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Existence and stability results for a system of hybrid fractional differential problems

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Abstract

In this paper, we are concerned with a system of hybrid fractional differential equations. We begin by establishing the existence and uniqueness results using the Banach contraction principle. Then, we derive sufficient conditions for proving an existence results by using of Leray-Schauder fixed point theorem. Also, we discuss the Ulam-Hyres stability for the mentioned hybrid system. At the end, we present an example to illustrate one of our main results.

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1. Introduction

Differential problems with fractional calculus arise in many scientific disciplines, see [10, 13, 14, 15, 26]. By applying some fractional derivative operators, many authors have established uniqueness and existence results for some types of problems with arbitrary order, for instance see [5, 12, 23, 24, 25, 30]. Several scholars have studied the existence, uniqueness,of solutions of

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fractional systems. For more details, one can consult [1, 4, 8, 20, 21, 22] and the references cited therein. Also, fractional hybrid differential problems have been studied by many researchers, see for instance [6, 7, 9, 17, 18, 31]. Recently, some scholars have discussed the stability of solutions of hybrid differential equations involving fractional calculus [2, 3, 16, 27] and the references cited therein. The standard form of hybrid differential problem [11] is given by the following differential equation

$$\begin{cases} D^1\left[\frac{q(\tau)}{f(t,q(\tau))}\right] = h(\tau, q(\tau)), \tau \in \Delta, \\ \\ q(\tau_0) = q_0, q_0 \in \mathbb{R}, \end{cases}$$

where $\Delta = [0, T], f \in C(\Delta \times \mathbb{R}, \mathbb{R} - \{0\})$ and $h \in C(\Delta \times \mathbb{R}, \mathbb{R})$. Several scholars have studied fractional types of the above hybrid differential equation. For example in [31], the authors considered a the hybrid fractional differential equations with Riemann-Liouville differential operators

$$\begin{cases} D^\alpha\left[\frac{q(\tau)}{f(\tau,q(\tau))}\right] = h(\tau, q(\tau)), \tau \in \Delta, 0 < \alpha < 1, \\ \\ q(0) = 0, \end{cases}$$

where $f: \Delta \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $h: \Delta \times \mathbb{R} \Rightarrow \mathbb{R}$. In [19], the authors considered a fractional coupled hybrid problems of the forme

$$\begin{cases} D^{\alpha_1}\left[\frac{q(\tau)}{f_1(\tau,q(\tau),p(\tau))}\right] = h_1(\tau, q(\tau), p(\tau)), \\ \\ D^{\alpha_2}\left[\frac{p(t)}{f_2(\tau,q(\tau),p(\tau))}\right] = h_2(\tau, q(\tau), p(\tau)), \\ \\ q(0) = q(1) = 0, p(0) = p(1) = 0, \\ t \in \Phi := [0,1], 1 < \alpha_1, \alpha_2 \leq 2, \end{cases}$$

where $f_1, f_2: \Phi \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $h_1, h_2: \Phi \times \mathbb{R} \to \mathbb{R}$. In the current work, we study the existence, uniqueness and stability of solutions for the following system:

$$\begin{cases} D^{\alpha_1}\left(\frac{q_1(\tau)}{f_1(\tau,q_1(\tau),q_2(\tau),\ldots,q_k(\tau))}\right) = h_1(\tau, q_1(\tau), q_2(\tau), \ldots, q_k(\tau)), \\ \\ D^{\alpha_2}\left(\frac{q_2(\tau)}{f_2(\tau,q_1(\tau),q_2(\tau),\ldots,q_k(\tau))}\right) = h_2(\tau, q_1(\tau), q_2(\tau), \ldots, q_k(\tau)), \\ \vdots \\ D^{\alpha_k}\left(\frac{q_k(\tau)}{f_k(\tau,q_1(\tau),q_2(\tau),\ldots,q_k(\tau))}\right) = h_k(\tau, q_1(\tau), q_2(\tau), \ldots, q_k(\tau)), \\ \\ q_i(0) = \theta_i \int_0^{\beta_i} \varphi_i(r) q_i(r) dr, \\ t \in \Phi, 0 < \alpha_i < 1, 0 < \beta_i < 1, \theta_i \in \mathbb{R}, i \in \{1,2,\ldots,k\}, \end{cases}$$

where, D^{α_i} is the derivative in the sense of Caputo, $\varphi_i \in C([0,\beta_i])$, $f_i: \Phi \times \mathbb{R}^k \to \mathbb{R} - \{0\}$ and $h_i: \Phi \times \mathbb{R}^k \to \mathbb{R}, i \in \{1,2,\dots,k\}$.

We describe some basic notations of fractional integrals and derivatives in this section 2. Section 3 is devoted to discuss the uniqueness result for (1) using the Banach contraction principle. We also prove an existence result by means of Leray Schauder alternative in this section. In Section 4, we present the Hyers-Ulam stability of system. In last section, an example is given.

2. Preliminaries

We present some useful definitions and lemmas [28, 29]. As notation, we put: $\overline{q} = (q_1, q_2, \dots, q_k)$ and $\overline{q}(\tau) = (q_1(\tau), q_2(\tau), \dots, q_k(\tau))$. We have:

Definition 1: The Riemann-Liouville fractional integral operator of order $\vartheta \geq 0$, for $\psi \in C([a,b])$ is defined as:

$$I^{\vartheta}[\psi(\tau)] = \frac{1}{\Gamma(\vartheta)}\int_a^{\tau} (\tau - s)^{\vartheta-1}\psi(s)ds, \vartheta > 0,$$
$$I^{0}[\psi(\tau)] = \psi(\tau),$$

where $\Gamma(\vartheta) := \int_0^{\infty} e^{-u}u^{\vartheta-1}du$.

Definition 2: Let $\psi \in C^n([a,b]), n-1 < \vartheta < n, n \in N^*$. The Caputo fractional derivative for ψ is given by:

$$D^{\vartheta}[\psi(\tau)] = \frac{1}{\Gamma(n-\vartheta)}\int_a^{\tau} (\tau - s)^{n-\vartheta-1}\psi^{(n)}(s)ds.$$

Furthermore, we recall the following lemmas [25]:

Lemma 3: *Suppose that $\vartheta > 0$ and $n-1 < \vartheta < n, n \in N^*$. Thus, general solution of $D^{\vartheta}x(t) = 0$ is given by*

$$w(\tau) = e_0 + e_1\tau + e_2\tau^2 + \dots + e_{n-1}\tau^{n-1},$$

such that $e_i \in \mathbb{R}, i = \overline{0, n-1}$.

Lemma 4: *Let $\vartheta > 0$. Then*

$$I^{\vartheta}\left[D^{\vartheta}[w(\tau)]\right] = w(\tau) + e_0 + e_1\tau + e_2\tau^2 + \dots + e_{n-1}\tau^{n-1},$$

for some $e_i \in \mathbb{R}, i = \overline{0, n-1}$.

Lemma 5: *For $i = 1, \dots, k$, let $f_i \in C(\Phi \times \mathbb{R}^k, \mathbb{R} - \{0\})$ and $g_i \in C(\Phi, \mathbb{R})$. Then, the solution of the problem*

$$D^{\alpha_i}\left(\frac{q_i(\tau)}{f_i(\tau, \overline{q}(\tau))}\right) = g_i(\tau), \tau \in \Phi, 0 < \alpha_i < 1, \tag{2}$$

with

$$q_i(0) = \theta_i \int_0^{\beta_i} \varphi_i(s)q_i(s)ds, 0 < \beta_i < 1, i = 1,2,\dots,k, \tag{3}$$

with $\theta_i \int_0^{\beta_i} f_i(r, \overline{q}(r))\varphi_i(r)dr \neq f_i(0, \overline{q}(0))$, is given by $\overline{q}(\tau)$, such that

$$q_i(t) = f_i\big(t,\overline{q}(\tau)\big) \times \Big(\frac{1}{\Gamma(\alpha_i)}\int_0^{\tau} (\tau-s)^{\alpha_i-1} g_i(s)ds$$
$$+\frac{\theta_i}{\left[f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i} f_i(r,\overline{q}(r))\varphi_i(r)dr\right]} \tag{4}$$
$$\times \int_0^{\beta_i} f_i\big(r,\overline{q}(r)\big)\varphi_i(r)\left[\frac{1}{\Gamma(\alpha_i)}\int_0^{r} (r-\tau)^{\alpha_i-1} g_i(\tau)d\tau\right]dr\Big).$$

Proof: Let q_i $i \in \{1,2,\dots,k\}$ be a solution for the system. By using Lemma 4, we have

$$\frac{q_i(\tau)}{f_i(\tau,\overline{q}(\tau))} = I^{\alpha_i} g_i(\tau) + e_0, \tag{5}$$

where $e_0 \in \mathbb{R}$.

Alternatively, we have

$$q_i(\tau) = f_i\big(\tau,\overline{q}(\tau)\big)(I^{\alpha_i}[g_i(\tau)] + e_0), \tau \in \Lambda, i = 1,2,\dots,k. \tag{6}$$

Multiplying (6) by $\theta_i\varphi_i(r)$, we can write

$$\theta_i\varphi_i(r)x_i(r) = \theta_i f_i\big(r,q(r)\big)\varphi_i(r)I^{\alpha_i}[g_i(r)] + e_0\theta_i f_i\big(r,q(r)\big)\varphi_i(r). \tag{7}$$

Integrating (7) with respect to r over $(0,\beta_i)$, yields

$$\theta_i\int_0^{\beta_i} \varphi_i(r)q_i(r)dr \tag{8}$$
$$= \theta_i\int_0^{\beta_i} f_i\big(r,q(r)\big)\varphi_i(r)I^{\alpha}[g_i(r)]dr + e_0\theta_i\int_0^{\beta_i} f_i\big(r,q(r)\big)\varphi_i(r)dr.$$

Using (3) and (5), we can obtain

$$c_0\left(f_i\big(0,\overline{w}(0)\big) - \theta_i\int_0^{\beta_i} f_i\big(r,\overline{q}(r)\big)\varphi_i(r)dr\right)$$
$$= \theta_i\int_0^{\beta_i} f_i\big(r,\overline{q}(r)\big)\varphi_i(s)I^{\alpha_i}[g_i(r)]dr. \tag{9}$$

This implies that

$$c_0 = \frac{\theta_i}{\left(f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i} f_i(r,\overline{q}(r))\varphi_i(r)dr\right)}$$
$$\times\int_0^{\beta_i} f_i\big(r,\overline{q}(r)\big)\varphi_i(r)I^{\alpha}[g_i(r)]dr. \tag{10}$$

Taking into account the value of e_0 (6), we obtain (4).

3. Existence and uniqueness results

For $i \in \{1,2,\dots,k\}$, we introduce the spaces

$$W_i = \{q_i(t), i = 1,2,\dots,k : q_i \in C(\Phi,R)\}$$

endowed the norm defined by

$$\|q_i\|_{W_i} = \sup\{|q_i(t)| : \tau \in \Phi\}.$$

It is clear that for each $i \in \{1,2,\dots,k\}$, $\big(W_i, \|.\|_{W_i}\big)$ is a Banach space. The product space $\left(\prod_{i=1}^{k} W_i, \|.\|_{\prod_{i=1}^{k} W_i}\right)$ is also a Banach space with norm $\|\overline{q}\|_{\prod_{i=1}^{k} W_i} = \|q_1\|_{W_1} + \|q_2\|_{W_2} + \dots + \|q_k\|_{W_k}$.

Now, we define $\phi: \prod_{i=1}^{k} W_i \to \prod_{i=1}^{k} W_i$ by:

$$\phi\overline{q}(\tau) = \left(\phi_1\overline{q}(\tau), \phi_2\overline{q}(\tau), \ldots, \phi_k\overline{q}(\tau)\right),$$

where,

$$\begin{aligned}\phi_i\overline{q}(\tau) = f_i\left(\tau, \overline{q}(\tau)\right) \times \Big(&\tfrac{1}{\Gamma(\alpha_i)}\int_0^\tau (\tau - s)^{\alpha_i - 1} h_i\left(s, \overline{q}(s)\right)ds \\ &+ \tfrac{\theta_i}{\left[f_i(0,\overline{q}(0)) - \theta_i \int_0^{\beta_i} f_i(r,\overline{q}(r))\varphi_i(r)dr\right]} \\ &\times \int_0^{\beta_i} f_i\left(r, \overline{q}(r)\right)\varphi_i(r)\left[\tfrac{1}{\Gamma(\alpha_i)}\int_0^r (r - s)^{\alpha_i - 1} h_i\left(s, \overline{q}(s)\right)ds\right]dr\Big),\end{aligned} \tag{11}$$

for $i \in \{1,2,\ldots,k\}$.

Now, we consider the following conditions.

(A_1): There exist nonnegative constants $q_{ij}, i, j = 1,2,\ldots,k,$ such that for all $\tau \in \Phi$ and $\overline{q}, \overline{p} \in \mathbb{R}^k$, we have

$$|h_i(\tau, \overline{q}) - h_i(\tau, \overline{p})| \leq \sum_{j=1}^k \omega_{ij}\left|q_j - p_j\right|.$$

(A_2): $f_i: \Phi \times \mathbb{R}^k \to \mathbb{R} - \mathbb{R} - \{0\}$ are continuous and there exist constants $N_i > 0$ which $\tau \in \Lambda$ and $\overline{w} \in \mathbb{R}^k$, $|f_i(\tau, \overline{w})| \leq N_i, i \in \{1,2,\ldots,k\}$.

(A_3): Let for $\sigma_i > 0$ and $\delta_{ij} \geq 0$ $(i, j = 1,2,\ldots,k)$ such that, $|h_i(\tau, \overline{w})| \leq \sigma_i + \sum_{j=1}^k \delta_{ij}\left|w_j\right|$ for all $\tau \in \Phi, \overline{w} \in \mathbb{R}^k$.

Then, we set the following quantities:

$$\Sigma_i := \frac{N_i}{\Gamma(\alpha_i+1)} + \frac{N_i\beta_i^{\alpha_i+1}|\theta_i|\sup_{s\in J}|\varphi_i(r)|}{\left|f_i(0,\overline{q}(0)) - \theta_i\int_0^{\beta_i} f_i(s,\overline{q}(r))\varphi_i(r)dr\right|\Gamma(\alpha_i+2)}, \tag{12}$$

where $i \in \{1,2,\ldots,k\}$.

Theorem 6: *Let $h_i: \Phi \times \mathbb{R}^k \to \mathbb{R}, i \in \{1,2,\ldots,k\}$ be continuous functions satisfying the hypotheses (A_1) and (A_2).*

If

$$\sum_{i=1}^k \left(\Lambda_i \sum_{j=1}^k \omega_{ij}\right) < 1, \tag{13}$$

then, (1) admits only one solution on Φ.

Proof: Put $L_i := \sup_{\tau\in\Phi}|h_i(\tau, 0,0,\ldots,0)| < \infty, i = 1,2,\ldots,k$ and then define r, such that

$$\mu \geq \frac{\sum_{i=1}^k \Lambda_i L_i}{1 - \sum_{i=1}^k \Lambda_i \sum_{j=1}^k \omega_{ij}}.$$

We first show that $\phi B_\mu \subset B_\mu$, where $B_\mu = \left\{\overline{q} \in \prod_{i=1}^k W_i : \|\overline{q}\|_{\prod_{i=1}^k W_i} \leq \mu\right\}$. So thanks to (A_1), for $\overline{q} \in B_\mu$, we observe that

$$\begin{aligned}\left|h_i\left(\tau, \overline{q}(\tau)\right)\right| &\leq \left|h_i\left(\tau, \overline{q}(\tau)\right) - h_i(\tau, 0,0,\ldots,0)\right| + |h_i(\tau, 0,0,\ldots,0)| \\ &\leq \sum_{j=1}^k \omega_{ij}\mu + L_i.\end{aligned} \tag{14}$$

Hence, we get

$$\|\phi_i(\overline{q})\|_{W_i} \leq N_i \times \sup_{\tau\in\Phi}\Big\{\Big(\frac{1}{\Gamma(\alpha_i)}\int_0^\tau (\tau-s)^{\alpha_i-1}\big|h_i\big(s,\overline{q}(s)\big)\big|ds$$
$$+\frac{|\theta_i|}{\big|f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i}f_i(r,\overline{q}(r))\varphi_i(r)dr\big|} \tag{15}$$
$$\times\int_0^{\beta_i}|\varphi_i(r)|\Big[\frac{1}{\Gamma(\alpha_i)}\int_0^r (r-s)^{\alpha_i-1}\big|h_i\big(s,\overline{q}(s)\big)\big|ds\Big]dr\Big)\Big\}.$$

Using (14), we obtain

$$\|\phi_i(\overline{q})\|_{W_i}$$
$$\leq \left[\frac{N_i}{\Gamma(\alpha_i+1)}+\frac{N_i\beta_i^{\alpha_i+1}|\theta_i|\sup_{r\in\Phi}|\varphi_i(r)|}{\big|f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i}f_i(r,\overline{q}(r))\varphi_i(r)dr\big|\Gamma(\alpha_i+2)}\right] \tag{16}$$
$$\times\Big(\sum_{j=1}^k\omega_{ij}\mu+L_i\Big)$$
$$= \Lambda_i\Big(\sum_{j=1}^k\omega_{ij}\mu+L_i\Big), i=1,2,\ldots,k.$$

This implies that

$$\|\phi_i(\overline{q})\|_{W_i}\leq\Lambda_i\Big(\sum_{j=1}^k\omega_{ij}\mu+L_i\Big), i=1,2,\ldots,k. \tag{17}$$

Consequently,

$$\|\phi\overline{q}\|_{\prod_{i=1}^k W_i}\leq\mu, \tag{18}$$

which implies that $\phi B_\mu\subset B_\mu$.

Next, for $\overline{q},\overline{p}\in W$ and, for all $\tau\in\Lambda$, we get

$$\|\phi_i(\overline{q})-\phi_i(p)\|_{W_i}$$
$$\leq N_i\times\sup_{\tau\in\Phi}\Big\{\Big(\frac{1}{\Gamma(\alpha_i)}\int_0^\tau(\tau-s)^{\alpha_i-1}\big|h_i\big(s,\overline{q}(s)\big)-h_i\big(s,\overline{p}(s)\big)\big|ds$$
$$+\frac{|\theta_i|}{\big|f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i}f_i(r,\overline{q}(r))\varphi_i(r)dr\big|}\int_0^{\beta_i}|\varphi_i(r)| \tag{19}$$
$$\times\Big[\frac{1}{\Gamma(\alpha_i)}\int_0^r(r-s)^{\alpha_i-1}\big|h_i\big(s,\overline{q}(s)\big)-h_i\big(s,\overline{p}(s)\big)\big|ds\Big]dr\Big)\Big\}.$$

Thanks to (A_1), we can write

$$\|\phi_i(\overline{q})-\phi_i(\overline{p})\|_{W_i}$$
$$\leq \sum_{j=1}^k\omega_{ij}N_i\times\sup_{\tau\in\Phi}\Big\{\Big(\frac{1}{\Gamma(\alpha_i)}\int_0^\tau(\tau-s)^{\alpha_i-1}\sum_{j=1}^k\|q_j-p_j\|ds$$
$$+\frac{|\theta_i|}{\big|f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i}f_i(r,\overline{q}(r))\varphi_i(r)dr\big|}\int_0^{\beta_i}|\varphi_i(r)|$$
$$\times\Big[\frac{1}{\Gamma(\alpha_i)}\int_0^r(r-s)^{\alpha_i-1}\sum_{j=1}^k\|q_j-p_j\|ds\Big]dr\Big)\Big\} \tag{20}$$
$$\leq \sum_{j=1}^k\omega_{ij}\Big[\frac{N_i}{\Gamma(\alpha_i+1)}$$
$$+\frac{N_i\beta_i^{\alpha_i+1}|\theta_i|\sup_{r\in\Phi}|\varphi_i(r)|}{\big|f_i(0,\overline{q}(0))-\theta_i\int_0^{\beta_i}f_i(r,\overline{q}(r))\varphi_i(r)dr\big|\Gamma(\alpha_i+2)}\Big]\Big(\sum_{j=1}^k\|q_j-p_j\|\Big)$$
$$= \sum_{j=1}^k\omega_{ij}\Lambda_i\times\Big(\sum_{j=1}^k\|q_j-p_j\|\Big),$$

for $i = 1,2,\dots,k$.

This implies that

$$\|\phi(\overline{q}) - \phi(\overline{p})\|_{\prod_{i=1}^{k} W_i} \leq \sum_{i=1}^{k} \left(\Lambda_i \sum_{j=1}^{k} \omega_{ij}\right) \times \left(\sum_{j=1}^{k} \|q_j - p_j\|\right). \tag{21}$$

Since $\sum_{i=1}^{k} \left(\Lambda_i \sum_{j=1}^{k} \omega_{ij}\right) < 1$, by the given assumption, therefore ϕ is a contractive. Hence it follows, by Banach's fixed point theorem ϕ has a unique fixed point, which is a solution of system (1).

The following existence result for system (1) is based on Lemma of [15].

Theorem 7: *Let $h_i: \Lambda \times \mathbb{R}^k \to \mathbb{R}, i = 1,2,\dots,k$ be continuous functions satisfying the hypotheses (A_2) and (A_3) and also*

$$\sum_{i=1}^{k} \left(\Lambda_i \sum_{j=1}^{k} \delta_{ij}\right) < 1,$$

then (1) admits one solution at least defined on Φ.

Proof: We begin by showing that ϕ is completely continuous. The application ϕ is continuous.

Let $\Omega \subset \prod_{i=1}^{k} W_i$ be bounded. Then we can find M_i $(M_i > 0)$, such that

$$\left|h_i\left(\tau, \overline{q}(\tau)\right)\right| \leq M_i, \forall \overline{q} \in \Omega.$$

Then for any $\overline{q} \in \Omega$ and for $i = 1,2,\dots,k$, we have

$$\begin{aligned}\|\phi_i \overline{q}\|_{W_i} \leq N_i \times \Bigg(&\frac{1}{\Gamma(\alpha_i)} \int_0^{\tau} (\tau - s)^{\alpha_i - 1} \left|h_i\left(s, \overline{q}(s)\right)\right| ds \\ &+ \frac{|\theta_i|}{\left|f_i(0, \overline{q}(0)) - \theta_i \int_0^{\beta_i} f_i(r, \overline{q}(r)) \varphi_i(r) dr\right|} \\ &\times \int_0^{\beta_i} |\varphi_i(r)| \left[\frac{1}{\Gamma(\alpha_i)} \int_0^{r} (r - s)^{\alpha_i - 1} \left|h_i\left(s, \overline{q}(s)\right)\right| ds\right] dr\Bigg),\end{aligned} \tag{22}$$

Hence, for all $i = 1,2,\dots,k$, we obtain

$$\begin{aligned}&\|\phi_i \overline{q}\|_{W_i} \\ \leq \quad & M_i \times \left[\frac{N_i}{\Gamma(\alpha_i+1)} + \frac{N_i |\theta_i| \beta_i^{\alpha_i+1} \sup_{r \in \Phi} |\varphi_i(r)|}{\left|f_i(0, \overline{q}(0)) - \theta_i \int_0^{\beta_i} f_i(r, \overline{q}(r)) \varphi_i(r) dr\right| \Gamma(\alpha_i+2)}\right] \\ : \; = \; & M_i \Lambda_i,\end{aligned} \tag{23}$$

and so

$$\|\phi \overline{q}\|_{\prod_{i=1}^{k} W_i} \leq \sum_{i=1}^{k} M_i \Lambda_i := M. \tag{24}$$

Thus, it follows from (24), that $\|\phi \overline{q}\|_{\prod_{i=1}^{k} W_i} < \infty$.

Next, we demonstrate that ϕ is equicontinuous. Let $\tau_1, \tau_2 \in \Upsilon$ with $\tau_1 < \tau_2$. Then

$$\begin{aligned}|\phi_i \overline{q}(\tau_2) - \phi_i \overline{w}(\tau_1)| \leq & \frac{N_i}{\Gamma(\alpha_i)} \int_0^{\tau_1} |(\tau_2 - s)^{\alpha_i - 1} (\tau_1 - s)^{\alpha_i - 1}| \left|h_i\left(s, \overline{w}(s)\right)\right| ds \\ & \frac{N_i}{\Gamma(\alpha_i)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha_i - 1}| \left|h_i\left(s, \overline{w}(s)\right)\right| ds.\end{aligned} \tag{25}$$

for $i = 1,2,\dots,k$. Hence, we have

$$|\phi_i\overline{q}(\tau_2) - \phi_i\overline{q}(\tau_1)| \leq \frac{N_iM_i}{\Gamma(\alpha_i+1)}\left[(\tau_2-\tau_1)^{\alpha_i} + \left|\tau_2^{\alpha_i} - \tau_1^{\alpha_i}\right|\right], i = 1,2,\dots,k. \quad (26)$$

Thus,

$$\|\phi\overline{q}(\tau_2) - \phi\overline{q}(\tau_1)\|_{\prod_{i=1}^k W_i} \leq \sum_{i=1}^k \frac{N_iM_i}{\Gamma(\alpha_i+1)}\left[(\tau_2-\tau_1)^{\alpha_i} + \left|\tau_2^{\alpha_i} - \tau_1^{\alpha_i}\right|\right]. \quad (27)$$

The right hand side of (27) tend to zero independently of $\overline{q}$ as $\tau_2 \to \tau_1$. Therefore, the operator ϕ is completely continuous.

Finally, we show the set $\Upsilon = \left\{\overline{q} \in \prod_{i=1}^k W_i, \overline{q} = \lambda\phi\overline{q}, 0 < \lambda < 1\right\}$ is bounded. Let $\overline{x} \in \Upsilon$. Then, for each $\tau \in \Phi$, we have

$$q_i(t) = \lambda\phi_i\overline{q}, i = 1,2,\dots,k. \quad (28)$$

Then, for $i = 1,2,\dots,k$, we obtain

$$\begin{aligned}|q_i(t)| \leq \lambda N_i \times \Big(&\frac{1}{\Gamma(\alpha_i)}\int_0^\tau (\tau-s)^{\alpha_i-1}\left|h_i\left(s,\overline{q}(s)\right)\right|ds \\ &+ \frac{|\theta_i|}{\left|f_i(0,\overline{q}(0)) - \theta_i\int_0^{\beta_i} f_i(r,\overline{q}(r))\varphi_i(r)dr\right|} \\ &\times \int_0^{\beta_i} |\varphi_i(r)|\left[\frac{1}{\Gamma(\alpha_i)}\int_0^r (r-s)^{\alpha_i-1}\left|h_i\left(s,\overline{q}(s)\right)\right|ds\right]dr\Big).\end{aligned} \quad (29)$$

Thanks to (A_3), we can write

$$\begin{aligned}|q_i(t)| \leq N_i \times &\left(\frac{1}{\Gamma(\alpha_i+1)} + \frac{|\theta_i|\beta_i^{\alpha_i+1}\sup_{r\in\Phi}|\varphi_i(r)|}{\left|f_i(0,\overline{q}(0)) - \theta_i\int_0^{\beta_i} f_i(r,\overline{q}(s))\varphi_i(r)dr\right|\Gamma(\alpha_i+2)}\right. \\ &\left.\times\left(\sigma_i + \sum_{j=1}^k \delta_{ij}\|q_j\|\right)\right.\end{aligned} \quad (30)$$

Hence, we have

$$\|q_i\|_{W_i} \leq \Lambda_i\sigma_i + \Lambda_i\sum_{j=1}^k \delta_{ij}\|q_j\|, i = 1,2,\dots,k. \quad (31)$$

This implies that

$$\sum_{i=1}^k \|q_i\|_{W_i} \leq \sum_{i=1}^k \Lambda_i\sigma_i + \sum_{i=1}^k \Lambda_i\sum_{j=1}^k \delta_{ij}\|q_j\|, \quad (32)$$

and so

$$\|\overline{q}\|_{\prod_{i=1}^k W_i} \leq \frac{\sum_{i=1}^k \Lambda_i\sigma_i}{\Pi}, \quad (33)$$

where $\Pi = \min_{1\leq j\leq n}\left(1 - \sum_{i=1}^k\left(\Lambda_i\sum_{j=1}^k \delta_{ij}\right)\right)$. This shows that Υ is bounded. Consequently, system (1) admits a solution.

4. Stability results

In this part, we will define and study the Ulam-Hyers stability (UH-stability) of the fractional hybrid system (1). For $\tau \in \Phi$, we have:

$$\begin{cases} \left| D^{\alpha_1} \left(\frac{q_1(\tau)}{f_1(\tau, w_1(\tau), w_2(\tau), \dots, w_k(\tau))} \right) - h_1(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)) \right| \le \upsilon_1, \\ \left| D^{\alpha_2} \left(\frac{q_2(\tau)}{f_2(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau))} \right) - h_2(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)) \right| \le \upsilon_2, \\ \vdots \\ \left| D^{\alpha_k} \left(\frac{q_k(\tau)}{f_k(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau))} \right) - h_k(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)) \right| \le \upsilon_k, \end{cases} \quad (34)$$

where $\upsilon_i, i = 1,2\dots, k.$ are positive real numbers.

Definition 8: Fractional hybrid system (1) is UH-stable if $\forall\ \upsilon > 0$ which $\upsilon = \max\{\upsilon_i, i = 1,2\dots, k\} > 0$ and for each solution $\overline{p} \in \prod_{i=1}^{k} W_i$ of (34), there exists a solution $\overline{q} \in \prod_{i=1}^{k} W_i$ of the system (1) and $\exists\ \eta_{f_i,h_i} > 0$ with

$$\|\overline{p} - \overline{q}\|_{\prod_{i=1}^{k} W_i} \le \eta_{f_i,h_i}\upsilon, i = 1,2\dots, k.$$

Definition 9: Fractional hybrid system (1) is generalized UH-stable if there exists $\psi_{f_i,h_i} \in C(\mathbb{R}_+, \mathbb{R}_+), \psi_{f_i,h_i}(0) = 0, i = 1,2\dots, k$ such that for each solution $\overline{p} \in \prod_{i=1}^{k} W_i$ of (34) there exists a solution $\overline{q} \in \prod_{i=1}^{k} W_i$ of the system (1) with

$$\|\overline{p} - \overline{q}\|_{\prod_{i=1}^{k} X_i} \le \psi_{f_i,h_i}(\upsilon), i = 1,2\dots, k.$$

Theorem 10: *Assume that hypotheses* $(A_j), j = 1,2$ *hold. If*

$$\sum_{i=1}^{k} \frac{N_i}{\Gamma(\alpha_i+1)} < \frac{1}{\sum_{j=1}^{k} \omega_{ij}}, \quad (35)$$

then the fractional hybrid system (1) is UH-stable and consequently, generalized UH-stable.

Proof: Let $\overline{p} \in \prod_{i=1}^{k} W_i$ is a solution of the inequality (34) and let $\overline{q} \in \prod_{i=1}^{k} W_i$ represent the unique solution of the problem

$$\begin{cases} D^{\alpha_1} \left(\dfrac{q_1(\tau)}{f_1(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau))} \right) = h_1(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)), \\ D^{\alpha_2} \left(\dfrac{q_2(\tau)}{f_2(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau))} \right) = h_2(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)), \\ \vdots \\ D^{\alpha} \left(\dfrac{q_k(\tau)}{f_k(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau))} \right) = h_k(\tau, q_1(\tau), q_2(\tau), \dots, q_k(\tau)), \\ q_i(0) = p_i(0), i = 1,2, \dots, k, \end{cases}$$

Applying Lemma 5, we can write

$$q_i(\tau) = f_i(\tau, \overline{q}(\tau))\left(I^{\alpha_i}[g_i^{q_i}(\tau)] + c_0\right), i = 1,2, \dots, k,$$

where $g_i^{q_i}(\tau) = h_i(t, q_1(\tau), q_2(\tau), \dots, q_k(\tau)), i = 1,2, \dots, k.$

By integration of the (34), we obtain

$$\left|p_i(t) - f_i\big(\tau,\overline{q}(\tau)\big)\big(I^{\alpha_i}\big[g_i^{q_i}(\tau)\big] + d_0\big)\right| \leq \frac{\upsilon_i \tau^{\alpha_i}}{\Gamma(\alpha_i+1)} \leq \frac{\upsilon_i}{\Gamma(\alpha_i+1)}, i = 1,2,\dots,k.$$

Thanks to $(A_j), j = 1,2$, we can write

$$|p_i(\tau) - q_i(\tau)| \leq \left|p_i(\tau) - f_i\big(\tau,\overline{q}(\tau)\big)\big(I^{\alpha_i}\big[g_i^{q_i}(\tau)\big] + d_0\big)\right|$$

$$+\left|f_i\big(\tau,\overline{q}(\tau)\big)\right| I^{\alpha_i}\left[\left|g_i^{p_i}(\tau) - g_i^{q_i}(\tau)\right|\right]$$

$$\leq \quad \frac{\upsilon_i}{\Gamma(\alpha_i+1)} + \frac{N_i}{\Gamma(\alpha_i+1)} \sum_{j=1}^{k} \omega_{ij} \left|p_j(s) - q_j(s)\right| ds, i = 1,2,\dots,k,$$

This implies that

$$\sum_{i=1}^{k} \|p_i - q_i\|_{W_i} \leq \sum_{i=1}^{k} \frac{\upsilon_i}{\Gamma(\alpha_i+1)} + \sum_{i=1}^{k} \frac{N_i}{\Gamma(\alpha_i+1)} \sum_{j=1}^{k} \omega_{ij} \left\|p_j - q_j\right\|_{W_i}.$$

Consequently, we obtain

$$\|\overline{p} - \overline{q}\|_{\prod_{i=1}^{k} W_i} \leq \frac{\sum_{i=1}^{k} \frac{1}{\Gamma(\alpha_i+1)}}{1 - \sum_{i=1}^{k} \frac{N_i}{\Gamma(\alpha_i+1)} \sum_{j=1}^{k} \omega_{ij}} \upsilon =: \eta_{f_i,h_i} \upsilon.$$

Hence, (1) is UH-stable.

If we put $\psi_{f_i,h_i}(\theta) = \eta_{f_i,h_i}\theta, \psi_{f_i,h_i}(0) = 0$, then hybrid system (1) is generalized UH-stable.

5. Application

Consider the following hybrid fractional system:

$$\begin{cases} D^{\alpha_1}\left(\dfrac{q_1(t)}{\frac{|\cos q_1(\tau)|+|\sin q_2(\tau)|+|\cos q_3(\tau)|+3}{5}}\right) = \dfrac{\cos w q_1(\tau)+\sin\big(q_2(\tau)+q_3(\tau)\big)}{\pi e^{\tau}+15} + \ln(1+\tau^2), \tau \in \Phi, \\ D^{\alpha_2}\left(\dfrac{q_2(\tau)}{\frac{|\sin q_1(\tau)|+\left|\cos\big(q_2(\tau)+q_3(\tau)\big)\right|}{13}}\right) = \dfrac{e^{-\tau^2}\Big(e^{-|q_1(\tau)|}+\cos q_2(\tau)+\sin\big(2\pi q_2(\tau)\big)\Big)}{19\pi+e^{\tau^2}} + \cosh(2+\tau^2), \tau \in \Phi, \\ D^{\alpha_3}\left(\dfrac{q_3(\tau)}{\frac{|\cos q_1(\tau)|+\left|\sin\big(q_2(\tau)+q_3(\tau)\big)\right|+2}{13}}\right) = \dfrac{\sin\big(2\pi q_1(\tau)\big)}{32\pi} + \dfrac{\tan^{-1}\big(q_2(\tau)\big)+q_3(\tau)}{16} + \cos^2\tau + \tau^2 + 3, \tau \in \Phi, \\ q_i(0) = \theta_i \int_0^{\beta_i} e^{ir} q_i(r) dr, i = 1,2,3. \end{cases} \tag{36}$$

Here $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}, \alpha_3 = \frac{2}{3}, \theta_1 = 5, \theta_2 = 7, \theta_3 = \frac{3}{5}, \beta_i = \frac{4}{5}, (i = 1,2,3), \varphi_i(\tau) = e^{i\tau}, \tau \in \Phi = [0,1]$ and

$$\begin{aligned} f_1(\tau,q_1,q_2,q_3) &= \frac{|\cos q_1(\tau)|+|\sin q_2(\tau)|+|\cos q_3(\tau)|+3}{5}, \\ f_2(\tau,q_1,q_2,q_3) &= \frac{|\sin q_1(\tau)|+\left|\cos\big(q_2(\tau)+q_3(\tau)\big)\right|}{13}, \\ f_3(\tau,q_1,q_2,q_3) &= \frac{|\cos q_1(\tau)|+\left|\sin\big(q_2(\tau)+q_3(\tau)\big)\right|+2}{13}, \\ h_1(\tau,q_1,q_2,q_3) &= \frac{\cos q_1(\tau)+\sin\big(q_2(\tau)+q_3(\tau)\big)}{\pi e^{\tau}+15} + \ln(1+\tau^2), \end{aligned}$$

$$
\begin{aligned}
h_2(\tau,q_1,q_2,q_3) &= \frac{e^{-\tau^2}\left(e^{-|q_1(\tau)|}+\cos w_2(\tau)+\sin\left(2\pi q_2(\tau)\right)\right)}{19\pi+e^{\tau^2}}+\cosh(2+\tau^2),\\
h_3(\tau,q_1,q_2,q_3) &= \frac{\sin\left(2\pi q_1(\tau)\right)}{32\pi}+\frac{\tan^{-1}\left(q_2(\tau)\right)+q_3(\tau)}{16}+\cos^2\tau+\tau^2+3.
\end{aligned}
$$

For $(q_1,q_2,q_3),(p_1,p_2,p_3)\in\mathbb{R}^3$ and $\tau\in[0,1]$

$$
|f_1(\tau,q_1,q_2,q_3)|\le\frac{6}{5},\ |f_2(\tau,q_1,q_2,q_3)|\le\frac{2}{13},\ |f_3(\tau,q_1,q_2,q_3)|\le\frac{4}{13},
$$

and

$$
\begin{aligned}
|h_1(\tau,q_1,q_2,q_3)-h_1(\tau,p_1,p_2,p_3)| &\le \frac{1}{\pi+15}(|q_1-p_1|+|q_2-p_2|+|q_3-p_3|),\\
|h_2(\tau,q_1,q_2,q_3)-h_2(\tau,p_1,p_2,p_3)| &\le \frac{1}{19\pi+1}(|q_1-p_1|+|q_2-p_2|+|q_3-p_3|),\\
|h_3(\tau,q_1,q_2,q_3)-h_3(\tau,p_1,p_2,p_3)| &\le \frac{1}{16}(|q_1-p_1|+|q_2-p_2|+|q_3-p_3|).
\end{aligned}
$$

We found $\Lambda_1; 1.192\,3, \Lambda_2; 0.971\,27, \Lambda_3; 1.131\,5$ and we take $\omega_{11}=\omega_{12}=\omega_{13}=\frac{1}{\pi+15}, \omega_{21}=\omega_{23}=\omega_{23}=\frac{1}{19\pi+1}, \omega_{31}=\omega_{32}=\omega_{33}=\frac{1}{16}$.

Hence, we obtain

$$
\sum_{i=1}^{3}\left(\Lambda_i\sum_{j=1}^{3}\omega_{ij}\right); 0.457\,33<1,
$$

and

$$
\frac{\sum_{i=1}^{k}\frac{1}{\Gamma(\alpha_i+1)}}{1-\sum_{i=1}^{k}\frac{N_i}{\Gamma(\alpha_i+1)}\sum_{j=1}^{k}\omega_{ij}}=4.477\,0.
$$

Thanks to Theorem 6, the problem (36) admits a unique solution on $[0,1]$. By Theorem 11, problem (36) is UH-stable with

$$
\|\overline{p}-\overline{q}\|_{\prod_{i=1}^{3}W_i}\le 4.477\,\theta, \upsilon>0.
$$

References

[1] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, "Existence results for a system of coupled hybrid fractional differential equations," *The Scientific World Journal*, vol. 2014, pp. 1-7 (2014).

[2] A. Ali, K. Shah, and R. A. Khan, "Existence of solution to a coupled system of hybrid fractional differential equations," *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 1, pp. 9-18 (2017).

[3] J. Alzabut, M. Houas, and M. I. Abbas, "Application of fractional quantum calculus on coupled hybrid differential systems within the sequential Caputo fractional *q*-derivatives," *Demonstratio Mathematica*, vol. 56, no. 1, pp. 1-16 (2023).

[4] C. Bai and J. X. Fang, "The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 150, pp. 611-621 (2004).

[5] Z. Bai and H. Lu, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, pp. 495-505 (2005).

[6] D. Baleanu, S. Etemad, S. Pourrazi, and Sh. Rezapour"On the new fractional hybrid boundary value problems with three-point integral hybrid conditions," *Advances in Difference Equations*, Art. no. 473, pp. 1-21 (2019).

[7] T. Bashiri, S. M. Vaezpour, and C. Park, "Existence results for fractional hybrid differential systems in Banach algebras," *Advances in Difference Equations*, Art. no. 2016, pp. 1-13 (2016).

[8] S. Belarbi and Z. Dahmani, "Solvability for a nonlinear coupled system of fractional differential equations," *Matematika*, vol. 30, pp. 123-133 (2014).

[9] C. Derbaz, H. Hammouche, M. Benchohra, and Y. Zhou, "Fractional hybrid differential equations with three-point boundary hybrid conditions," *Advances in Difference Equations*, Art. no. 125, pp. 1-11 (2019).

[10] B. C. Dhage and S. K. Ntouyas, "Existence results for boundary value problems for fractional hybrid differential inclusions," *Topological Methods in Nonlinear Analysis*, vol. 44, no. 1, pp. 229-238 (2014).

[11] B. C. Dhage and V. Lakshmikantham, "Basic results on hybrid differential equations, *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 3, pp. 414-424 (2010).

[12] Z. Dahmani and S. Belarbi, "New results for fractional evolution equations using Banach fixed point theorem," *International Journal of Nonlinear Analysis and Applications*, vol. 5, pp. 22-30 (2014).

[13] D. Delbosco and L. Rodino, "Existence and uniqueness for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 204, pp. 429-440 (1996).

[14] K. Diethelm and N. J. Ford, "Analysis of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 265, pp. 229-248 (2002).

[15] A. M. A. El-Sayed, "Nonlinear functional differential equations of arbitrary orders," *Nonlinear Analysis*, vol. 33, pp. 181-186 (1998).

[16] S. Ferraoun and Z. Dahmani, "Existence and stability of solutions of a class of hybrid fractional differential equations involving RL-operator," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 4, pp. 885-903 (2020).

[17] M. A. E. Herzallah and D. Baleanu, "On fractional order hybrid differential equations," *Abstract and Applied Analysis*, vol. 2014, pp. 1-8 (2014).

[18] K. Hilal and A. Kajouni, "Boundary value problems for hybrid differential equations with fractional order," *Advances in Difference Equations*, Art. no. 183, pp. 1-19 (2015).

[19] M. Houas, "Solvability of a system of fractional hybrid differential equations," *Communications in Optimization Theory*, vol. 2018, pp. 1-9 (2018).

[20] M. Houas and A. Saadi, "Existence and uniqueness results for a coupled system of nonlinear fractional differential equations with two fractional orders," *Journal of Interdisciplinary Mathematics*, vol. 23, no. 6, pp. 1047-1064 (2020).

[21] M. Houas, "Existence results for a coupled system of fractional Caputo-Langevin equations involving two fractional orders," *Journal of Interdisciplinary Mathematics*, vol. 27, no. 1, pp. 1-16 (2024).

[22] M. Houas, "Solvability and stability for fractional differential equations involving two Riemann-Liouville fractional orders," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 8, pp. 1699-1715 (2023).

[23] M. Houas, "Existence of solutions for nonlinear Hadamard differential equations with nonlocal conditions," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 3, pp. 593-612 (2021).

[24] M. Houas and Z. Dahmani, "On existence of solutions for fractional differential equations with nonlocal multi-point boundary conditions," *Lobachevskii Journal of Mathematics*, vol. 37, pp. 120-127 (2016).

[25] A. A. Kilbas and S. A. Marzan, "Nonlinear differential equation with the Caputo fractional derivative in the space of continuously differentiable functions," *Differential Equations*, vol. 41, pp. 84-89 (2005).

[26] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis*, vol. 69, pp. 2677-2682 (2008).

[27] M. Paknazar and M. D. L. Sen, "Fractional coupled hybrid Sturm-Liouville differential equation with multi-point boundary coupled hybrid condition," *Axioms*, vol. 10, no. 2, pp. 1-26 (2021).

[28] L. Podlubny, *Fractional Differential Equations*. New York, NY, USA: Academic Press (1999).

[29] G. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Amsterdam, The Netherlands: Gordon and Breach (1993).

[30] G. Wang, W. Liu, and C. Ren, "Existence of solutions for multi-point nonlinear differential equations of fractional orders with integral boundary conditions," *Electronic Journal of Differential Equations*, no. 54, pp. 1-10 (2012).

[31] Y. Zhao, S. Sun, Z. Han, and Q. Li, "Theory of fractional hybrid differential equations," *Computers & Mathematics with Applications*, vol. 62, pp. 1312-1324 (2011).

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