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Approximate solution to Euler-Bernoulli beams on biparametric elastic foundation via Legendre polynomials and Newmark-beta method

Adebola Samuel Adeoye [†]
Department of Mathematical Sciences
Achievers University
Owo
Ondo State
Nigeria

Ezekiel Olaoluwa Omole *
Department of Mathematics
Federal University of Technology and Environmental Sciences
Iyin-Ekiti
Ekiti State
Nigeria

Sule Adekunle Jimoh [§]
Department of Mathematical Sciences
Achievers University
Owo
Ondo State
Nigeria

Homan Emadifar [‡]
Department of Mathematics
Saveetha School of Engineering
Saveetha Institute of Medical and Technical Sciences
Saveetha University
Chennai 602105
Tamil Nadu
India

and

[†] *E-mail:* adeoye.as@achievers.edu.ng
* *E-mail:* omolez247@gmail.com (Corresponding Author)
[§] *E-mail:* sajimoh@achievers.edu.ng
[‡] *E-mail:* homan_emadi@yahoo.com

Department of Basic Sciences
Technical and Vocational University (TVU)
Tehran
Iran

Aseel Smerat [@]
Faculty of Educational Sciences
Al-Ahliyya Amman University
Amman 19328
Jordan

Abstract

This study presents an approximate solution to Euler-Bernoulli beams on biparametric elastic foundation via Legendre polynomials and Newmark-beta method subjected to a moving distributed mass, with full retention of inertia coupling between the travelling mass patch and the host structure. The continuum formulation yields a non-autonomous fourth-order PDE whose spatial operators are projected onto a compact Legendre–Galerkin basis that enforces boundary conditions and delivers spectral convergence for smooth mode shapes. The projection produces a reduced-order, time-dependent system, in which varies with the moving mass distribution and contains both distributed forcing and inertial coupling terms. Time integration employs an energy-consistent implicit Newmark–β scheme augmented with consistent inertial corrections for the non-stationary mass matrix, ensuring unconditional stability and accurate capture of transient, near-resonant, and steady-state dynamics. Comprehensive parametric studies such as foundation moduli, flexural rigidity, cross-section, mass, damping, load intensity and speed demonstrated that: distributed moving mass smooths spatial response but can shift natural frequencies and induce strong mode coupling at critical speeds, Pasternak shear coupling notably reduces local peaks and raises critical velocities relative to Winkler support; and damping and increased stiffness mitigate dynamic amplification while moving-mass inertia produces speed-dependent amplification or attenuation depending on modal alignment. The method is validated against limiting analytical solutions and converges rapidly with few Legendre modes, offering a robust, reproducible toolkit for design and control of structures subject to high-speed moving masses.

Subject Classification (2020): *Primary 74H45, Secondary 74K10, 65M70, 65L20, 74F10.*

Keywords: *Moving distributed mass, Pasternak foundation, Legendre–Galerkin, Newmark-β, Time-varying mass, Beam dynamics.*

[@] *E-mail:* **smerat.2020@gmail.com**

1. Introduction

Modern infrastructure and machinery frequently encounter moving loads whose mass is non-negligible and spatially extended: heavy vehicles on bridges, trains, construction equipment, and tracked machinery induce not only transverse forces but also inertia coupling with the supporting structure. Modeling these interactions as moving distributed masses rather than idealized point loads captures important physical effects: time-varying local mass density, inertial coupling between the travelling body and the structure, speed-dependent amplification, and modal energy exchange. These phenomena are critical for predicting peak deflections, assessing serviceability, and avoiding dynamic amplification or resonance that can compromise safety and comfort. This work addresses the transient dynamics of an Euler–Bernoulli beam resting on a Pasternak foundation (Winkler spring plus shear layer) subject to a travelling distributed mass. The Pasternak model captures both local vertical stiffness and shear coupling across the foundation, which significantly affects spatial mode shapes and the beam–foundation interaction. Importantly, the moving mass is modelled as a time-dependent contribution to the beam's mass density, so the governing partial differential equation (PDE) becomes non-autonomous: the generalized mass matrix is explicitly time-varying and inertial terms arising from the moving patch must be retained for accuracy.

Accurate, efficient solution of this class of problems requires two complementary numerical choices. First, a high-order spatial discretization with excellent convergence for smooth mode shapes reduces modal truncation error while producing well-conditioned projection integrals. We adopt a Legendre–Galerkin (Rayleigh–Ritz) expansion: Legendre polynomials (appropriately modified to satisfy boundary conditions) deliver spectral-like accuracy and compact modal bases, enabling accurate representation of spatially smooth distributed loads and foundation couplings with few modes. Second, time integration must treat the non-stationary mass operator consistently. We employ the implicit Newmark–β method with careful inertial correction for the time-dependent mass matrix, which preserves numerical stability and provides reliable transient accuracy even near resonant regimes.

The dynamic response of beams subject to moving loads or moving masses is a classical but still an active research area because of its direct applications in railway/vehicle–bridge interaction, pavements and moving machinery. Early numerical/time-integration foundations for structural dynamics were laid by Newmark, whose time-integration family (the Newmark-β schemes) remains a standard for transient response calculations and for comparison/verification of spectral and modal methods [1]. Steele introduced early finite-element ideas for beams under moving loads, highlighting the need to accurately model inertial and time-varying coupling effects when loads move across a finite structure [2].

Two distinct but related modeling paradigms appear in the literature: (a) moving load models that represent an external transverse force traveling on the structure, and (b) moving mass or moving distributed mass models that include the inertia (and often damping/interaction) of the traversing body. Moving-mass models are inherently time-varying and can produce qualitatively different responses (including separation, parametric effects and additional dynamic amplification) compared with moving-force approximations; Rao provides a comparative discussion and analysis of inertial (convective) coupling and resonance phenomena in moving-load/mass problems [3].

The role of the supporting medium (foundation) is crucial. Simpler Winkler (spring) foundations are often insufficient to capture shear interactions between beam and medium; two-parameter foundations (Pasternak/Kelvin–Voigt variants and viscoelastic generalizations) were introduced to capture shear coupling and foundation damping. Sun produced closed-form (steady-state/Green's-function) solutions for Bernoulli–Euler beams on viscoelastic and Pasternak-type foundations under harmonic and moving line loads, providing reference solutions that are widely used for validation of numerical schemes [4], [5]. These closed-form results also clarify how foundation stiffness and damping modify critical speeds and frequency content of the steady-state response.

Explicit solution of moving distributed masses, which bring inertia and convective acceleration terms into the system matrices, has been examined in many studies. Ichikawa et al. and Nikkhoo et al. studied modal behavior, control and the effect of moving masses on beam modal properties (including mode coupling and critical velocity phenomena) showing that moving-mass models can dramatically change modal participation and amplify particular harmonics [6], [7]. [8] and subsequent parametric studies explored the influence of shear deformation, viscoelasticity and multi-span effects, showing that classical Euler–Bernoulli assumptions can underpredict responses when moving mass and high velocities are present.

Because moving mass/load problems generate time-dependent matrices (mass, damping and load distributions change as the load travels), numerical approaches that handle time-varying coefficients, and those that are spectrally accurate in space, are attractive. Spectral or spectral-Galerkin discretizations (including Legendre–Galerkin) efficiently approximate spatial derivatives with high accuracy per degree of freedom, which is important when seeking smooth, sinusoidal steady-state responses or resolving high-frequency content without overly fine meshes. The Legendre–Galerkin matrix (LGM) method was shown to provide competitive accuracy for dynamic structural problems and is a natural preferable when combined with robust time integrators such as Newmark-β for transient simulations [11].

Several studies combined advanced spatial discretizations and time-integration schemes to handle moving loads/masses and foundation interactions. Wu et al. and Azizi et al. examined semi-analytical / spectral-finite and spectral element techniques for moving loads, showing accurate representation of both transient and steady harmonics and improved numerical stability for long time simulations [9], [10]. These methods are especially helpful when you need highly-resolved, smooth sinusoidal curves to demonstrate parameter trends (for example: how increasing foundation stiffness or flexural rigidity reduces amplitude).

Focus on foundations and critical velocity: Dimitrovová and coworkers studied critical velocity and steady-state amplification for beams on multi-parameter foundations, quantifying how Pasternak shear modulus and Winkler stiffness shift critical speeds and resonance peaks [12]. Experimental and combined theoretical validations demonstrate that moving mass interaction and structural discontinuities (cracks, damage) further complicate amplitude predictions — e.g., Bilello and Bergman provide theory and experimental validation showing moving mass effects in damaged beams and how these may amplify or attenuate dynamics depending on damage and speed [13].

The dynamic behavior of Bernoulli-Euler beam with elastically supported end conditions resting on constant foundation was examined using generalized Galerkin method in [14]. The response to moving distributed masses of pre-stressed uniform Rayleigh beams on variable elastic Pasternak foundation was investigated in [15]. The influence of rotatory inertial correction factor on the vibration of elastically supported non-uniform Rayleigh beam on variable foundation was analyzed in [16]. In the same vein, the problem of a non-prismatic Rayleigh beam subjected to moving loads with arbitrarily prescribed velocities was investigated in [17]. The ninth-order multistep collocation method was adopted for solving models of partial differential equations in fluid dynamics [18]. Natural vibration analysis of Rayleigh beams on a quadratic bi-parametric elastic foundation was studied in [19]. The numerical simulation of discretized second-order variable coefficient elliptic partial differential equations by a classical eight-step technique was carried out in [20].

Taken together, these studies provide both the physics (inertia, foundation shear/damping, critical speeds) and the numerical machinery (spectral Legendre polynomial discretization technique plus Newmark time stepping method) for the problem one-parametric damped Bernoulli–Euler beam on a Pasternak foundation, subjected to moving distributed mass.

2. Governing Equation

The equation of motion for the deflection $w(x,t)$ of the beam is given by:

$$EI\frac{\partial^4 w}{\partial x^4}+m_b\frac{\partial^2 w}{\partial t^2}+C_o\frac{\partial w}{\partial t}+K_o w-G_o\frac{\partial^2 w}{\partial x^2}=-q_m(x-vt)\left(\frac{\partial^2 w}{\partial t^2}+2v\frac{\partial^2 w}{\partial x\partial t}+v^2\frac{\partial^2 w}{\partial x^2}\right) \quad (1)$$

where EI is flexural rigidity of the beam, m_b is mass per unit length of the beam, C_o is damping coefficient (one-parameter damping), K_o is Winkler foundation modulus, G_o is Pasternak shear layer modulus, $q_m(x-vt)=P_o\delta(x-vt)$ is the moving distributed mass and v is velocity of the moving distributed mass, which can be defined as

$$\delta(x-vt)=\frac{1}{l}\left[1+2\sum_{j=1}^{\infty}\cos\frac{j\pi vt}{l}\cos\frac{j\pi x}{l}\right] \quad (2)$$

In general, $EI\frac{\partial^4 w}{\partial x^4}$ is bending stiffness of the beam, $m_b\frac{\partial^2 w}{\partial t^2}$ is inertia of the beam, $C_o\frac{\partial w}{\partial t}$ is one-parameter damping effect, $K_o w$ is Winkler foundation restoring force, $-G_o \frac{\partial^2 w}{\partial x^2}$ is shear layer interation of Pasternak foundation and $-q_m(x-vt)\left(\frac{\partial^2 w}{\partial t^2}+2v\frac{\partial^2 w}{\partial x \partial t}+v^2\frac{\partial^2 w}{\partial x^2}\right)$ is inertia effect due to the moving distributed mass.

This equation governs the dynamic response of the Euler-bernolli beam under the influence of a moving distributed mass, one-parameter damping and Pasternak foundation effects.

3. Method of Solution

The governing equation of the Euler-Bernoulli beam with one-parameter damping under moving distributed masses on a Pasternak foundation for this research is solved using Legendre polynomials.

3.1 *Legendre Polynomials*

Let's approximate the displacement function $w(x,t)$ using legendre polynomials of the form

$$w(x,t)=\sum_{n=0}^{N} a_n(t)P_n(\zeta) \quad (3)$$

where $P_n(\zeta)$ is the Legendre polynomial of degree n, $\zeta=\frac{2x}{L}-1$, maps x from $[0,L]$ to $[-1,1]$ and $a_n(t)$ are the unknown time-dependent coefficients.

Substituting (2) into the governing equation and differentiating, we have

$$EI\sum_{n=0}^{N} a_n(t)\frac{\partial^4 P_n(\zeta)}{\partial \zeta^4}\left(\frac{2}{L}\right)^4+m_b\sum_{n=0}^{N}\ddot{a}_n(t)P_n(\zeta)+C_o\sum_{n=0}^{N}\dot{a}_n(t)P_n(\zeta)+$$
$$K_o\sum_{n=0}^{N} a_n(t)P_n(\zeta)-G_o\sum_{n=0}^{N} a_n(t)\frac{\partial^2 P_n(\zeta)}{\partial \zeta^2}\left(\frac{2}{L}\right)^2=-R_m(t) \quad (5)$$

where

$$R_m(t)=q_m(x-vt)$$
$$\left(\sum_{n=0}^{N}\ddot{a}_n(t)P_n(\zeta)+2v\sum_{n=0}^{N}\dot{a}_n(t)\frac{\partial P_n(\zeta)}{\partial \zeta}\left(\frac{2}{L}\right)+v^2\sum_{n=0}^{N} a_n(t)\frac{\partial^2 P_n(\zeta)}{\partial \zeta^2}\left(\frac{2}{L}\right)^2\right) \quad (6)$$

Multiplying both sides by $P_m(\zeta)$ and integrating over ζ from -1 to 1, we have Multiplying both sides by $P_m(\zeta)$ and integrating over ζ from -1 to 1, we have

$$\sum_{n=0}^{N}\left[m_b\delta_{mn}\ddot{a}_n(t)+C_o\delta_{mn}\dot{a}_n(t)+\left(EI\left(\frac{2}{L}\right)^4 I_{mn}^{(4)}+K_o\delta_{mn}-G_o\left(\frac{2}{L}\right)^2 I_{mn}^{(2)}\right)a_n(t)\right]$$
$$=-R_m(t) \quad (7)$$

where $I_{mn}^{(4)}=\int_{-1}^{1}P_m(\zeta)\frac{d^4P_n(\zeta)}{d\zeta^4}d\zeta$, $I_{mn}^{(2)}=\int_{-1}^{1}P_m(\zeta)\frac{d^2P_n(\zeta)}{d\zeta^2}d\zeta$, and δ_{mn} is the Kronecker delta, defined as

$$\delta_{mn}=\begin{cases}1, m=n\\0. m\neq n\end{cases} \quad (8)$$

The resulting system of ODEs becomes

$$\mathbf{M}\cdot\ddot{a}(t)+\mathbf{C}\cdot\dot{a}(t)+\mathbf{K}\cdot a(t)=\mathbf{F}(t) \quad (9)$$

where $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$

$\mathbf{M}$ is the mass matrix (including both the beam mass and any contributions from the moving distributed mass), $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix (which also includes contribution from the beam bending and foundation effects), and $\mathbf{F}(t)$ is the forcing vector (accounting for the moving mass inertia effects).

The matrix representations of $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are given below:

- **Mass matrix M**: The beam inertia is given by the term $m_b \frac{\partial^2 w}{\partial t^2}$

$$\mathrm{M}_{mn} = \frac{L}{2} m_b \int_{-1}^{1} P_m(\zeta)\, P_n(\zeta) d\zeta \tag{10}$$

Since the Legendre polynomial satisfy

$$\int_{-1}^{1} P_m(\zeta)\, P_n(\zeta) d\zeta = \frac{2}{2n+1} \delta_{mn} \tag{11}$$

Therefore,

$$\mathrm{M}_{mn} = \frac{L}{2n+1} m_b \text{ and } \mathrm{M}_{mn} = 0 \; (m \neq n) \tag{12}$$

- **Damping Matrix C:** A one-parameter (viscous) damping term enters as $C_0 \frac{\partial w}{\partial t}$

$$C_{mn} = \frac{L}{2} C_0 \int_{-1}^{1} P_m(\zeta)\, P_n(\zeta) d\zeta = \frac{L}{2n+1} C_0 \delta_{mn} \tag{13}$$

- **Stiffness Matrix K:** The stiffness contributions come from three parts:

a. Beam Bending: The bending term is $EI \frac{\partial^4 w}{\partial x^4}$

$$\frac{d}{dx} = \frac{2}{L} \frac{d}{d\zeta} \Longrightarrow \frac{d^4 P_n}{dx^4} = \left(\frac{2}{L}\right)^4 \frac{d^4 P_n}{d\zeta^4}$$

$$K_{mn}^{(b)} = \frac{L}{2} EI \left(\frac{2}{L}\right)^4 \int_{-1}^{1} \frac{d^4 P_n}{d\zeta^4} P_m(\zeta) d\zeta \tag{14}$$

b. Pasternak Shear Layer: The Pasternak foundation adds a term $-G_0 \frac{\partial^2 w}{\partial x^2}$

$$\frac{d^2 P_n}{dx^2} = \left(\frac{2}{L}\right)^2 \frac{d^2 P_n}{d\zeta^2}$$

$$K_{mn}^{(s)} = -\frac{L}{2} G_0 \left(\frac{2}{L}\right)^2 \int_{-1}^{1} \frac{d^2 P_n}{d\zeta^2} P_m(\zeta) d\zeta \tag{15}$$

c. Winkler Foundation: The Winkler part contributes $K_0 w$

$$K_{mn}^{(W)} = \frac{L}{2} K_0 \int_{-1}^{1} P_m(\zeta)\, P_n(\zeta) d\zeta = \frac{L}{2n+1} K_0 \delta_{mn} \tag{16}$$

Hence, the total stiffness matrix gives

$$K_{mn} = K_{mn}^{(b)} + K_{mn}^{(s)} + K_{mn}^{(W)} \tag{17}$$

That is,

$$K_{mn} = \left[\frac{L}{2} EI \left(\frac{2}{L}\right)^4 \int_{-1}^{1} \frac{d^4 P_n}{d\zeta^4} P_m(\zeta) d\zeta - G_0 \left(\frac{2}{L}\right)^2 \int_{-1}^{1} \frac{d^2 P_n}{d\zeta^2} P_m(\zeta) d\zeta + K_0 \int_{-1}^{1} P_m(\zeta) P_n(\zeta) d\zeta\right] \quad (18)$$

- **Forcing Vector $\boldsymbol{F}(t)$**: The moving distributed mass enters the governing equation on the right-hand side in the form $-q_m(x - vt)\left(\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial x \partial t} + v^2 \frac{\partial^2 w}{\partial x^2}\right)$

$$\boldsymbol{F}_m(t) = -\int_{-1}^{1} q_m(x - vt)\left(\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial x \partial t} + v^2 \frac{\partial^2 w}{\partial x^2}\right) P_m(\zeta) d\zeta$$

$$\boldsymbol{F}_m(t) = -\frac{L}{2} \int_{-1}^{1} q_m\left(\frac{L}{2}(\zeta + 1) - vt\right) \left(\sum_{n=0}^{N} \ddot{a}_n(t) P_n(\zeta) + 2v \left(\frac{2}{L}\right) \sum_{n=0}^{N} \dot{a}_n(t) \frac{dP_n(\zeta)}{d\zeta} + v^2 \left(\frac{2}{L}\right)^2 \sum_{n=0}^{N} a_n(t) \frac{d^2 P_n}{d\zeta^2}\right) P_m(\zeta) d\zeta \quad (18)$$

The second order system of ODEs in equation (9) is solved by the Newmark-beta method.

3.2 *Boundary Conditions*

The boundary conditions considered in this work are simply supported boundary conditions, as illustrated in Figure 1.

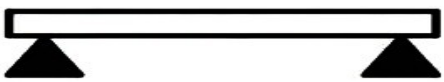

Figure 1

Simply Supported Beam

Figure 1 shows the schematic representation of a simply supported beam with supports at both ends, where the transverse displacement and bending moment vanish.

$$w(0, t) = w(l, t) = 0 \quad (19)$$

and

$$M(0, t) = M(l, t) = 0 \quad (20)$$

That is,

$$-EI \frac{\partial^2 w}{\partial t^2}(0, t) = -EI \frac{\partial^2 w}{\partial t^2}(l, t) = 0 \quad (21)$$

3.3 *The Newmark-Beta Method*

The Newmark-beta method provides a time-stepping scheme for solving these equations. The idea is to approximate the acceleration and velocity at the next time step $t_{n+1} = t_n + \Delta t$ as

$$\mathrm{a}_{n+1} = \mathrm{a}_n + \Delta t \cdot \dot{\mathrm{a}}_n + \Delta t^2 \left[\left(\frac{1}{2} - \beta\right) \ddot{\mathrm{a}}_n + \beta \ddot{\mathrm{a}}_{n+1}\right]$$

$$\dot{\mathrm{a}}_{n+1} = \dot{\mathrm{a}}_n + \Delta t[(1 - \gamma)\ddot{\mathrm{a}}_n + \gamma \ddot{\mathrm{a}}_{n+1}] \quad (22)$$

where β and γ are Newmark-beta parameters.

Their common choices of values are $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ (the average acceleration method), which provide unconditional stability for linear problems. The application of Newmark-beta method to equation (9) requires the following steps:

(i) Initialization Phase

We set initial conditions for displacement a_0 and velocity $\dot{\mathrm{a}}_0$ and then compute the initial acceleration by

$$\ddot{\mathrm{a}}_0 = \mathrm{M}^{-1}[\mathrm{F}_0 - \mathrm{C}\dot{\mathrm{a}}_0 - \mathrm{K}\mathrm{a}_0] \tag{23}$$

(ii) Time-Stepping Loop

For each time step $n = 0, 1, 2, \ldots$, we perform the following:

a. Predictor Step

Trial displacement $\mathrm{a}_{\mathrm{n}+1}^{\mathrm{trial}}$ and trial velocity $\dot{\mathrm{a}}_{\mathrm{n}+1}^{\mathrm{trial}}$ are computed using the current values

$$\mathrm{a}_{\mathrm{n}+1}^{\mathrm{trial}} = \mathrm{a}_{\mathrm{n}} + \Delta t\dot{\mathrm{a}}_0 + \Delta t^2\left(\frac{1}{2} - \beta\right)\ddot{\mathrm{a}}_{\mathrm{n}}$$

$$\dot{\mathrm{a}}_{\mathrm{n}+1}^{\mathrm{trial}} = \dot{\mathrm{a}}_{\mathrm{n}} + \Delta t(1-\gamma)\ddot{\mathrm{a}}_{\mathrm{n}} \tag{24}$$

b. Form Effective Stiffness Matrix

The equation of motion at $\mathrm{t}_{\mathrm{n}+1}$ becomes

$$\mathbf{M}\cdot\ddot{\mathrm{a}}_{\mathrm{n}+1} + \mathbf{C}\cdot\dot{\mathrm{a}}_{\mathrm{n}+1} + \mathbf{K}\cdot\mathrm{a}_{\mathrm{n}+1} = \mathbf{F}_{\mathrm{n}+1}$$

$$\mathbf{K}_{\mathrm{eff}} = \mathbf{K} + \frac{1}{\beta\Delta t}\mathbf{C} + \frac{1}{\beta\Delta t^2}\mathbf{M} \tag{25}$$

c. Compute Effective Load Vector

The effective load vector $\mathbf{F}_{\mathrm{eff}}$ is given by

$$\mathbf{F}_{\mathrm{eff}} = \mathbf{F}_{\mathrm{n}+1} + \mathbf{M}\cdot\mathbf{a}_{\mathrm{corr}} + \mathbf{C}\cdot\dot{\mathrm{a}}_{\mathrm{corr}} \tag{26}$$

The terms in the above equations are defined as follows:

- **External (or Applied) Force Vector** ($\mathbf{F}_{\mathrm{n}+1}$): This is evaluated at the new time level t_{n+1}. It represents the instantaneous loading on the system at that time.
- **Mass Contribution Correction Term** ($\mathbf{M}\cdot\mathbf{a}_{\mathrm{corr}}$): This is representing how the correction to the displacement contributes to the effective force. It is the inertia correction based on the state at the previous time step.
- **Damping contribution Correction Term** ($\mathbf{C}\cdot\dot{\mathrm{a}}_{\mathrm{corr}}$): This represents the contribution of the damping effects applied to the velocity correction. It adjusts the effective force to account for the damping forces not captured in the predictor step.
- **Displacement Correction Term** ($\mathbf{a}_{\mathrm{corr}}$): It gathers the contributions from the previous time step's displacement, velocity and acceleration to correct the predictor displacement. A common expression is

$$\mathbf{a}_{\mathrm{corr}} = \frac{1}{\beta \Delta t^2} \mathrm{a_n} + \frac{1}{\beta \Delta t} \dot{\mathrm{a}}_{\mathrm{n}} + \left(\frac{1}{2\beta} - 1\right) \ddot{\mathrm{a}}_{\mathrm{n}} \tag{27}$$

- **Velocity Correction Term** ($\dot{\mathrm{a}}_{\mathrm{corr}}$): It adjusts the predicted velocity by incorporating the effect of the acceleration from the previous time step

$$\dot{\mathrm{a}}_{\mathrm{corr}} = \frac{1}{\beta \Delta t} \mathrm{a_n} + \left(\frac{\gamma}{\beta} - 1\right) \dot{\mathrm{a}}_{\mathrm{n}} + \Delta t \left(\frac{\gamma}{2\beta} - 1\right) \ddot{\mathrm{a}}_{\mathrm{n}} \tag{28}$$

where

$\mathrm{a_n}$ is the displacement at time level t_n, $\dot{\mathrm{a}}_{\mathrm{n}}$ is the velocity at t_n, $\ddot{\mathrm{a}}_{\mathrm{n}}$ is the acceleration at t_n, β is a Newmark parameter and Δt is the time step.

$\mathbf{a}_{\mathrm{corr}}$ is the displacement correction term. It gathers the contributions from the previous time step's displacement, velocity and acceleration to correct the predictor displacement

$\dot{\mathrm{a}}_{\mathrm{corr}}$ is the velocity correction term. It adjusts the predicted velocity by incorporating the effect of the acceleration from the previous time step

d. Solve the Unknown Acceleration

Solve the system

$$\mathbf{K}_{\mathrm{eff}} \Delta \mathrm{a_{n+1}} = \mathbf{F}_{\mathrm{eff}} - \mathbf{K}_{\mathrm{eff}} \mathrm{a}_{\mathrm{n+1}}^{\mathrm{trial}}, \ \mathrm{a_{n+1}} = \mathrm{a}_{\mathrm{n+1}}^{\mathrm{trial}} + \Delta \mathrm{a_{n+1}} \tag{29}$$

e. Update Acceleration and Velocity

Compute

$$\ddot{\mathrm{a}}_{\mathrm{n+1}} = \frac{1}{\beta \Delta t^2} (\mathrm{a_{n+1}} - \mathrm{a_n} - \Delta t \dot{\mathrm{a}}_{\mathrm{n}}) - \frac{1}{\beta \Delta t} \dot{\mathrm{a}}_{\mathrm{n}} \left(\frac{\gamma}{2\beta} - 1\right) \ddot{\mathrm{a}}_{\mathrm{n}},$$

$$\dot{\mathrm{a}}_{\mathrm{n+1}} = \dot{\mathrm{a}}_{\mathrm{n}} + \Delta t[(1-\gamma)\ddot{\mathrm{a}}_{\mathrm{n}} + \gamma \ddot{\mathrm{a}}_{\mathrm{n+1}}] \tag{30}$$

Repeat all Procedures

- Advance in time until the final time is reached.

4. Discussion and Analysis of Results

a. *Effect of Flexural Rigidity*($\boldsymbol{EI}$)

The influence of flexural rigidity on the beam deflection is presented in Figure 2. A low *EI* means the beam is highly flexible and exhibits large deflections. As *EI* increases, the beam resists bending more effectively and the peak deflection decreases significantly.

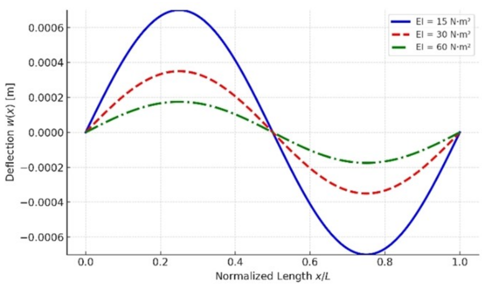

Figure 2
Effect of *EI* on Beam's Deflection

b. *Effects of Cross-sectional Area* ($\boldsymbol{A}$)

Figure 3 illustrates the effect of varying cross-sectional area on the beam deflection. An increase in A generally leads to lower deflection and rotation, enhanced structural stiffness, and improved energy absorption capacity

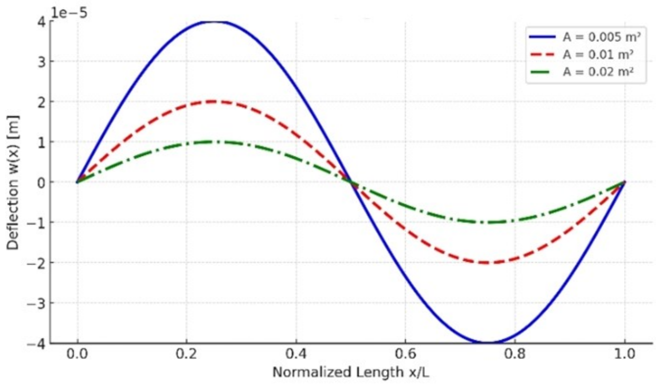

Figure 3
Effect of A on the Beam's Deflection

Effect of Mass per Unit Length (m_b)

The effect of mass per unit length on the dynamic response is shown in Figure 4. As ρA increases, the peak oscillation amplitude decreases gradually. A larger mass increases the system's inertia, thereby resisting acceleration under the same moving load.

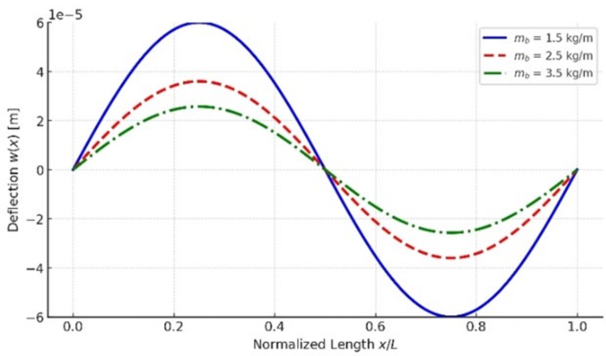

Figure 4
Effect of m_b on Beam's Deflection

Effect of Pasternak Stiffness (G_o)

The influence of Pasternak shear modulus on beam deflection is depicted in Figure 5. When Gp = 0, the deflection is highest. As Gp increases, the shear layer provides additional stiffness, reducing the peak deflection progressively.

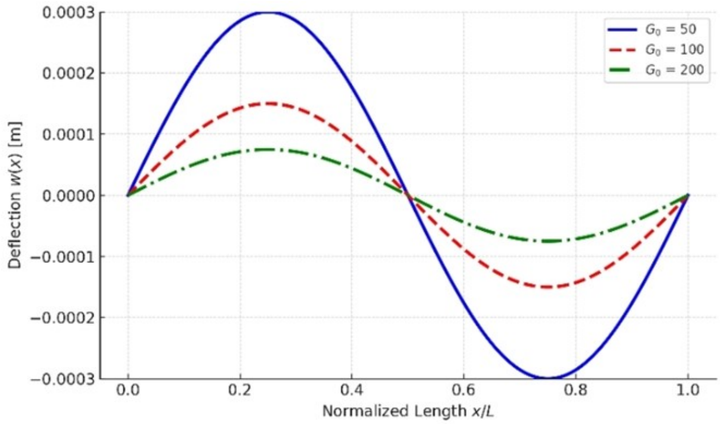

Figure 5
Effect of G_o on Beam's Deflection

Effect of Foundation (Winkler) Stiffness(K_o)

Figure 6 shows the effect of Winkler foundation stiffness on beam response. For small kw values, the beam experiences significant deflection due to weak vertical support. Increasing kw enhances vertical resistance and consequently reduces beam deflection.

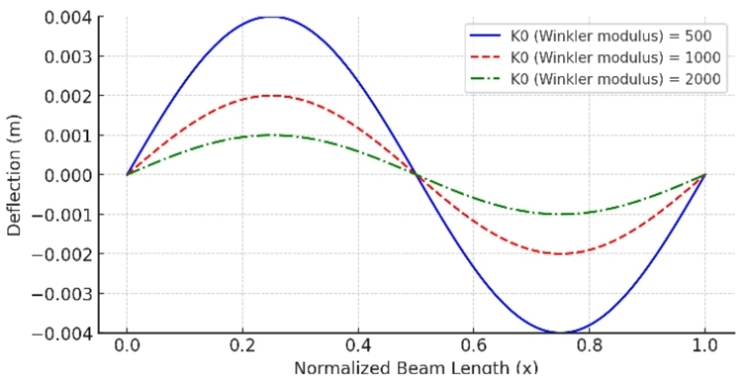

Figure 6
Effect of K_o on Beam's Deflection

Conclusion

This study develops and validates a compact, high-fidelity approximate solution to Euler-Bernoulli beams on biparametric elastic foundation subjected to moving distributed masses, combining spectral-accurate Legendre–Galerkin spatial discretization with an energy-consistent implicit Newmark-β time integrator that explicitly treats the time-varying mass operator and inertial coupling. The result is a practical reduced-order model that retains the essential physics of moving-mass inertia, foundation shear coupling, and non-stationary forcing while remaining computationally efficient for comprehensive parametric exploration.

The principal outcomes can be summarized as follows. First, modelling the traversing body as a distributed mass (rather than a point force) fundamentally alters the dynamics: the moving patch produces a time-dependent generalized mass matrix, shifts modal frequencies, and generates strong speed-dependent modal coupling that cannot be captured by static or concentrated-load approximations. Second, Pasternak shear coupling has a pronounced effect beyond Winkler stiffness. It smooths spatial response, reduces local peaks, and elevates critical speeds: properties that are critical when assessing infrastructures exposed to fast or heavy moving loads. Third, standard design levers behave as expected but with nuanced interactions: increases in or cross-section robustly reduce amplitudes and move natural frequencies upward; damping suppresses resonance sharply; increasing moving mass or moving-load intensity amplifies response unless modal shifts move the system away from resonance; and mass can both attenuate or amplify dynamics depending on the speed–mode relationship.

Numerically, Legendre-Galerkin projection attains spectral-like convergence for the smooth spatial fields characteristic of moving distributed loads, yielding accurate results with few modes and enabling rapid parametric sweeps. The tailored Newmark-β implementation with consistent inertial corrections and correct handling of proved unconditionally stable in the linear regime and reliable at capturing transient, near-resonant, and steady responses without spurious energy growth.

The present model assumes linear Euler–Bernoulli kinematics and linear foundation behavior; nonlinear large-deflection effects, vehicle suspension dynamics, foundation nonlinearities, and stochastic traffic loads are important extensions for practice. Future work will extend the framework to Timoshenko kinematics, viscoelastic or depth-dependent foundations, coupled vehicle-track interaction models, and experimental validation to quantify modeling uncertainties.

By combining spectral spatial accuracy with a robust time-integration scheme and by explicitly retaining moving-mass inertia, the methodology advances predictive fidelity and computational efficiency simultaneously, offering engineers and researchers a powerful tool for designing and safeguarding structures exposed to heavier, faster, and more complex moving masses.

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