and $(\widetilde{x^{(0)}}, \widetilde{y^{(0)}}) \in \text{int}(\Omega^1(x_3^*, y_3^*))$ such that

$$(x^{(0)}, y^{(0)}) \preceq_{NE} (\widetilde{x^{(0)}}, \widetilde{y^{(0)}}).$$

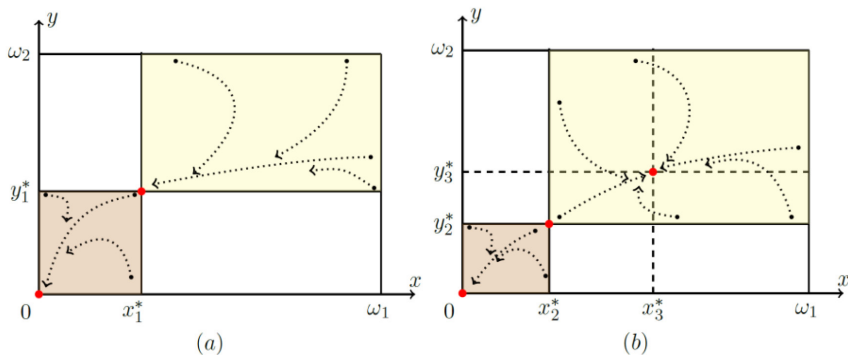


Figure 1

(a) Visual illustration of Theorem 3 when $\beta = \nu$, $\alpha = 0.4$, $\beta = \frac{9}{4}(2 + \sqrt{3})$, $\gamma = 1, d_0 = 0.5, \mu = 0.6$ in the case when (x_1^*, y_1^*) is a non-hyperbolic point. (b) Visual illustration of Theorem 4 when $\beta > \nu$, $\alpha = 0.4$, $\beta = \frac{9}{4}(2 + \sqrt{3})$, $\gamma = 1, d_0 = 0.5, \mu = 0.6$. The case when (x_2^*, y_2^*) is a repeller and $(0,0)$ and (x_3^*, y_3^*) are attracting points.

By monotonicity of W_0 , we have

$$W_0^n(\overline{x^{(0)}}, \overline{y^{(0)}}) \preceq_{NE} W_0^n(x^{(0)}, y^{(0)}) \preceq_{NE} W_0^n(\widetilde{x^{(0)}}, \widetilde{y^{(0)}})$$

which implies $W_0^n(x^{(0)}, y^{(0)}) \rightarrow (x_3^*, y_3^*)$ as $n \rightarrow \infty$. This completes the proof. See Fig.1(b) for numerical values and the visual illustration.

4.2 On the set $\Omega \setminus (\Omega_1(x^*, y^*) \cup \Omega_2(x^*, y^*))$

In the following examples, we show trajectories of initial points from the set $\Omega \setminus (\Omega_1 \cup \Omega_2)$.

Let us consider the operator W_0 with parameter values $\alpha = 0.4$, $\beta = \frac{9}{4}(2 + \sqrt{3})$, $\gamma = 1$, $d_0 = 0.5$, $\mu = 0.6$, satisfying the condition $\beta = \nu$. Then by (10), we get $(x_1^*, y_1^*) = (\frac{3\sqrt{3}}{5}, 9 - 5\sqrt{3})$.

Example 1: For initial points $z_1^{(0)} = (0.1, 0.5)$ and $z_2^{(0)} = (2.5, 0.1)$, the trajectory of the system (6) is shown in Fig.2(a). In this case, the system converges to the origin, i.e.,

$$\lim_{n \rightarrow \infty} x^{(n)} = 0, \lim_{n \rightarrow \infty} y^{(n)} = 0.$$

For initial points $z_3^{(0)} = (0.1, 0.6)$ and $z_4^{(0)} = (3, 0.1)$, the trajectory of the system (6) is also shown in Fig.2(a) (refer to the same figure for both cases). Here, the first coordinate of the trajectory converges to $\frac{3\sqrt{3}}{5}$, while the second coordinate has a limit point of $9 - 5\sqrt{3}$, i.e.

$$\lim_{n \rightarrow \infty} x^{(n)} = \frac{3\sqrt{3}}{5}, \lim_{n \rightarrow \infty} y^{(n)} = \frac{\alpha}{\mu} = 9 - 5\sqrt{3}.$$

We consider the operator W_0 with the following parameter values: $\alpha = 0.4$, $\beta = 9$, $\gamma = 1$, $d_0 = 0.5$, and $\mu = 0.6$. Notably, these parameters satisfy the condition $\beta > \nu$. By (10), we obtain the fixed points $(x_2^*, y_2^*) = (\frac{3}{5}, \frac{1}{4})$ and $(x_3^*, y_3^*) = (\frac{9}{5}, \frac{3}{7})$.

Example 2: Figure 2(b) illustrates the trajectories of system (6) for initial points $z_1^{(0)} = (0.1, 0.3)$ and $z_2^{(0)} = (1, 0.1)$. In both cases, the system converges to the origin.

Considering initial points $z_3^{(0)} = (0.5, 0.5)$, $z_4^{(0)} = (0.5, 0.66)$, $z_5^{(0)} = (2, 0.1)$, and $z_6^{(0)} = (4, 0.3)$, the trajectory of system (6) is shown in Fig. 2(b). In this case, the first coordinate of the trajectory appears to converge towards $\frac{9}{5}$, while the second coordinate appears to have a limit point of $\frac{3}{7}$.

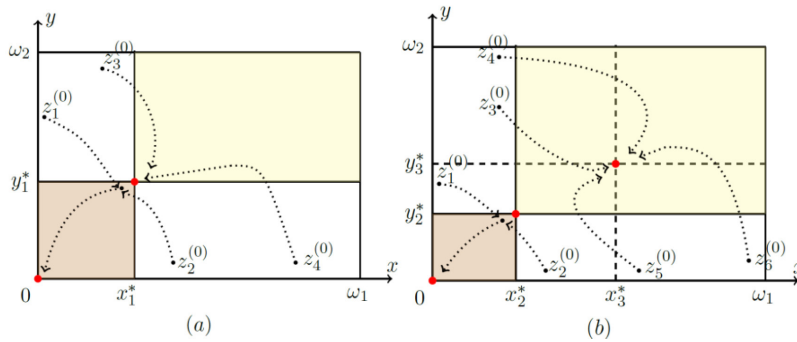


Figure 2

(a) Visual illustration of Example 1 when $\beta = \nu$, and initial points $z_1^{(0)} = (0.1, 0.5)$, $z_2^{(0)} = (2.5, 0.1)$, $z_3^{(0)} = (0.1, 0.6)$, $z_4^{(0)} = (3, 0.1)$. (b) Visual illustration of Example 2 when $\beta > \nu$, and initial points $z_1^{(0)} = (0.1, 0.3)$, $z_2^{(0)} = (1, 0.1)$, $z_3^{(0)} = (0.5, 0.5)$, $z_4^{(0)} = (0.5, 0.66)$, $z_5^{(0)} = (2, 0.1)$, $z_6^{(0)} = (4, 0.3)$.

5. Discussion and conclusion

We note that for continuous time system (2) the following results are known (see [17]): All fixed (equilibrium) points are found and their types are determined. Moreover, local behavior of the dynamical system in the neighborhood of the fixed point $(0, 0)$ is studied. In our discrete-time case, for $d_0 > 0$ we also determined types of all fixed points. Besides this we have been able to study global behavior of the system in neighborhood of $(0, 0)$. Also, we have found invariant sets and studied the dynamical system on the sets. The last results are not known for the continuous time.

References

- [1] A. Bilgin, A. Brett, M. R. S. Kulenovic, and E. Pilav, "Global dynamics of a cooperative discrete system in the plane," *International Journal of Bifurcation and Chaos*, vol. 28, no. 7 (2018), doi: 10.1142/S0218127418300227.
- [2] F. Bozkurt and A. Yousef, "Flip bifurcation and stability analysis of a fractional-order population dynamics with Allee effect," *Journal of Interdisciplinary Mathematics*, vol. 22, no. 6, pp. 1009–1029 (2019), doi: 10.1080/09720502.2019.1698403.

- [3] Z. S. Boxonov and U. A. Rozikov, "A discrete-time dynamical system of stage-structured wild and sterile mosquito population," *Nonlinear Studies*, vol. 28, no. 2, pp. 413–425 (2021).
- [4] Z. S. Boxonov and U. A. Rozikov, "Dynamical system of a mosquito population with distinct birth-death rates," *Journal of Applied Nonlinear Dynamics*, vol. 10, no. 4, pp. 807–816 (2021), doi: 10.5890/JAND.2021.12.015.
- [5] Z. S. Boxonov, "A discrete-time dynamical system of mosquito population," *Journal of Difference Equations and Applications*, vol. 29, no. 1, pp. 67–83 (2023), doi: 10.1080/10236198.2022.2160245.
- [6] Z. S. Boxonov, "Description of trajectories of an evolution operator generated by mosquito population," *Russian Journal of Nonlinear Dynamics*, vol. 20, no. 2, pp. 197–207 (2024), doi: 10.20537/nd240401.
- [7] Z. S. Boxonov, "On dynamical systems of quadratic stochastic operators constructed for bisexual populations," *Mathematical Notes*, vol. 114, no. 6, pp. 1134–1150 (2023).
- [8] Z. S. Boxonov, "Discrete dynamic system of mathematical model of coronavirus spread," *Nonlinear Studies*, vol. 27, no. 3, pp. 823–830 (2020).
- [9] E. Dancer and P. Hess, "Stability of fixed points for order preserving discrete-time dynamical systems," *Journal für die reine und angewandte Mathematik*, vol. 419, pp. 125–139 (1991).
- [10] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Westview Press (2003).
- [11] X. Fauvergue, "A review of mate-finding Allee effects in insects: from individual behavior to population management," *Ecological Entomology* (2012), doi: 10.1111/eea.12021.
- [12] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC (2005).
- [13] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Research Notes in Mathematics Series, vol. 247, Longman Scientific and Technical, Harlow; John Wiley and Sons, Inc., New York (1991).
- [14] M. R. S. Kulenovic and O. Merino, "Global bifurcation for competitive systems in the plane," *Discrete and Continuous Dynamical Systems Series B*, vol. 12, pp. 133–149 (2009).

- [15] Y. I. Lyubich, *Mathematical Structures in Population Genetics*, Springer-Verlag, Berlin (1992).
- [16] J. Li, "Malaria model with stage-structured mosquitoes," *Mathematical Biosciences and Engineering*, vol. 8, pp. 753–768 (2011).
- [17] J. Li, L. Cai, and Y. Li, "Stage-structured wild and sterile mosquito population models and their dynamics," *Journal of Biological Dynamics*, vol. 11, no. 2, pp. 79–101 (2017).
- [18] A. M. Lutambi, M. A. Penny, T. Smith, and N. Chitnis, "Mathematical modelling of mosquito dispersal in a heterogeneous environment," *Mathematical Biosciences*, vol. 241, no. 2, pp. 198–216 (2013).
- [19] L. Niu and S. Wang, "Global stability and period-doubling bifurcations of a discrete Kolmogorov predator-prey model with Ricker-type prey growth," *arXiv preprint*, arXiv:2405.11806v1 (2024).
- [20] U. A. Rozikov, *Population Dynamics: Algebraic and Probabilistic Approach*, World Scientific, Singapore (2020).
- [21] U. A. Rozikov and Z. S. Boxonov, "A discrete-time dynamical system of wild mosquito population with Allee effects," *Discontinuity, Nonlinearity, and Complexity*, vol. 13, no. 3, pp. 495–506 (2024), doi: 10.5890/DNC.2024.09.008.
- [22] U. A. Rozikov and M. V. Velasco, "A discrete-time dynamical system and an evolution algebra of mosquito population," *Journal of Mathematical Biology*, vol. 78, no. 4, pp. 1225–1244 (2019).
- [23] C. Wang and X. Li, "Stability and Neimark–Sacker bifurcation of a semi-discrete population model," *Journal of Applied Analysis and Computation*, vol. 4, no. 4, pp. 419–435 (2014).
- [24] M. Walker, J.C. Blackwood, V. Brown and L. M. Childs, "Modelling Allee effects in a transgenic mosquito population during range expansion," *Journal of Biological Dynamics*, vol. 13, no. S1, pp. 2–22 (2019), doi: 10.1080/17513758.2018.1464219.

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