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Exploring boundedness and oscillatory behaviour in impulsive stochastic fractional differential equations : Theoretical insights and practical applications

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Abstract

This research focuses on Impulsive Stochastic Fractional Differential Equations (ISFDEs) with order $\alpha \in (0, 1)$ and discusses the role played by impulsivity and stochasticity in the system. Impulsivity means the sudden changes in the state of the system at certain moments, usually because of external shocks. On the other hand, stochasticity represents the random nature in the system, often described by stochastic processes, for example, Brownian motions. Stochastic differential equations provide a powerful framework to model randomness in dynamic systems [15]. Impulsive fractional integro-delay systems have been studied to capture sudden state changes in dynamic systems [9]. The research discusses the boundedness, oscillatory properties, and the existence of the solution for the ISFDEs. The results provide a theory for the understanding of the role played by the combination of fractional derivatives, impulsivity, and stochasticity in the system's stability and long-term behavior. The results are applicable in control theory, biology, economics, and other fields, especially in the presence of sudden changes and random elements in the system.

Subject Classification: *34A08, 60H10, 34K45, 60H15, 34D20, 93E03.*

Keywords: *Impulsive stochastic fractional differential equations, Stochastic processes, Impulses, Fractional calculus, Boundedness of solutions, Oscillatory behavior, Mathematical modeling, Fractional differential equations, Random fluctuations, Impulsive systems, Stability and existence of solutions, Control theory applications, Biological systems modeling.*

1. Introduction

Fractional calculus is a generalization of classical calculus that permits differentiation and integration of arbitrary (non-integer) order, offering enhanced capability to describe memory and hereditary effects in complex dynamical systems [1]. Fractional differential equations have been extensively studied due to their strong modeling capabilities in memory-dependent systems. [4] Fractional calculus can be considered a generalization of the conventional calculus of functions of one real variable. The main difference between these concepts is that, in the conventional calculus of functions of one real variable, the order of differentiation or integration is taken to be integer-valued, whereas in the concept of fractional calculus, the order of differentiation or integration is allowed to be real-valued. In other words, the conventional calculus of functions of one real variable is based on the concepts of integer-order differentiation and integer-order integration, whereas the concept of fractional calculus is based on the concepts of real-order differentiation and real-order integration. The key concepts of

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fractional calculus are fractional derivatives, which are a generalization of the derivative concept to non-integer orders, and fractional integrals, which are inverse processes [2]. The concepts have found numerous applications in fields such as signal processing, physics (for example, modeling processes such as diffusion and visco-elasticity), finance (for example, modeling complex systems), and engineering (for example, analyzing systems with memory or long-range interactions)[3]. The mathematical framework of fractional calculus is complex and may involve advanced mathematical tools such as complex analysis, special functions, and so on. Nonetheless, it has been found that the fractional calculus is very efficient in modeling real-world phenomena that cannot be effectively described by integer calculus. The study of fractional differential equations has received considerable attention in recent times because of their numerous applications in science and engineering [4]. These applications include rheology, self-similar media, control theory, electroanalytical chemistry, chemical physics, economics, etc. [5], among other related fields [6].

Impulsive fractional differential equations (IFDEs) integrate two basic mathematical ideas impulsiveness and fractional calculus. The IFDEs describe processes where the dynamics are characterized by both fractional derivatives and impulses at certain instants of time[7].

Properties and Advantages of Fractional Calculus

Fractional calculus extends classical calculus to include non-integer orders for derivatives and integrals. The generalization of classical calculus to fractional calculus has proven to be a powerful tool in modeling complex systems, as it provides distinctive properties and advantages that classical calculus does not offer.

Fractional Calculus Properties

Non-locality and memory: Fractional derivatives, as opposed to integer-order derivatives, naturally take the system's history into consideration, making it possible to simulate processes with long-range interactions and memory effects.

Flexibility in Modeling: Systems that exhibit anomalous diffusion, viscoelasticity, or genetic characteristics can be better described by the use of fractional operators, which may describe dynamic behaviors on different time scales.

Generality in Mathematics: Fractional calculus is a flexible tool for constructing and solving problems in various fields by relating integral equations and integer-order calculus.

Advantages of Fractional Calculus

Increased Descriptive Power

Fractional calculus can be used to describe accurately some real-world phenomena, which cannot be described by traditional calculus.

Broad Applicability

Fractional calculus finds its application in a wide range of fields, including finance, where it is used to study and predict stock market trends based on historical data, biology, where it is used to study the firing of neurons and other time-dependent physiological systems, and control theory, where it is used to create robust controllers for memory-based systems.

Integration with modern methods

Fractional calculus integrates well with stochastic processes and impulsive systems, and it offers powerful tools for studying stochastic and impulsive systems, as shown in this study. The impulse causes an instantaneous change in the system at a certain time, denoted by t_k, and it is expressed by

$$y(t_k^+) = y(t_k^-) + z_k,$$

As $y(t_k^+)$ shows the state of the system after the impulse at time t_k, $y(t_k^-)$ shows the state of the system before the impulse at time t_k, and z_k represents the magnitude of the impulse. This equation is used for discrete disturbances happening exactly at certain time points. It is also used to represent living systems, for example, how neurons fire and how bodies operate, mixing continuous processes with instantaneous events. Systems with a bursty behavior, for example, systems with on-and-off controls or impulsive external influences, can be represented with the help of IFDEs. To comprehend the solution for these equations with instantaneous jumps and fractions, it is important to have a good knowledge of fractional mathematics and impulse theory. Solving IFDEs is not easy, for it involves complex fraction-based changes and instantaneous jumps simultaneously. It is not easy to comprehend the behavior of the solution for these equations, for it involves complex instantaneous jumps and continuous fraction changes simultaneously. Several recent studies have investigated controllability of fractional stochastic systems under Hilfer derivatives [8].The equations with sudden jumps and fractions give us a "math way" to represent systems that change a bit by bit over time, and systems with quick instant changes. These equations are good to represent systems in the real world where both the effects of the past and sudden changes play big roles in the behavior of a system. Equations with a mix of fractions, sudden jumps, and chances bring together three big ideas in math and modeling: fractional math, sudden changes, and randomness[10] and [11]. These equations represent systems that have fractional derivatives, sudden changes or jumps, and chances or uncertainties. These equations show abrupt instant changes or occurrences happening at certain moments [12] [13]. It includes a chance or unpredictability component, typically represented by random processes, e.g., Brownian motion, or arbitrary noise [14]. These formulas combine fractional derivatives, impulses, and random elements, which represent the constant change of a system, as well as quick changes from impulses and unpredictable variations.

A stochastic process can be represented as:

$$y(t) = y(0) + \int_0^t f(s, y(s))ds + \sigma(t)dB(t),$$

The position of the system at time t is given by the function $y(t)$, while the initial condition of the system is given by the function o_0. The function $f(s, y(s))$ gives the deterministic part of the system's behavior, describing the normal dynamic behavior of the system. The function $\sigma(t)$ is the diffusion coefficient that determines the level of randomness in the system. Moreover, $B(t)$ is the Brownian motion that gives the random process introducing the randomness into the system, while $dB(t)$ is the random disturbance. This equation describes continuous disturbances in the system through the stochastic process $B(t)$. The expressions of the impulsive fractional stochastic equations are given in terms of the fractional derivatives, impulse functions, and stochastic integrals or differential equations. These expressions include randomness in the system's dynamics. The solutions of these equations are stochastic processes that represent continuous fractional dynamics and impulse-type changes in the system. Modeling financial markets where the occurrence of sudden events like news releases causes instant changes in the market, and the memory effect is given through the fractional calculus.Stochasticity accounts for random fluctuations in prices. Modeling biological phenomena such as neuron firing patterns or physiological processes that involve both continuous fractional evolution and stochastic fluctuations. Describing systems where sudden unpredictable events (impulses) affect the system's dynamics, coupled with inherent stochastic disturbances [16]. Solving impulsive fractional stochastic equations often requires advanced mathematical techniques due to the complexity arising from fractional derivatives, impulses, stochastic elements and machine learning [17, 18, 19].

An impulsive stochastic process combines both continuous random disturbances (from stochastic processes) and discrete shocks (from impulses):

$$dy(t) = g(t, y(t))dt + \sigma(t, y(t))dB(t) + \sum_{k=1}^{\infty} I_k \delta(t - t_k)dt,$$

Where $g(t, y(t))dt$ represents the deterministic part of the system's evolution, $\sigma(t, y(t))dB(t)$ represents continuous stochastic disturbances modeled by Brownian motion, and $\sum_{k=1}^{\infty} I_k \delta(t - t_k)dt$ represents discrete impulses occurring at specific times t_k with magnitudes I_k. This equation describes both the continuous random fluctuations and the discrete jumps, modeling a system under the influence of both kinds of perturbations. The simulation of the solutions of these equations can be computationally intensive because of the combination of fractional calculus and stochastic processes. The interpretation of the solutions of these equations can be complex because of the combination of continuous fractional dynamics, impulsive dynamics, and stochastic processes. The impulsive fractional stochastic equations are an efficient modeling tool for systems undergoing continuous fractional evolution, impulsive dynamics, and stochastic processes. These equations are applied in different fields where memory effects, impulsive dynamics, and stochastic processes are crucial in the functioning of the system [20].

The main purpose and the aim of the current paper are to make an additional contribution to the already growing literature on the subject of IFDEs and IFSEs, Controllability of fractional stochastic delay systems has been widely explored in earlier works [33]. Recent works have extended controllability results to Hilfer fractional systems with nonlocal conditions [34]. The paper deals with the application of fractional derivatives in a way that the combination of fractional derivatives with impulses and randomness enables the modeling of the system in question in as much detail as possible. Moreover, the paper attempts to provide a general description of the mathematical methods used for the solution and the interpretation of the equations in question, with the emphasis on the latest achievements in the field. The main idea here is to gain better insight into the application of the sophisticated models in question in various fields, ranging from control theory and signal processing to biology and finance.", including the complex relationship between some fractional derivatives, impulsive terms, and stochastic processes, and outlines ways to handle them [21].

$$\begin{cases} {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}\left(y(t)+r(t)y(t)\right)\right]\right)+q(t)y(t)=0, t>t_0, t\neq t_k, \alpha\in(0,1), \\ y(t_k^+)=a_k y(t_k^-), {}_{t_k}D^{\alpha}y(t_k^+)=b_{t_{k-1}}D^{\alpha}y(t_k^-), k\in\mathbb{N} \end{cases}$$

$$D_{+,t}^{\beta}u(y,t)+a(t)D_{+,t}^{\beta-1}u(y,t)=b(t)\Delta u(y,t)+ \sum_{k=1}^{m} c_k(t)\Delta u(y,t-\tau_k)-G(y,t),$$

under the impulsive condition

$$D_{+,t}^{\beta-1}u(y,t_j^+)-D_{+,t}^{\beta-1}u(y,t_j^-)=\sigma(y,t_j)D_{+,t}^{\beta-1}u(y,t_j), \quad j=1,2..,(y,t)\in\Omega\times\mathbb{R}_+, \beta\in(0,1).$$

we use boundary conditions of two types

$$\begin{aligned} \frac{\partial u(y,t)}{\partial N}+G(y,t)u(y,t) &= 0,(y,t)\in\partial\Omega\times\mathbb{R}_+, t\neq t_j \\ u(y,t) &= 0,(y,t)\in\partial\Omega\times\mathbb{R}_+, t\neq t_j, \end{aligned}$$

and

$$\begin{cases} {}^{HC}D_{t_k}^{\alpha}y(t)\in G\left(t,y(t)\right), t\in J=(t_k,t_k+1), \\ y(t_k^+)=I_k\left(y(t_k^-)\right), k\in\mathbb{N}, \\ y(1)=y_{*}. \end{cases}$$

The fractional calculus has been applied to describe memory effects with anomalous diffusion and visco-elasticity. However, most of these models concentrate only on the fractional derivatives and ignore the sudden changes (impulses) and randomness. The earlier studies, such as [22], have investigated the impulsive fractional models but did not combine both impulsivity and stochasticity. The novelty of this study is that it combines the fractional derivatives, impulsiveness, and stochasticity, providing a more general framework. Although the earlier approaches, such as Caputo or Riemann-Liouville derivatives, are capable of describing memory effects, they fail to describe the complexity caused by both impulses and randomness. This paper extends these models by proposing new solution methods, filling a gap in the existing literature. The oscillatory solutions of the given fractional impulsive differential equation are investigated in this study, which is inspired by the earlier publications [23].

$$\begin{cases} {}^{C}D_{a}^{\delta}y(t) = A(t)y(t) + G\big(t, y(t)\big) + \sigma(t, y_t)\frac{dB(t)}{dt} \\ y(t) = \phi(t); -\omega \le t < 0 \\ \triangle\, y(t_k) = z_k; k \in \mathbb{N} \end{cases} \tag{1}$$

${}^{C}D_{a}^{\delta}$ is Caputo Fractional derivative.

$$y_t = \{y(t+\theta): \omega \le \theta \le 0\}$$

$\omega \in [0, +\infty)$ can also be viewed as a stochastic process $PC([\omega, 0];, ed,$ where $\delta: [t_0, T] \times PC([\omega, 0]; \Re^s) \to \Re^s$ and $\sigma: [t_0, T] \times PC([\omega, 0]; \Re^s) \to \Re^{s\times n}$. P(t) is $n - dimensional$ the standard Brownian motion. The path of the brownian motion is continuous but now not here differentiable. Hence, the discussion of the bounded nature of the derivative of the brownian motion in a classical sense is inappropriate since the brownian motion:

$$y_{t_0} = \phi = \{\phi(\theta): -\omega \le \theta \le 0\}$$

is an F_{t_0} measurable. Further $PC([-\omega, 0]; \Re^s)$-valued random variable such that

$$\phi \in M^2([-\omega, 0]; \Re^s) \text{ where } M^2([-\omega, 0]; \Re^s)$$

denotes the family of process $\{\phi(t)\}_{t_0 \le 0} \in \mathfrak{L}^p([-\omega, 0]; \Re^s)$ such that $\mathfrak{E}\int_{-\omega}^{0} |\phi(t)|^2 dt < \infty$. However, if we consider this issue from a generalized perspective, it is possible to understand the derivative of any other stochastic process, aside from Brownian motion, with the application of stochastic calculus, where notions such as the integral and stochastic derivatives can be used. As such, it is possible to manage certain derivative aspects of stochastic processes and Brownian motion with the aid of stochastic calculus. In stochastic calculus, the derivative of Brownian motion, as it changes over time, is given as $\frac{dB(t)}{dt}$. This derivative can be described as a stochastic process, referred to as a stochastic differential, written as $dB(t)$ or $dW(t)$. However, this stochastic differential is not considered a classical derivative as discussed in deterministic calculus, and its behavior does not necessarily correlate with notions of boundedness. The behavior of $dB(t)$ or $dW(t)$ has different characteristics than those encountered by derivative functions. It is mostly used in the stochastic differential equations, where it represents the randomness or infinitesimal change in a stochastic process. In conclusion, the Brownian motions lack the classical derivative since they are not differentiable. In stochastic calculus, the Brownian motions have stochastic derivatives that are unbounded [24]. The general form of $\sigma(t, y_t)$ commonly includes the volatility or the diffusive term of the modeled stochastic differential equations. In this case, y_t denotes the stochastic process, and $\sigma(t, y_t)$ denotes the volatility coefficient. The boundedness of $\sigma(t, y_t)$ commonly depends on the nature of the function σ it symbolically represents, as well as the stochastic process y_t involved. There are a few things to consider:

- **Properties of $\boldsymbol{\sigma}$**: The boundedness of $\sigma(t, y_t)$ may rely on the nature of the function σ if σ is a bounded function, meaning its values are limited within a certain range for all t and y_t then $\sigma(t, y_t)$ would also be bounded.

- **Properties of y_t**: The behavior of the stochastic process y_t can influence the boundedness of $\sigma(t, y_t)$. Here y_t is a bounded process itself (for instance, a process confined to a finite interval or a compact set), it might contribute to the boundedness of $\sigma(t, y_t)$.
- **Stochastic Process Behavior**:The behaviour of y_t might involve moments like variance or higher-order moments that could influence the boundedness of $\sigma(t, y_t)$.

2. Preliminaries

Definition 2.1: $\Re^s$-value stochastic process $y(t)$ defined on $t_0 - \omega < t < T$ is used to find the solution of equation (1) type equations with the initial conditions $y_{t_0} = \phi = \{\phi(\theta): -\omega \leq \theta \leq 0\}$

$y(t)$ is continuous and $\{y(t)\}_{t_0 \leq t \leq T}$ is

$$y(t) = \begin{cases} \phi_0 + \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} [A(s)y(s) + g(s, y(s)) + \\ \qquad \sigma(t, y_t) dB(s)], \forall t \in (t_0, t_1) \\ \phi_0 + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} [A(s)y(s) + g(s, y(s)) + \\ \sigma(t, y_t) dB(s)], \forall t \in (t_0, t_1) \end{cases}$$

This is the answer to equation (1) above. It is a special remedy. In the event that $\tilde{x}(t)$ is not different from $y(t)$, then $P\{y(t) = \tilde{o}(t)$ for any $t_0 - \omega < t \leq T\} = 1$

Lemma 2.1: ([25]) Suppose ε, β and p be greater than zero and have constant value in which $[P(\varepsilon - 1) + 1] > 0, [P(\beta - 1) + 1] > 0$. Then

$$\int_0^t (t-s)^{p(\varepsilon-1)} s^{p(\beta-1)} ds = t^{\mu} \mathcal{B}(P(\beta-1)+1, P(\varepsilon-1)+1), t \geq 0,$$

$$\mathcal{B}(\xi, \eta) = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds,$$

$$\mu > 0, \xi > 0 \text{ and } \mu = P(\varepsilon + \beta - 1) + 1$$

Definition 2.2: ([26]) Let $\delta \in (0,1]$ and $u: [0, \infty) \to X$. The fractional integral of order δ for function u with the:

$$I_t^{\delta} v(t) = \int_0^t g_{\delta}(t-s) v(s) ds, t > 0,$$

Assume that the right-hand side of the equation is point-wise defined on $[0, \infty)$, and g_δ denotes an RL kernel.

$$g_{\delta}(t) = \frac{t^{\delta-1}}{\Gamma(\delta)}, t > 0.$$

Definition 2.3: ([27]). The Caputo fractional derivative of order δ, shown by CD_t^{δ}, is given by:

$$CD_t^{\delta} v(t) = \frac{d}{dt}\left[I_t^{1-\delta}(v(t) - v(0))\right],$$

with $g_{1-\delta}(t-s)$ the kernel of the fractional operator, where $g_{1-\delta}$

$$CD_t^\delta v(t) = \frac{d}{dt}\left(\int_0^t g_{1-\delta}(t-s)(v(t)-v(0))\,ds\right), t > 0,$$

In general, for a given function $w: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, the Caputo fractional derivative of time is given by:

$$\partial_t^\delta w(t, o) = \partial_t\left(\int_0^t g_{1-\delta}(t-s)(w(t,o)-w(0,s))\,ds\right), t > 0.$$

More generally, for $u: [0, \infty] \times \mathbb{R}^n \to \mathbb{R}^n$. Caputo fractional derivative with respect to time of function w can be written as

$$\partial_t^\delta w(t, s) = \partial_t\left(\int_0^t g_{1-\delta}(t-s)(w(t,s)-w(0,o))ds\right), t > 0.$$

Definition 2.4: The Mittag-Leffler function is a generalization of the exponential function, and it is commonly employed in the study of the fractional calculus because of the strong relation that exists between the FDE and the Mittag-Leffler function, which is given in the following relation For two parameters $(\alpha > 0)$ and $(\beta \in \mathbb{R})$, the Mittag-Leff

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, z \in \mathbb{C},$$

where $(\Gamma(\cdot))$ is the Gamma function.

Special Cases: For $(\alpha = 1)$ and $(\beta = 1)$, the Mittag-Leffler function is used to reduces the exponential function:

$$E_{1,1}(z) = e^z.$$

For $\beta = 1$, it simplifies to:

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

Relevance: The Mittag-Leffler function is of primary importance for the representation of the solutions of the fractional differential equations. In the current paper, the Mittag-Leffler function is used for the analysis of the behavior of the solutions for the Impulsive Stochastic Fractional Differential Equations.

$$E_\delta(-t^\delta A) = \int_0^\infty \mathfrak{M}_\delta(s)e^{-st^\delta A}ds,$$
$$E_{\delta,\delta}(-t^\delta A) = \int_0^\infty \delta s\mathfrak{M}_\delta(s)e^{-st^\delta A}ds,$$

where $\mathfrak{M}_\delta(\lambda)$ stands for the Mainardi Wright type function with the following

$$\mathfrak{M}_\delta(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!\Gamma(1-\delta(1+n))}.$$

Proposition 2.2:

(i) $E_{\delta,\delta}(-t^\beta A) = \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\delta,\delta}(-t^\delta \mu)(\mu I + A)^{-1}d\mu;$

(ii) $A^\delta E_{\delta,\delta}(-t^\delta A) = \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^\delta E_{\delta,\delta}(-t^\delta \mu)(\mu I + A)^{-1}d\mu.$

Proof:

(i) From the perspective of $\int_0^\infty \delta s\mathfrak{M}_\delta(s)e^{-st}ds = E_{\delta,\delta}(-t)$

$$\begin{aligned} E_{\delta,\delta}(-t^{\delta}A) &= \int_0^{\infty} \delta s \mathfrak{M}_{\delta}(s) e^{-st^{\delta}A} ds \\ &= \frac{1}{2\pi i}\int_0^{\infty} \delta s \mathfrak{M}_{\delta}(s) \int_{\Gamma\theta} e^{-\mu s t^{\delta}} (\mu I + A)^{-1} d\mu ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\delta,\delta}(-t^{\delta}\mu)(\mu I + A)^{-1} d\mu, \end{aligned}$$

here Γ_{θ} is appropriate integral route.

(ii) Similarly

$$\begin{aligned} A^{\alpha}E_{\delta,\delta}(-t^{\delta}A) &= \int_0^{\infty} \delta s \mathfrak{M}_{\delta}(s) A^{\alpha} e^{-st^{\delta}A} ds \\ &= \frac{1}{2\pi i}\int_0^{\infty} \delta s \mathfrak{M}_{\delta}(s) \int_{\Gamma\theta} \mu^{\alpha} e^{-\mu s t^{\delta}} (\mu I + A)^{-1} d\mu ds \\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^{\alpha} E_{\delta,\delta}(-\mu t^{\delta})(\mu I + A)^{-1} d\mu. \end{aligned}$$

The results immediately arise generally equivalent.

Lemma 2.3: ([18]) *The operators* $E_{delta}(-t^{delta}A)$ *and* $E_{delta,delta}(-t^{delta}A)$. *The continuity of the solution operators is guaranteed when* $t > 0$ *in the uniform operator topology. Further, the uniformly continuous on* $[r, infty)$.

Lemma 2.4: ([19]) *Let* $0 < \delta < 1$. *The following properties hold for the Mittag-Leffler function* $E_{\delta}(-t^{\delta}A)$*: For all* $uinX$*:*

$$\lim_{t\to 0^+} E_{\delta}(-t^{\delta}A)u = u.$$

for all $u \in D(A)$ *and* $t > 0$*:*

$$CD_t^{\delta} E_{\delta}(-t^{\delta}A)u = -AE_{\delta}(-t^{\delta}A)u.$$

For any $uinX$:

$$E'\delta(-t^{\delta}A)u = -t^{\delta-1}AE\delta,\delta(-t^{\delta}A)u.$$

For all $t > 0$:

$$E_{\delta}(-t^{\delta}A)u = I_t^{1-\delta}\left(t^{\delta-1}E_{\delta,\delta}(-t^{\delta}A)u\right).$$

3. Existence

Lemma 3.1: *The Mittag-Leffler function is important in the expression of the solution of Let* $\delta \in (0,1)$ *and* $\{A(t)y(t) + G(t, y(t)) + \sigma(t, y_t)\frac{dB(t)}{dt}\}: J \to \mathbb{R}$

$$\begin{cases} {}^{C}D_a^{\delta} y(t) = A(t)y(t) + G\big(t, y(t)\big) + \sigma(t, y_t)\frac{dB(t)}{dt} \\ y(t) = \phi(t); -\omega \le t < 0 \\ \triangle\, y(t_k) = z_k; k \in \mathbb{N} \end{cases} \tag{2}$$

By applying RL integral on both side of equation and put conditions we get following results

$$y(t) = \begin{cases} \phi_0 + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}[A(s)y(s) + g(s, y(s)) + \\ \qquad \sigma(t, y_t)dB(s)], \forall t \in (t_0, t_1) \\ \phi_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}[A(s)y(s) + \\ \qquad g(s, y(s)) + \sigma(t, y_t)dB(s)], \forall t \in (t_0, t_1) \end{cases} \tag{3}$$

is the solution of above equation with $x(0) = \phi(0) = \phi_0$. *It's a special kind of answer. While* $\tilde{x}(t)$ *is not different from y(t) then the function x, provided by* $P\{y(t) = \tilde{o}(t)$ *for any* $t_0 - \omega < t \leq T\} = 1$ *This is expected, a solution is considered oscillating whenever it exhibits neither positive nor negative situations. If not, it is referred to be non-oscillating. A solution is considered oscillatory if, for every* $\{\xi_k\}k \in \mathbb{N} \subset [\theta_0, \infty)$ *there's an increasing divergent sequence, That is to say* $x(\xi_k^+)x(\xi_k^-) \leq 0 \ \forall \ k \in \mathbb{N}$.

Lemma 3.2:

$$\begin{cases} {}^{C}D_a^{\delta}y(t) = A(t)y(t) + G\big(t, y(t)\big) + \sigma(t, y_t)\frac{dB(t)}{dt} \\ y(t) = \phi(t); -\omega \leq t < 0 \\ \triangle\, y(t_k) = z_k; k \in \mathbb{N} \end{cases}$$

satisfying solution.

$$y(t) = \begin{cases} \phi_0 + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}[A(s)y(s) + \\ \qquad g(s, y(s)) + \sigma(t, y_t)dB(s)], \forall t \in (t_0, t_1) \\ \phi_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}[A(s)y(s) + \\ \qquad g(s, y(s)) + \sigma(t, y_t)dB(s)], \forall t \in (t_0, t_1) \end{cases}$$

$$y(t) \ = \ \phi_0 + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)y(s)ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}g(s, y(s))ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\sigma(t, y_t)dB(s)$$

Taking Laplace on both sides

$$y(\lambda) = \frac{\phi_0}{\lambda} + \frac{1}{\lambda^{\delta}}[A(\lambda)y(\lambda)] + \frac{1}{\lambda^{\delta}}G(\lambda, u(\lambda)) + \frac{1}{\lambda^{\delta}}\sigma(\lambda, y_\lambda)dB(\lambda)$$

$$\lambda^{\delta}y(\lambda) \ = \ \phi_0\lambda^{\delta-1} + [A(\lambda)y(\lambda)] + G(\lambda, y(\lambda)) + \sigma(\lambda, y_\lambda)dB(\lambda)$$

$$(\lambda^{\delta} + A)y(\lambda) \ = \ \phi_0\lambda^{\delta-1} + G(\lambda, y(\lambda)) + \sigma(\lambda, y_\lambda)dB(\lambda)$$

$$y(\lambda) \ = \ (\lambda^{\delta} + A)^{-1}\phi_0\lambda^{\delta-1} + (\lambda^{\delta} + A)^{-1}G(\lambda, y(\lambda)) + (\lambda^{\delta} + A)^{-1}\sigma(\lambda, y_\lambda)dB(\lambda)$$

Considering the inverse of Laplace on each side

$$y(t) \ = \ \phi_0\int_a^t E_{\delta}(-(t-s)^{\delta}A)ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^{\delta}A)G(\lambda, y(\lambda))ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^{\delta}A)\sigma(\lambda, y_\lambda)dB(\lambda).$$

Similarly,

$$y(t) \ = \ (\phi_0 + \sum_{i=1}^{k} y_i)\int_a^t E_{\delta}(-(t-s)^{\delta}A)ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^{\delta}A)G(\lambda, y(\lambda))ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^{\delta}A)\sigma(\lambda, y_\lambda)dB(\lambda).$$

4. Bounded Behavior

Theorem 4.1: *Assume the* $0 < \delta < 1, P > 1, \beta > 0, P(\delta - 1) + 1 > 0, P(\beta - 1) + 1 > 0, q = \frac{p}{p-1}$ *with continuous mapping* $\{A(t)y(t) + G(t, y(t)) + \sigma(t, y_t)\frac{dB(t)}{dt}\}: J \to \mathbb{R}$ *exists such that*

$$\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)y(s)|ds \leq d \tag{4}$$

is bounded $\forall t \geq a$. *The given function f(t,x) satisfies the following conditions: For any given two parameters* $(\alpha > 0)$ *and* $(\beta \in \mathbb{R})$, *the* (P_1): *Continuity of f(t,x) exists in domain Mittag-Leffler function* $(E_{alpha,beta}(z))$ *as:* (P_2): *Both g and h are continuously and non-negative functions exists such that* $g, h: \mathbb{R}^+ := [a, \infty) \to \mathbb{R}$ *also g is increasing function.Take* $0 < \beta \leq 2 - \delta - \frac{1}{P}$

$$|G(y,t)| \leq t^{\beta-1}h(t)g(\frac{|x|}{t}), t > a, (t,o) \in D, \tag{5}$$

and

$$\int_a^\infty s^{\frac{\mu q}{p}} h^q(s)ds < \infty, \tag{6}$$

where $\mu := P(\delta + \beta - 2) + 1 \leq 0.$ (P_3):

$$\int_a^\infty \frac{d\eta}{g^q(\eta)} \to \infty. \tag{7}$$

Impulsive values satisfy the following prerequisite. (P_4): *The value* $\bar{\mathcal{M}}$ *exists in order to*

$$|\sum_{i=1}^k y_i| < \bar{\mathcal{M}}, k \in \mathbb{N} \tag{8}$$

if $y(t)$ *is a solution to (1)*

$$\limsup_{t\to\infty} \frac{|y(t)|}{t} < \infty. \tag{9}$$

(P_5) *From the above conditions of boundedness we can find bounded value of stochastic function value*

$$|\sigma(t, y_t)\frac{dB(t)}{dt}| \leq Mt^{\beta-1}h^*(t)g^*(\frac{|y_t|}{t}), t > a, (t,o) \in D \tag{10}$$

$$\int_a^\infty s^{\frac{\mu q}{p}} h^{*q}(s)ds < \infty,$$

where $\mu := P(\delta + \beta - 2) + 1 \leq 0.$

$$\int_a^\infty \frac{d\eta}{g^{*q}(\eta)} \to \infty,$$

$$\limsup_{t\to\infty} \frac{|y_t|}{t} < \infty.$$

$$\begin{aligned}|y(t)| \leq\ & |\phi_0| + |\sum_{i=1}^k y_i| + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|A(s)y(s)|ds \\ &+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|g(s, y(s))|ds \\ &+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|\sigma(s, y_s)dB(s)|, t \in (t_k, t_{k+1}].\end{aligned}$$

Apply (5), Then obtain

$$|y(t)| \leq |\phi_0| + |\sum_{i=1}^{k} y_i| + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|A(s)y(s)|ds$$
$$+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h(s)g(\frac{|y(s)|}{s})ds$$
$$+ \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h^*(s)g^*(\frac{|y_s|}{s})ds, t \in (t_k, t_{k+1}].$$

From (4) we find $\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)y(s)|ds \leq d \forall t \geq a$, Suppose

$$\mathfrak{D}(k) = |x_0| + |\sum_{i=1}^{k} y_i| + \frac{d}{\Gamma(\delta)},$$
$$|y(t)| \leq \mathfrak{D}(k)t + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h(s)g(\frac{|y(s)|}{s})ds$$
$$+\frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h^*(s)g^*(\frac{|y_s|}{s})ds,$$
$$\leq \mathfrak{D}(k)t + \frac{1}{\Gamma(\delta)}(t-a)\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g(\frac{|y(s)|}{s})ds$$
$$+ \frac{M}{\Gamma(\delta)}(t-a)\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*(\frac{|y_s|}{s})ds,$$

That results from inequality,

$$\frac{|y(t)|}{t} \leq \mathfrak{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g(\frac{|y(s)|}{s})ds$$
$$+ \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*(\frac{|y_s|}{s})ds.$$

by using

$$\mathcal{S}(t) = \mathfrak{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g\left(\frac{|y(s)|}{s}\right)ds,$$
$$\mathcal{S}^*(t) = \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*(\frac{|y_s|}{s})ds,$$
$$\frac{|y(t)|}{t} \leq \mathcal{S}(t,k), t \in (t_k, t_{k+1}]. \tag{11}$$
$$\frac{|y(t)|}{t} \leq \mathcal{S}^*(t,k), t \in (t_k, t_{k+1}]. \tag{12}$$

here both g^* *and* g *are increasing functions*

$$g\left(\frac{|y(t)|}{t}\right) \leq g(\mathcal{S}(t,k)), t \in (t_k, t_{k+1}].$$
$$g^*\left(\frac{|y(t)|}{t}\right) \leq g^*(\mathcal{S}^*(t,k)), t \in (t_k, t_{k+1}].$$

$$\begin{aligned} \mathcal{S}(t,k) &\leq 1 + \mathcal{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\varepsilon-1}s^{\gamma-1}h(s)g(\mathcal{S}(s,k))ds, \\ \mathcal{S}^*(t,k) &\leq \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\beta-1}s^{\gamma-1}h^*(s)g^*(\mathcal{S}^*(s,k))ds, t \in (t_k, t_{k+1}]. \end{aligned} \tag{13}$$

Take $\mu < \varepsilon = \delta - 1 < 1$

$$\frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\varepsilon-1}s^{\beta-1}h(s)g(\mathcal{S}(s,k))ds + \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\varepsilon-1}s^{\beta-1}h^*(s)g^*(\mathcal{S}^*(s,k))ds,$$
$$\leq \frac{1}{\Gamma(\delta)}(\int_a^t (t-s)^{p(\varepsilon-1)}s^{p(\beta-1)}ds)^{\frac{1}{p}}(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds)^{\frac{1}{q}},$$
$$+\frac{M}{\Gamma(\delta)}(\int_a^t (t-s)^{p(\varepsilon-1)}s^{p(\beta-1)}ds)^{\frac{1}{p}}(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\delta)}(\mathcal{B}t^{\mu})^{\frac{1}{p}}(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds)^{\frac{1}{q}} + \frac{M}{\Gamma(\delta)}(\mathcal{B}t^{\mu})^{\frac{1}{p}}(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds)^{\frac{1}{q}},$$

$$= \frac{1}{\Gamma(\delta)}(\mathcal{B}t^{\mu})^{\frac{1}{p}}[(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds)^{\frac{1}{q}}$$

$$+M(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds)^{\frac{1}{q}}]. t \in (t_k, t_k+1],$$

where $\mathcal{B} := \mathcal{B}(P(\beta-1)+1, P(\varepsilon-1)+1))$, $\mu = P(\delta+\beta-2)+1 \leq 0$.

By applying $\mu \leq 0$ *with* $t > s \geq a$, *it gives,*

$$\int_a^t (t-s)^{\varepsilon-1}s^{\beta-1}h(s)g(\mathcal{S}(s,k))ds \leq \frac{1}{\Gamma(\delta)}\mathcal{B}^{\frac{1}{p}}[(\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds)^{\frac{1}{q}}$$

$$+ M(\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds)^{\frac{1}{q}}], \forall t \in (t_k, t_{k+1}]. \tag{14}$$

using (14) we have,

$$(o+y)^q \leq 2^{q-1}(o^q+y^q), o, y \geq 0, q > 1.$$

For $t \in (t_k, t_{k+1}]$, *we obtain from (13) that*

$$\mathcal{S}^q(t,k) \leq 2^{q-1}((1+\mathfrak{D}(k)^q) + (\mathcal{B}^{\frac{1}{p}}\frac{1}{\Gamma(\delta)})^q[(\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds)^{\frac{1}{q}}$$

$$+M(\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds)^{\frac{1}{q}}]).$$

Setting $\mathfrak{P}_1(k) = 2^{q-1}[(1+\mathfrak{D}(k)^q)]$, $\mathfrak{Q}_1 = 2^{q-1}(\mathcal{B}^{\frac{1}{p}}\frac{1}{\Gamma(\delta)})^q$, $\mathfrak{R}_1 = 2^{q-1}M$. Then

$$\mathcal{S}^q(t,k) \leq \mathfrak{P}_1(k) + \mathfrak{Q}_1\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds + \mathfrak{R}_1\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds,$$

Denote

$$\psi(\eta) = g^q(\eta), \quad \Psi(\xi) = \int_{s_k}^{\xi}\frac{d\eta}{\psi(\eta)}, s_k = s(t_k^+, k). \tag{15}$$

$$\psi^*(\eta) = g^{*q}(\eta), \quad \Psi^*(\xi) = \int_{s_k}^{\xi}\frac{d\eta}{\psi^*(\eta)}, s_k = s(t_k^+, k). \tag{16}$$

As we know that

$$\Psi(\mathcal{S}(t,k)) = \int_{s_k}^{s(t,k)}\frac{d\eta}{\psi(\eta)}$$

$$\Psi^*(\mathcal{S}(t,k)) = \int_{s_k}^{s(t,k)}\frac{d\eta}{\psi^*(\eta)},$$

P_3 *implies that*

$$\lim_{\mathcal{S}(t,k))\to\infty}\Psi(\mathcal{S}(t,k)) = \infty$$

$$\lim_{\mathcal{S}(t,k))\to\infty}\Psi^*(\mathcal{S}(t,k)) = \infty.$$

According to Bihari Lemma ([28])

$$\mathcal{S}^q(t,k) \le L(k) := \Psi^{-1}(\Psi\mathfrak{P}_1(k) + \mathfrak{Q}_1 \int_a^t s^{\frac{\mu q}{p}} h^q(s)ds),$$
$$\mathcal{S}^q(t,k) \le L^*(k) := \Psi^{*-1}(\mathfrak{R}_1 \int_a^t s^{\frac{\mu q}{p}} h^{*q}(s))ds,$$

From axiom (P_4) *and boundedness of* $\mathfrak{P}_1(k)$. *And from* P_2 *and (15) with (16), we find result L(k),* $k \in \mathbb{N}$ *is bounded. Then*

$$\mathcal{S}^q(t,k) \le L = \sup_{k\ge1} L(k),$$
$$\mathcal{S}(t,k) \le L^{\frac{1}{q}},$$
$$\mathcal{S}(t,k) \le L^{*\frac{1}{q}},$$

From (11) and (12), we have

$$\frac{|y(t)|}{t} \le L^{\frac{1}{q}},$$
$$\frac{|y_t|}{t} \le L^{*\frac{1}{q}},$$

This leads to the conclusion that

$$\limsup_{t\to\infty} \frac{|y(t)|}{t} < \infty.$$
$$\limsup_{t\to\infty} \frac{|y_t|}{t} < \infty.$$

System stability in the face of unpredictability and abrupt changes is shown by boundedness. This is essential for applications like as preserving safe drug concentrations in biological systems, preventing economic models from going bankrupt, and guaranteeing the dependability of engineering procedures. Our study improves on conventional models by adding memory effects through the integration of fractional derivatives, giving a more accurate representation of stability and these concludes the demonstration.

5. Oscillatory Behavior

Theorem 5.1: *Let us suppose* δ, p, q, β *are constants and* μ *be defined in Theorem 4.1, and conditions are fulfilled from* $(P_1) - (P_5)$. *If for any constant* $\overline{m} \in (M\Gamma(\delta), 1 + M\Gamma(\delta))$, *then (1) refers oscillatory behavior if following conditions meet*

$$\liminf_{t\to\infty} [\overline{m}t + \int_a^t (t-s)^{\delta-1} A(s)y(s)ds] = -\infty. \quad (17)$$
$$\limsup_{t\to\infty} [\overline{m}t + \int_a^t (t-s)^{\delta-1} A(s)y(s)ds] = \infty. \quad (18)$$
$$\lim_{t\to\infty} \int_t^\infty s^{\frac{\mu q}{p}} h^q(s) g^q\left(\frac{y(s)}{s}\right) ds = 0,$$
$$\lim_{t\to\infty} \int_t^\infty s^{\frac{\mu q}{p}} h^{*q}(s) g^{*q}\left(\frac{y_s}{s}\right) ds = 0,$$

So, there is $d_1 \ge d_o$ *that satisfies*

$$0 < \int_{d_1}^{\infty} s^{\frac{\mu q}{p}} h^q(s) g^q\left(\frac{y(s)}{s}\right) ds < 1, \tag{19}$$

$$0 < \int_{d_1}^{\infty} s^{\frac{\mu q}{p}} h^{*q}(s) g^{*q}\left(\frac{y_s}{s}\right) ds < 1, \tag{20}$$

One can deduce while losing generalization as $a \le d_0 \le d_1 < t_1$.

Proof: Following the same steps as in the Theorem 4.1 evidence that one get to

$$\begin{aligned} y(t) \ \le\ & \phi_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} A(s) y(s) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} g(s, y(s)) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} \sigma(s, y_s) dB(s), t \in (t_k, t_{k+1}], k \ge 1. \end{aligned}$$

$$\begin{aligned} y(t) \ \le\ & \phi_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} A(s) y(s) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} g(s, y(s)) ds \\ & + \frac{1}{\Gamma(\delta)}\int_{d_1}^t (t-s)^{\delta-1} s^{\beta-1} h^q(s) g^q\left(\frac{y(s)}{s}\right) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} \sigma(s, y_s) dB(s) \\ & + M\frac{1}{\Gamma(\delta)}\int_{d_1}^t (t-s)^{\delta-1} s^{\beta-1} h^{*q}(s) g^{*q}\left(\frac{y_s}{s}\right) ds, \end{aligned}$$

$$\begin{aligned} y(t) \ \le\ & \phi_0 + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} A(s) y(s) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} g(s, y(s)) ds \\ & + \frac{1}{\Gamma(\delta)} t \int_{d_1}^t (t-s)^{\delta-2} s^{\beta-1} h^q(s) g^q\left(\frac{y(s)}{s}\right) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} \sigma(s, y_s) dB(s) \\ & + M\frac{1}{\Gamma(\delta)} t \int_{d_1}^t (t-s)^{\delta-2} s^{\beta-1} h^{*q}(s) g^{*q}\left(\frac{y_s}{s}\right) ds, t \in (t_k, t_{k+1}], k \ge 1 \end{aligned}$$

$$\begin{aligned} y(t) \ \le\ & \phi_0 + \bar{\mathcal{M}} + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1} A(s) y(s) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} g(s, y(s)) ds + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} \sigma(s, y_s) dB(s) \\ & + \frac{1}{\Gamma(\delta)} t \mathcal{B}^{\frac{1}{p}} \left[\left(\int_{d_1}^t s^{\frac{\theta q}{p}} h^q(s) g^q\left(\frac{y(s)}{s}\right) ds\right)^{\frac{1}{q}} + M\left(\int_{d_1}^t s^{\frac{\theta q}{p}} h^{*q}(s) g^{*q}\left(\frac{y_s}{s}\right) ds\right)^{\frac{1}{q}}\right], \end{aligned} \tag{21}$$

$$y(t) \le m + \frac{1}{\Gamma(\delta)}\left(\bar{m}t + \int_a^t (t-s)^{\delta-1} A(s) y(s) ds\right)t, \tag{22}$$

where

$$\begin{aligned} m = \ & \phi_0 + \bar{M} + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} g(s, y(s)) ds \\ & + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1} \sigma(s, y_s) dB(s). \end{aligned}$$

and

$$\bar{m}:=\bar{\mathcal{M}}\Gamma(\delta)+\Gamma(\delta)\mathcal{B}^{\frac{1}{p}}[(\int_{d_1}^{t} s^{\frac{\theta q}{p}}h^q(s)g^q(\frac{y(s)}{s})ds)^{\frac{1}{q}}$$
$$+M(\int_{d_1}^{t} s^{\frac{\theta q}{p}}h^{*q}(s)g^{*q}(\frac{y_s}{s})ds)^{\frac{1}{q}}].$$

Here m and $\bar{m}$ are constants. From (19) and (20), we have $\bar{\mathcal{M}}\Gamma(\delta)<\bar{m}<1+\bar{\mathcal{M}}\Gamma(\delta)$.

Take limit $t\to\infty$ to (22) and utilizing (17) runs opposed to the claim that $y(t)$ is going to end up positive. In the event that $y(t)<0$, by taking $y=-x$. It is simply observed, y fulfills (1) by substituting $-A(t)y(t)$ for $y(t)$ and $-f(t,-y) for\ f(t,x)$. Since the current case's proof is identical to that of the previous one, it is excluded. Oscillatory behavior reflects periodic patterns in systems, crucial for modeling phenomena like neural firing in biology, market cycles in economics, or waveform designs in signal processing. Our approach incorporates impulsive, stochastic, and fractional dynamics, providing a complete picture, which goes beyond the effects of individual models. That whole demonstration.

6. Example

Here we discuss some examples on the Impulsive Stochastic Fractional Differential equation.

Example 6.1: Consider (ISFDE) with $\frac{1}{2}$th-order

$$\begin{cases} {}^{C}D_a^{\delta}y(t)=A(t)y(t)+t^{\beta-1}h(t)g\left(\frac{|x|}{t}\right)+Mt^{\beta-1}h^*(t)g^*\left(\frac{|yt|}{t}\right), J:=[3,\infty), \delta=\frac{1}{2} \\ y(3)=\phi_0; -\omega\le t<0 \\ \Delta y(t_k)=\frac{1}{k(k+1)}; k\in\mathbb{N} \end{cases}$$

Here $\phi_0=1$, $h(t)=(t+2)^{\frac{-4}{3}}$, $g(\eta)=\frac{1}{\eta^{1/q}}$, $A(t)y(t)=-t^{\frac{-3}{2}}$, $s\ge 3$, $p=\frac{3}{2}$, $\beta=\frac{5}{6}$, M=1, a=3, from these values our satisfied results are $p(\delta-1)+1=\frac{1}{4}>0$, $p(\beta-1)+1=\frac{3}{4}>0$, $q=\frac{p}{p-1}=3$, and $\mu=p(\delta+\beta-2)+1=0$. So that results will be,

$$\begin{aligned}\frac{1}{t}\int_a^t(t-s)^{\delta-1}|A(s)y(s)|ds &= \frac{1}{t}\int_3^t (t-s)^{\frac{-1}{2}}|s^{\frac{-1}{2}}|ds \\ &= \frac{1}{t}\int_3^t t^{-1/2}(1-\frac{s}{t})^{-1/2}s^{\frac{-1}{2}}ds\end{aligned}\tag{24}$$

By setting $\theta:=\frac{s}{t}$ so above equation (24) becomes

$$\begin{aligned}\frac{1}{t}\int_a^t(t-s)^{\delta-1}|A(s)y(s)|ds &= \frac{1}{t}\int_{\frac{3}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}t^{\frac{1}{2}}t^{\frac{-1}{2}}td\theta \int_{\frac{3}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta \\ &\le \int_0^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta=\mathcal{B}(\tfrac{1}{2},\tfrac{3}{2}), t\ge 3.\end{aligned}$$

$$\int_a^{\infty} s^{\frac{\mu q}{p}}h^q(s)ds=\int_3^{\infty}(s+2)^{-4}ds=-\frac{1}{64}, and \int_a^{\infty}\frac{d\eta}{g^q(\eta)}=\int_3^{\infty}\eta d\eta\to\infty.$$

$$|\sum_{i=1}^{k} y_i| < 1, k \in \mathbb{N}$$
$$\lim_{t\to\infty}\sup\frac{|y(t)|}{t} < \infty.$$

Similarly, for the stochastic term, we select $h^*(t) = (2t+3)^{\frac{-2}{3}}$, $g^*(\eta) = \frac{1}{(2\eta)^{1/q}}$ the result is

$$\int_3^{\infty} s^{\frac{\mu q}{p}} h^{*q}(s)ds = -\frac{1}{18},$$
$$\int_a^{\infty} \frac{d\eta}{g^{*q}(\eta)} = \int_3^{\infty} 2\eta d\eta \to \infty.$$
$$\lim_{t\to\infty} Sup\frac{|y_t|}{t} < \infty.$$

$$\bar{m}t + \int_a^t (t-s)^{\delta-1}A(s)y(s)ds = \bar{m}t - t\int_{\frac{2}{t}}^{1} \theta^{-1/2}(1-\theta)^{-1/2}d\theta$$
$$= t(\bar{m} - \int_{\frac{2}{t}}^{1} \theta^{-1/2}(1-\theta)^{-1/2}d\theta)$$
$$\lim_{t\to\infty}\int_{\frac{2}{t}}^{1} \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}}d\theta = \mathcal{B}\left(\frac{1}{2},\frac{3}{2}\right).$$

From above results $M\Gamma(\delta) < \bar{m} < 1 + M\Gamma(\delta)$, we have

$$\bar{m} < \mathcal{B}\left(\frac{1}{2},\frac{3}{2}\right),$$
$$\bar{m} - \int_{\frac{2}{t}}^{1} \theta^{-1/2}(1-\theta)^{-1/2}d\theta < 0,$$

for sufficiently larger t. So,

$$\lim_{t\to\infty}(\bar{m}t + \int_a^t (t-s)^{\delta-1}A(s)y(s)ds) \to -\infty.$$

Hence, it is concluded that all necessary conditions $(P_1) - (P_5)$ with equation (17) are satisfied by equation (23). Hence the solution is oscillatory.

Examples to Verify the Hypothesis

The current section provides numerical examples which demonstrate the validity of the hypotheses established in the paper through their examination of the boundedness and oscillatory behavior of impulse stochastic fractional differential equations (ISFDEs). The hypotheses are as follows:

- **Hypothesis 1: Boundedness of Solutions.** A solution $y(t)$ of the ISFDE is said to be bounded if there exists a constant M such that:

$$|y(t)| \le M, \text{ for all } t \ge t_0.$$

- **Hypothesis 2: Oscillatory Behavior of Solutions.** The solution $y(t)$ of the ISFDE exhibits oscillatory behavior if there exists a sequence $\{t_k\}$ such that:

$$y(t_k^+)\cdot y(t_k^-) \le 0, \text{for all } k \in \mathbb{N}.$$

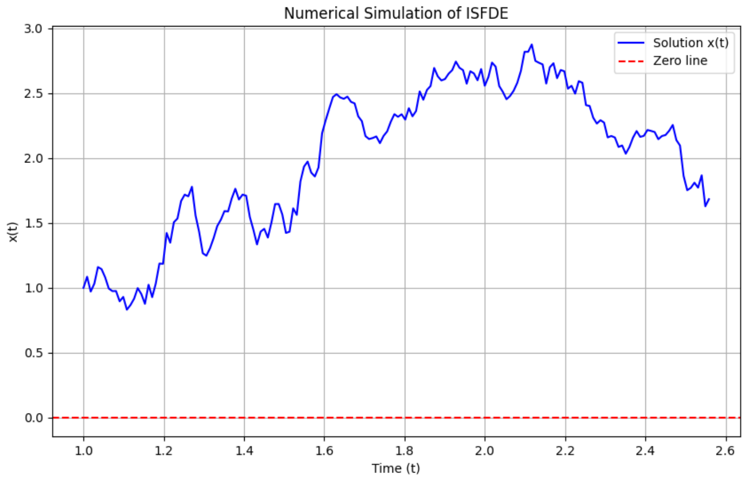

Figure 1
Numerical simulation of ISFDE

Example 6.2: Verifying Boundedness

Consider the ISFDE given by:

$$cD_a^{\delta}y(t) = A(t)y(t) + f(t, y(t)) + \sigma(t, y_t)\frac{dB(t)}{dt},$$

with:

$$A(t) = -t^{-\frac{3}{2}}, f(t, y(t)) = t^{\beta}h(t)g\left(\frac{|y(t)|}{t}\right), \sigma(t, y_t) = 1.$$

Figure (1) shows that the numerical simulation of the ISFDE confirms the boundedness for all $t \geq t_0$. We have checked the boundedness of the function $A(t) = -t^{-\frac{3}{2}}$, which is bounded for $t \geq 1$, and the solution $y(t)$ remains within a fixed range, i.e. $|y(t)| \leq M$ for all $t \geq t_0$. The solution consistently remains within a finite range. This result highlights the stability of the modeled system under stochastic influences.

Example 6.3: Verifying Oscillatory Behavior

$$cD_a^{\delta}y(t) = A(t)y(t) + \sigma(t, y_t)\frac{dB(t)}{dt},$$

with:

$$A(t) = -t^{-\frac{1}{2}}, \sigma(t, y_t) = t^{\beta-1}h(t)g\left(\frac{|y(t)|}{t}\right).$$

As can be seen from Figure (2), the solutions are oscillatory, meaning that the solutions are taking alternating positive and negative values. The above results confirm that ISFDEs can be efficiently used to model systems where the stability of the solutions is important. In order to confirm the above results, it has been checked that the solutions are satisfying the following condition: $y(t_k^+) \cdot$

$y(t_k^-) \leq 0$ for all $k \in \mathbb{N}$ and that the solutions are taking alternating positive and negative values.

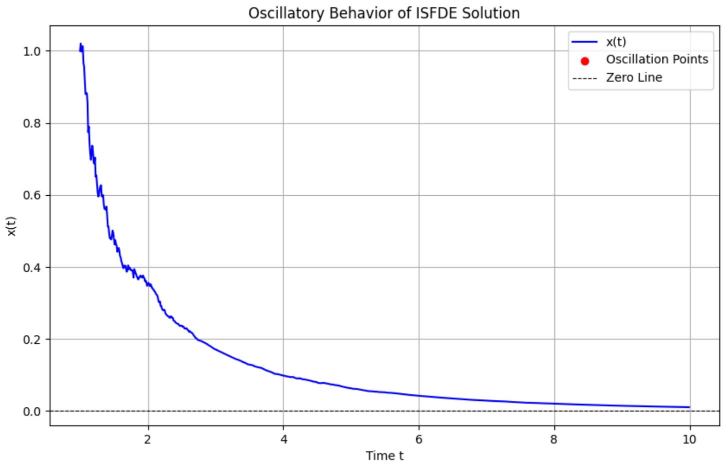

Figure 2
Oscillatory behavior of solution Consid er the ISFDE given by:

7. Limitations and future work

This study examines the boundedness and oscillatory properties of Impulsive Stochastic Fractional Differential Equations, providing important theoretical results for the theory of fractional calculus, stochastic effects, and impulsive effects. Nevertheless, there exist certain limitations to this study, providing room for improvement. The results obtained in this study are subject to certain assumptions related to the continuity, boundedness, and growth properties of the functions $f(t, y(t))$ and $\sigma(t, y_t)$, as discussed in theorem (4.1). The stochastic effects in the ISFDEs under study are restricted to standard Brownian motion or LÃ©vy processes. Adding these stochastic effects to the equations will make the model more complete and versatile. Assumption of Deterministic Impulsive Effects This study has assumed deterministic impulsive effects with given impulse times and magnitudes, which can be generalized in the future. However, in practical problems, the impulse times and magnitudes are often of stochastic nature. Although the present study has focused only on theoretical results, we have been able to obtain simulations to solve the ISFDEs, which will increase the practicality of the present study.

Future Directions

To address these limitations, our future research will focus on:

- Relaxing the assumptions related to the properties of the functions in order to generalize the results.

- Extension of the proposed framework to accommodate non-standard stochastic processes and stochastic impulses.
- Design of robust numerical methods and simulation of the models in order to validate the theoretical results.
- Extension of the ISFDEs into higher dimensions and the inclusion of different boundary conditions.

Comparison with Recent Works

This study develops a new method for Impulsive Stochastic Fractional Differential Equations, incorporating aspects of fractional calculus, impulses, and stochastic processes. In contrast to [28], which emphasizes the controllability of stochastic delay systems, this study examines the oscillations and boundedness of ISFDEs, providing a more generalized model for systems with memory and impulsive effects. As opposed to [29], which attempts to address optimal control for fractional systems, this study improves upon it by incorporating stochastic effects and employing the Mittag-Leffler function for a generalized solution. In contrast to [30], which examines the stability of impulsive systems, this study examines the oscillations of the solution.Lastly, [31, 32] deals with the study of Wiener and Fractional Brownian Motion, while our study deals with stochastic impulsive fractional equations and offers a wider perspective. The present study is in accordance with recent trends in the approximate controllability of stochastic systems of fractional order, including the application of fractional calculus and stochastic analysis. Unlike previous studies, our study deals with stochastic impulsive fractional equations using Mittag-Leffler functions and Caputo derivatives to prove sufficient conditions for the existence of solution and controllability of stochastic impulsive fractional systems, while previous studies have used Sadovskiis fixed point theorem. Moreover, our study offers a wider perspective by considering the interaction of impulsiveness, stochasticness, and fractality, which offers a wider perspective of the application of our results. Similar to Clarkes subdifferential, our study deals with Mittag-Leffler functions and Caputo derivatives to explore impulsiveness and stochasticness. Our work differs in the use of the Mittag-Leffler function and the Caputo derivative, as opposed to the Banach fixed point theorem and semigroup methods, and the Null controllability of nonlocal Hilfer fractional stochastic differential equations [35-36]. Fixed point theorems, fractional calculus, and semigroup theory are used in the investigation and derivation of sufficient conditions for the existence and controllability of fractional stochastic differential equations. The results are for mild solutions in Banach spaces and boundary controllability, with applications in fractional integropartial differential equations and nonlocal conditions [37, 38]

Conclusion

We worked with Impulse Stochastic Fractional Differential Equations (ISFDEs), which included a combination of fractional calculus, impulsive effects, and stochastic effects. The major findings of our research indicate that the equations have a unique solution, which exists throughout its domain.

Additionally, we were able to understand the nature of the solution, whether it stayed within limits or changed in a regular manner. The findings of our research indicate that it is possible to use fractional calculus as a powerful tool for understanding a combination of effects, which can be found in real-world applications. For example, understanding boundedness indicates that it is possible to use fractional calculus as a means of predicting long-term stability, as seen in a number of applications, including control systems or ecosystems. The oscillating nature of the solution is similar to periodic events, which include biological or economic patterns, as a result of external impulsive effects. The findings of our research will be beneficial in creating a better understanding of how a system can be made resistant to random effects, as well as being able to handle impulsive effects.

Our research will improve current concepts of fractional impulsive systems, and our contribution will be a part of the field of fractional calculus, as random and sudden change will be taken into account. By using the Mittag-Leffler function and Caputo's fractional derivative, we created a more flexible way of studying this system, which will help us understand its stability and dynamics.

Availability of data and materials

N/A (the study from this manuscript entails neither data generation nor analysis).

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