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Exploring generalized *h*-open sets

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Abstract

This work introduces a new class of sets, called h-open sets, defined through an operator γ and denoted by h_γ-open sets, within the framework of a topological structure. The analysis focuses on the connection between these sets and those defined as γ-open, identifying precise criteria under which both families coincide. We also investigate properties of h_γ-open sets, such as stability under unrestricted set unions and intersections involving only a limited number of components. Furthermore, we explore the connections between various notions of γ-open sets and h-open sets. Finally, we revisit a lemma from the literature that attempts to

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characterize h-open sets and provide counterexamples, as well as identify specific conditions under which the lemma holds true.

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1. Introduction

The study of generalized open subsets within abstract topological frameworks has been an active area of research in recent years, playing a fundamental role in the development of General Topology, Mathematical Analysis, and their applications. This paper focuses on extending a family of sets known as h-open sets, introduced by Abbas [1], to generalized h-open sets using an operator γ on the power set of a topological structure (W, Γ). Open-set concepts have traditionally occupied a central place in topology, and many generalizations have emerged over time, significantly enriching the field. For instance, in 1963, Levine [9] introduced semi-open sets, followed by Mashhour, El-Monsef, and Hasanein [10] studying pre-open sets in 1984. Andrijevic further expanded these concepts by defining semi-pre-open sets in 1986 [2] and b-open sets in 1996 [3]. Building upon this rich history, in 2023, Abbas [1] proposed a novel class known as h-open sets. The study of generalized open sets has also been explored in broader topological contexts. For instance, Jamil [7] introduced generalized open sets within the framework of micro topological spaces, while Imran, Tanak, and Al-Obaidi [6] did it in neutrosophic spaces, expanding the understanding of openness in non-classical topologies.

Our work aims to further generalize the concept of h-open sets by defining h_γ-open sets and investigating their properties and relationships with other well-known open sets in the literature.

The paper is organized as follows: Section 2 introduces preliminary notions necessary to understand the development of the work, such as the concept of operators on a set W and the corresponding γ-open sets. Section 3 introduces the notion of h_γ-open subsets and presents their basic properties, such as closure under arbitrary unions and finite intersections. In Section 4, we explore the relationships between different formulations of openness involving γ-sets and h-sets. In Section 5, we revisit a lemma from the literature that attempts to characterize h-open sets, provide counterexamples, and identify specific conditions under which the lemma holds true.

Throughout the paper, we provide examples to illustrate the concepts and results, with proofs of theorems included in their respective sections. Unless explicitly stated otherwise, (W, Γ) always denotes a topological space without assumed separation axioms. Given a subset $M \subseteq W$, we use $c(M)$ and $\iota(M)$ to represent its closure and interior, respectively. These fundamental concepts will be used extensively in our exploration of generalized h-open sets.

2. Preliminaries

Suppose W is a set with at least one element, and let $\gamma: \mathcal{P}(W) \to \mathcal{P}(W)$ be an application that preserves inclusions, in the sense that $\gamma(M) \subseteq \gamma(N)$ if M is contained in N. From this point onward, we will refer to a monotone application γ as an operator on W. According to [4], a subset M of W is said to be γ-open if M is contained in its image, that is, $M \subseteq \gamma(M)$. The family of γ-open subsets of W generates a generalized topology [5] on W, in the sense that it contains the empty set and is stable under arbitrary unions. We will denote the entire collection of γ-open sets by μ_γ.

If W is a topological structure, c and ι stand for the closure and interior operators, respectively. Compositions such as $\gamma = \iota c$, $\iota c \iota$, $c\iota$, $c\iota c$, and $c\iota \cup \iota c$ generate notable subclasses of γ-open sets. These γ-open sets are known as α-open [11], pre-open [10], β-open [2], b-open [3], and semi-open [9] sets, respectively. The collection of all semi-open sets (respectively α-open, pre-open, β -open and b -open sets) is denoted by $SO(W)$, (respectively αO(W),PO(W),SPO(W),bO(W)). An application $\gamma: \mathcal{P}(W) \to \mathcal{P}(W)$ is closed under finite intersections if $\gamma(M \cap N) = \gamma(M) \cap \gamma(N)$.

Example 2.1: Suppose W is a set with at least one element and $f: W \to W$ be a function. Then the application $\gamma: \mathcal{P}(W) \to \mathcal{P}(W)$ defined as $\gamma(N) = f(f^{-1}(N))$ for all subsets N of W is an operator on W. If $W = \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$, then

(1) The set $M = [0,4]$ is γ-open.

(2) The set $N = (-4,4)$ is not γ-open.

Let W carry a topological structure, and let γ be an operator acting on W. A subset M of W is called an h-open set [1] if for all $U \in \Gamma$, $U \neq \emptyset$ and UÜW, we have $M \subseteq \iota(M \cup U)$. The family consisting of all h-open sets forms a topological structure on W that is finer than Γ.

3. *h*-Sets via Operators

In what follows, we explore the concept of h_γ-openness within the context of a topological framework. Building on the foundational definitions established in the previous sections, we define h_γ-open sets as subsets of a topological space (W, Γ) that satisfy specific inclusion properties with respect to an operator γ.

Definition 3.1: Consider a topological structure (W, Γ), and an operator γ defined on W. A subset M of W is called an h_γ-open whenever $M \subseteq \gamma(M \cup v)$ holds for all $v \in \Gamma$ with $v \neq \emptyset$ and $v \subsetneq W$.

We denote by Γ_{h_γ} the family consisting of those subsets of W that satisfy the h_γ-openness condition. If in Definition 3.1, the operator γ is equal to the interior operator, then we obtain the definition of h-open sets given by Abbas in [1]. The following theorem shows the relation between the h_γ-open sets and the γ-open sets.

Theorem 3.2: Consider a topological structure (W, Γ), and an operator γ defined on W. If a set $M \subseteq W$ fails to be h_γ-open, then it cannot be γ-open.

Proof: Suppose that M is a γ-open set and take $U \in \Gamma$ such that $U \neq \emptyset$ and $U \subsetneqq W$. Then $M \subseteq \gamma(M) \subseteq \gamma(M \cup U)$.

The following example shows the existence of an h_γ-open set that is not a γ-open set.

Example 3.3: Consider the set of real numbers $\mathbb{R}$ endowed with the cofinite topology, and consider the operator on $\mathbb{R}$ defined by $\gamma = c\iota$. Then the set $\mathbb{Q}$ is not γ-open since $\gamma(\mathbb{Q}) = \emptyset$, but $\mathbb{Q}$ is h_γ-open because given any open subset $U \subseteq \mathbb{R}$ with $U \neq \emptyset$, we obtain $\gamma(\mathbb{Q} \cup U) = \mathbb{R}$.

Theorem 3.4: *Consider a topological structure (W, Γ), and an operator γ defined on W. If $M \subsetneqq W$ is an h_γ-open set and $\iota(M) \neq \emptyset$, then M is a γ-open set.*

Proof: Suppose that $M \subsetneqq W$ is an h_γ-open set and $\iota(M) \neq \emptyset$. Then,

$$M \subseteq \gamma(M \cup i(M)) = \gamma(M).$$

We return to the Theorem 3.2, in order to find conditions where the generalized topologies Γ_{h_γ} and μ_γ are the same.

Theorem 3.5: *Let (W, Γ) be a topological space and γ be an application closed under finite intersections. If W is a Hausdorff space, then $\mu_\gamma = \Gamma_{h_\gamma}$.*

Proof: Let $M \in \Gamma_{h_\gamma}$, if $M = W$ or $M = \emptyset$, then $M \in \Gamma_d$. Suppose that $M \in \Gamma_{h_\gamma}$ and take $x \in M$ and $y \in W \backslash M$. In this case, one can find disjoint open subsets V and U such that $x \in U$ and $y \in V$. Therefore,

$$\begin{aligned} M &\subseteq \gamma(M \cup V) \cap \gamma(M \cup U) \\ &= \gamma((M \cup V) \cap (M \cup U)) \\ &= \gamma(M \cup (V \cap U)) = \gamma(M). \end{aligned}$$

The condition that W be a Hausdorff space cannot be omitted in the previous theorem. Example 3.3 provides an instance of a non-Hausdorff space where the collection of γ-open sets is properly contained in the collection of h_γ-open sets.

The following result shows that any union of h_γ-open sets remains h_γ-open, and it establishes conditions on the operator γ that guarantee stability of the h_γ-open structure under finite intersections.

Theorem 3.6: *Consider a topological structure (W, Γ), together with an operator γ defined on W.*

(1) *The collection Γ_{h_γ} of all h_γ-open sets is a generalized topology on W.*

(2) *If γ is closed under finite intersections, then the collection Γ_{h_γ} of all h_γ-open sets is closed under finite intersections.*

Proof:

(1) It is easy to see that $\emptyset \in \Gamma_{h_\gamma}$. Let $\{M_\alpha\}_{\alpha\in J}$ be an arbitrary collection of elements in Γ_{h_γ} and let U be a nonempty, proper open set. Then

$\bigcup_{\alpha\in J} M_\alpha \subseteq \bigcup_{\alpha\in J} \gamma(M_\alpha \cup U) \subseteq \gamma\left(\bigcup_{\alpha\in J} M_\alpha \cup U\right)$.

(2) Let M, N be two elements of Γ_{h_γ} and U be a nonempty, proper open set, then

$$M \cap N \subseteq \gamma(M \cup U) \cap \gamma(N \cup U) \subseteq \gamma((M \cap N) \cup U).$$

4. Relation between some γ-open sets and h-open sets

In topology, many investigations are based on finding relationships between different notions of open sets and their properties. In this section, we provide conditions on the operator γ to find relationships between various notions of open sets known in the literature and the h-open sets [1]. Recall that h-open sets are obtained from h_γ-open sets by considering $\gamma = \iota$. The idea is to use an operator γ on W and find conditions on γ that allow us to determine when a γ-open set is also an h-open set.

Theorem 4.1: Consider a topological structure (W, Γ) and an operator γ acting on subsets of W which satisfies the condition that, for every set $M \subseteq W$, the image $\gamma(M)$ is contained in the interior of M. In this setting, any subset $N \subseteq W$ that fails to be h-open cannot be γ-open.

Proof: It is straightforward

An example of an operator that satisfies the previous condition is the following: For a subset M of W, let $\gamma(M)$ be defined as the greatest open set, distinct from the empty set, that is included in M; then $\gamma(M) \subseteq \iota(M)$. In general, if the condition $\gamma(M) \subseteq \iota(M)$ for all $M \subseteq W$ is not required, the families of γ-open sets and h-open sets may exhibit no meaningful connection, as demonstrated in the following examples.

Example 4.2: Let (W, Γ) be a topological space with $W \neq \emptyset$, Γ be indiscrete topology and $\gamma = c\iota \cup \iota c$, then $\Gamma_h = \{\emptyset, W\}$ and the collection of γ-open sets is $\mathcal{P}(W)$. Due to $\gamma = c\iota \cup \iota c$, the collections of γ-open sets is the collecction of the b-open sets.

Example 4.3: Let $W = \{xa, xb, xc, xd\}$ and $\Gamma = \{\emptyset, W, \{xa, xb\}\}$.

(1) If $\gamma = c\iota \cup \iota c$ then the γ-open sets are

$bO(W) = \{\emptyset, W, \{xa, xb\}, \{xa\}, \{xb\},$
$\{xa, xc\}, \{xa, xd\}, \{xb, xc\}, \{xb, xd\},$
$\{xa, xb, xc\}, \{xa, xb, xd\}, \{xa, xc, xd\}, \{xb, xc, xd\}\};$

(2) If $\gamma = \iota c$ then the γ-open sets are
$PO(W) = \{\emptyset, W, \{xa, xb\}, \{xa\}, \{xb\},$
$\{xa, xc\}, \{xa, xd\}, \{xb, xc\}, \{xb, xd\},$
$\{xa, xb, xc\}, \{xa, xb, xd\}, \{xa, xc, xd\}, \{xb, xc, xd\}\};$

(3) If $\gamma = \iota c \iota$ then the γ-open sets are
$\alpha O(W) = \{\emptyset, W, \{xa, xb\}, \{xa, xb, xc\}, \{xa, xb, xd\}\};$

(4) If $\gamma = c \iota c$ then the γ-open sets are
$SPO(W) = \{\emptyset, W, \{xa, xb\}, \{xa\}, \{xb\},$
$\{xa, xc\}, \{xa, xd\}, \{xb, xc\}, \{xb, xd\},$
$\{xa, xb, xc\}, \{xa, xb, xd\}, \{xa, xc, xd\}, \{xb, xc, xd\}\};$

(5) The collection of h-open sets is
$\Gamma_h = \{\emptyset, W, \{xa\}, \{xb\}, \{xa, xb\}, \{xc, xd\}, \{xa, xc, xd\}, \{xb, xc, xd\}\}.$

Note that none of the operators $\gamma = c\iota$, $\iota c \iota$, ιc, c and $c\iota \cup \iota c$ satisfies the condition that $\gamma(B) \subseteq \iota(B)$ for all $B \subseteq W$. Consequently, no direct relationship exists between γ-open sets and h-open sets.

5. Some remark on h-open sets in topological space

In the article [1], page 4, the author presents Lemma 2.23, which attempts to characterize h-open sets. However, upon closer examination, we have discovered that this lemma, as stated, is not universally true. To demonstrate this, we have constructed a counterexample that disproves the general validity of the lemma. Nevertheless, the core idea behind Lemma 2.23 is not entirely without merit. We have identified specific conditions under which the lemma does hold true. These conditions involve additional constraints on the nature of the h-open sets themselves.

Lemma 5.1: ([1] Lemma 2.23). *Consider a topological space* (W, Γ)*, and take* $M \subset W$*. The following statements are equivalent:*

(1) M *is* h*-open.*

(2) *There exists an open set* $V \in \Gamma$ *such that* $M \subseteq V \subseteq c(M)$.

The following example shows us a set M that is not an h-open set; however, there exists an open set U in W such that $M \subseteq U \subseteq c(M)$.

Example 5.2: Let $W = \{xa, xb, xc, xd\}$ and $\Gamma = \{\emptyset, W, \{xa, xb\}\}$. Take $M = \{xa, xb, xd\}$, note that $M \subseteq W \subseteq c(M)$, but $M \notin \Gamma_h$. Now, note that $B = \{xc, xd\}$, is an h-open set, but there not exist $U \in \Gamma$, such that $M \subseteq W \subseteq c(M)$.

Under the assumption that $\iota(M) \neq \emptyset$, the direct implication of Lemma 5.1 follows from Lemma 3.4. Nonetheless, the converse does not hold, as demonstrated by the next example.

Example 5.3: Let $W = \{xa, xb, xc, xd\}$ and $\Gamma = \{\emptyset, W, \{xa, xb\}\}$. Take $M = \{xa, xb, xd\}$, $\iota(M) \neq \emptyset$, then for $U = W$, we have that $M \subseteq W \subseteq c(M)$, but M is not an h-open set.

Looking for conditions under which Lemma 5.1 holds true, we discovered the following result.

Theorem 5.4: *Consider a topological structure (W, Γ) and let M be a subset of W. If M is h-open and $\iota c(M) \neq \emptyset$, then we can find an open set U in W satisfying $M \subseteq U \subseteq c(M)$.*

Proof: Suppose that M is an h-open set and take $U = \iota(M \cup \iota c(M))$, then U is a nonempty open set such that $M \subseteq U \subseteq c(M)$.

In the previous result, the condition that $\iota c(M) \neq \emptyset$ cannot be removed, as shown in the example below.

Example 5.5: Consider the set of real numbers $\mathbb{R}$ endowed with the cofinite topology. For any finite set M, we have that $\iota c(M) = \emptyset$, and M is an h-open set. However, there does not exist an open set U such that $M \subseteq U \subseteq c(M)$.

The converse implication of Lemma 5.1 is by no means true. It is immediate to see that a subset M of W is pre-open if and only if there exists an open set U in W such that $M \subseteq U \subseteq c(M)$ (see [8]). Therefore, a set satisfying the condition given in Lemma 5.1 qualifies as pre-open according to the standard definition; however, it fails to be h-open by virtue of Theorem 4.1.

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