Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 7, pp. 2217–2232
DOI : 10.47974/JIM-2386

An extended numerical method for solving a version of fractional Zakharov-Kuznetsov equation

Ahmed Anber *
Department of Mathematics
University of Science and Technology of Oran
Oran 31000
Algeria

and

Laboratory of Discrete and Applied Mathematics and Operations Research
University of Blida 1
Algeria

Zoubir Dahmani †
Laboratory of Discrete and Applied Mathematics and Operations Research
University of Blida 1
09000 Blida
Algeria

Abstract

In this paper, an extension in a general x-space for the (G'/G)-method is used to find new traveling wave solutions for a version of the time-space fractional Zakharov Kuznetsov equation (FZKE, for short). Several cases are discussed to illustrate the efficiency of proposed method.

Subject Classification: *26A33, 35C07.*

Keywords: *Fractional derivative, Zakharov Kuznetsov equation, (G'/G)-expansion method, Traveling wave solution.*

* *ORCID-ID:* **0009-0004-9769-9467**
† *ORCID-ID:* **0000-0003-4659-0723**

* *E-mail:* **ah.anber@gmail.com (Corresponding Author)**
† *E-mail:* **zzdahmani@yaoo.fr**

1. Introduction

The Zakharov-Kuznetsov equation is a fundamental mathematical model that describes various physical phenomena, such as Rossby waves in rotating atmospheres and drift vortices in three-dimensional plasmas [16]. The classical ZK equation is given by:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\Delta u) = 0,$$

where Δ represents the Laplacian operator.

In recent years, the FZKE equation has attracted significant attention due to its ability to model complex systems with memory effects and non-local interactions [3, 15, 6]. Fractional derivatives, particularly the Caputo fractional derivative [14], provide a robust mathematical framework for analyzing such phenomena.

The primary objective of this work is to apply the (G'/G)-expansion method to obtain new traveling wave solutions for the FZKE equation, which is expressed as:

$$D_t^{\alpha}u + a_1 u^{\gamma} D_{x_1}^{\beta_1}u + a_2 D_{x_1}^{3\beta_1}u + a_3 D_{x_1}^{\beta_1}\left(D_{x_2}^{2\beta_2}u + D_{x_3}^{2\beta_3}u\right) + a_4 D_{x_1}^{5\beta_1}u = 0,$$

where $0 < \alpha, \beta_i \leq 1$, D^{β_i} denotes the Caputo fractional derivative, γ is a positive constant, and a_i are real constants.

When $\alpha = \beta_i = \gamma = 1$, this equation is reduced to the well known ZKE equation:

$$u_t + a_1 u u_x + a_2 u_{xxx} + a_3(u_{xyy} + u_{xzz}) + a_4 u_{xxxxx} = 0.$$

Significant progress has been done in solving the FZKE equation using various methods, such as the modified Adomian decomposition method [15], the undetermined coefficients method [9], Lie symmetry method [8], the δ-homotopy perturbation transform method and q-homotopy analysis transform method [1], the multiple exp-function method [11], and the projected differential transform method [10], there remains a need for more efficient and versatile analytical techniques. The (G'/G)-expansion method has proven to be a powerful tool for solving nonlinear partial differential equations [2, 4, 5], but its application to the fractional ZK equation has not been fully explored.

Several recent studies have addressed new aspects of the FZKE equation; For instance, Alam and Tunc [3] investigated optical soliton solutions for a conformable version of the equation, while Rashid, Kubra and Guirao [15] proposed approximate analytical solutions using the Adomian decomposition method. Other studies, such as those by Ganie, Mofarreh, and Khan [6] and Mohammed et al. [12], explored solutions for stochastic and quantum versions of the equation. However, these methods often face limitations in terms of computational complexity or generality. This work aims to fill the gap of these limitations by extending the (G'/G)-expansion method to the FZKE equation and deriving new exact solutions. The proposed method is not only computationally efficient but also provides deeper insights into the wave dynamics governed by the FZKE equation.

2. Preliminaries

In this part, we present some basic definitions and properties, of Caputo derivatives, that will be used later. Fore more details, see [14, 13].

Definition 1: The RL-integral operator of order $\lambda \geq 0$, for a continuous function on $[0,a], a > 0$ is defined by

$$\begin{aligned} J^{\lambda}f(s) &= \frac{1}{\Gamma(\lambda)}\int_0^s (s-\xi)^{\lambda-1}f(\xi)d\xi, \lambda > 0, s > 0, \\ J^0 f(s) &= f(s). \end{aligned} \tag{2.1}$$

Definition 2: The Caputo derivative $f \in C^m$ on $[0,a]$ in given by

$$D^{\lambda}f(s) = \begin{cases} \frac{1}{\Gamma(m-\lambda)}\int_0^s (s-\xi)^{m-\lambda-1}f^{(m)}(\xi)d\xi, m-1 < \lambda < m, m \in \mathbb{N}^*, \\ \frac{d^m}{ds^m}f(s),\ \lambda = m. \end{cases} \tag{2.2}$$

We need the properties:

$$J^{\lambda_1}J^{\lambda_2}f(s) = J^{\lambda_1+\lambda_2}f(s) = J^{\lambda_2}J^{\lambda_1}f(s) \tag{2.3}$$

$$D^{\lambda_1}K = 0, J^{\lambda_1}(K) = \frac{K}{\Gamma(\lambda_1+1)}s^{\lambda_1} \quad K \text{ is a constant,} \tag{2.4}$$

$$D^{\lambda_1}s^{\lambda_2} = \begin{cases} \frac{\Gamma(\lambda_2+1)}{\Gamma(\lambda_2-\lambda_1+1)}s^{\lambda_2-\lambda_1}, \lambda_2 > \lambda_1 - 1 \\ 0, \lambda_2 \leq \lambda_1 - 1 \end{cases} \tag{2.5}$$

and

$$J^{\lambda_1}\left(s^{\lambda_2}\right) = \frac{\Gamma(\lambda_2+1)}{\Gamma(\lambda_1+\lambda_2+1)}s^{\lambda_1+\lambda_2} \tag{2.6}$$

3. Description of the (G'/G)-expansion method

In this section, we present the main steps of the proposed extension for the (G'/G)-expansion method. Fore more details on this important approach, see [2, 4, 5]

We first consider the equation:

$$F\left(u, D_t^{\beta}u, D_{x_i}^{\alpha_i}u, D_{x_i}^{\alpha_i}(D_{x_j}^{\alpha_j}u), D_{x_i}^{2\alpha_i}u, D_{x_i}^{3\alpha_i}u, \dots\right) = 0, i, j \in \{1, \dots, n\}, \tag{3.1}$$

where u is an unknown function of $x_1, x_2, \dots, x_n$ and t, F is a polynomial of u and its partial fractional derivatives and $D_t^{\beta}u, D_{x_i}^{\alpha_i}u$ $(i = 1,2,\dots,n)$ are partial fractional derivatives of u, with $0 < \beta, \alpha_i \leq 1$.

To aply the (G'/G)-method for the above equation, we use the transformation:

$$U(\xi) = u(t, x_1, x_2, \dots x_n), \xi = \frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)} + \dots + \frac{k_nx_n^{\alpha_n}}{\Gamma(1+\alpha_n)} - \frac{ct^{\beta}}{\Gamma(1+\beta)}, \tag{3.2}$$

such k_i and c are constants.

Thanks to [7], Corollary 4.1, we can write

$$D_x^{\alpha}(f(u(x))) = f_u' D_x^{\alpha}(u).$$

So (3.1) can be converted to the following ODE:

$$H(U, U', U'', U''', \ldots) = 0, \tag{3.3}$$

where, H is a function of U and its derivatives.

The main steps of the proposed method are given as follows:

Step one: We put for the solutions of (3.3) like:

$$U(\xi) = \sum_{k=0}^{m} A_k \left(\frac{G'(\xi)}{G(\xi)}\right)^k + \sum_{k=1}^{m} B_k \left(\frac{G'(\xi)}{G(\xi)}\right)^{-k} \tag{3.4}$$

where, A_k, B_k are constants to be determined later, m is a positive integer to be determined, and G is a solution of the linear second order differential equation

$$G'' + \lambda G' + \mu G = 0, \mu \in \mathbb{R}, \tag{3.5}$$

where λ and μ are constants determined as well.

The general solution of (3.5) is a combination of $sinh$ and $cosh$ or of sin and cos functions if $\Delta = \lambda^2 - 4\mu > 0$ or $\Delta = \lambda^2 - 4\mu < 0$, respectively.

In the present work, we consider only the first case and so,

$$G(\xi) = e^{-\frac{\lambda\xi}{2}}\left(\Theta_1 \sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + \Theta_2 \cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right), \tag{3.6}$$

where $\Theta1, \Theta2$ are arbitrary constants.

So, we can find

$$\frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{\Delta}}{2}\left(\frac{\Theta_1 \sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + \Theta_2 \cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{\Theta_1 \cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + \Theta_2 \sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right)}\right) - \frac{\lambda}{2} \tag{3.7}$$

Step two: Determining m, we balancing concept using (3.3).

Step three: We replace (3.4) and (3.5) in (3.3) and using Maple, we find λ, μ, A_k and B_k. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$ the solutions of (3.5) are well known. So, we obtain exact solutions of (3.1).

Step four: We replace the obtained values into (3.4), we find the solutions of (3.1).

4. Applications

In this part, we apply the proposed method to search wave solutions for the FZKE equation.

We begin by considering the following equation

$$D_t^{\beta} u + a_1 (u)^{\gamma} D_{x_1}^{\alpha_1} u + a_2 D_{x_1}^{3\alpha_1} u + a_3 D_{x_1}^{\alpha_1}\left(D_{x_2}^{2\alpha_2} u + D_{x_3}^{2\alpha_3} u\right) + a_4 D_{x_1}^{5\alpha_1} u = 0 \tag{4.1}$$

We consider

$$u(t, x_1, x_2, x_3) = U(\xi), \xi = \frac{k_1 x_1^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{k_2 x_2^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{k_3 x_3^{\alpha_3}}{\Gamma(1+\alpha_3)} - \frac{c t^{\beta}}{\Gamma(1+\beta)}, \tag{4.2}$$

so, we have

$$
\begin{aligned}
&D_t^{\beta} u(t,x_1,x_2,x_3) = -cU_\xi \\
&D_{x_1}^{\alpha_1} u(t,x_1,x_2,x_3) = k_1 U_\xi \\
&D_{x_2}^{2\alpha_2} u(t,x_1,x_2,x_3) = (k_2)^2 U_{\xi\xi} \\
&D_{x_3}^{2\alpha_3} u(t,x_1,x_2,x_3) = (k_3)^2 U_{\xi\xi}
\end{aligned} \tag{4.3}
$$

Substituting (4.3) in (4.1), we get

$$-cU_\xi + a_1 k_1 (U)^\gamma U_\xi + \left(a_2(k_1)^3 + a_3 k_1((k_2)^2 + (k_3)^2)\right)U_{\xi\xi\xi} + a_4(k_1)^5 U_{\xi\xi\xi\xi\xi} = 0,$$

so, it yields

$$-cU + \underbrace{\frac{1}{\gamma+1} a_1 k_1}_{K_1} U^{\gamma+1} + \underbrace{\left(a_2(k_1)^3 + a_3 k_1((k_2)^2 + (k_3)^2)\right)}_{K_2} U_{\xi\xi} + \underbrace{a_4(k_1)^5}_{K_3} U_{\xi\xi\xi\xi} = 0. \tag{4.4}$$

where

$$K_1 = \frac{1}{\gamma+1} a_1 k_1, K_2 = a_2(k_1)^3 + a_3 k_1((k_2)^2 + (k_3)^2)$$
$$K_3 = a_4(k_1)^4$$

When $a_4 \neq 0$, balancing the nonlinear term $U_{\xi\xi\xi\xi}$, that has exponent $m+4$, with the highest order derivative term $U^{\gamma+1}$, that has the exponent $(\gamma+1)m$, this gives $m = \frac{4}{\gamma}$. To obtain closed-form solutions, m must be an integer, therefore, we choose $\gamma = 1,2$ and 4.

In the same way, when $a_4 = 0$ and ($a_3 \neq 0$ or $a_2 \neq 0$), we obtain $m = \frac{2}{\gamma}$ and we choose $\gamma = 1$ and 2.

4.1 *Solutions for* $a_4 \neq 0$ *and* $\gamma = 4$

For $\gamma = 4$, we have $m = 1$, then the (G'/G)-expansion method admits the use of the finite expansion

$$U(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right) + B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1}. \tag{4.5}$$

Substituting (4.5) into (4.4), setting coefficients of to zero, we obtain the following under determined system of algebraic equations:

$$
\begin{aligned}
&-cA_0 + (4A_1^2B_1^2A_0^5 + 25A_0A_1^2B_1^2 + 20A_0^3A_1B_1)K_1 + (\lambda\mu A_1 + \lambda B_1)K_2 \\
&+(\lambda^3\mu A_1 + 8\lambda\mu^2 A_1 + B_1\lambda^3 + 8\mu\lambda B_1)K_3 \\
&+\begin{pmatrix} (4A_0^4A_1 + 30A_1^2A_0^2B_1 + 4A_1^3B_1^2A_0^4 + 6A_1^3B_1^2)K_1 \\ +(\lambda^2A_1 + 2A_1\mu)K_2 + (\lambda^4A_1 + 22\lambda^2\mu A_1 + 16\mu^2A_1)K_3 - cA_1 \end{pmatrix}\left(\frac{G'}{G}\right)
\end{aligned}
$$

$$+\left((10A_0^3A_1^2 + 20A_0A_1^3B_1)K_1 + 3K_2\lambda A_1 + (15\lambda^3 A_1 + 60\lambda\mu A_1)K_3\right)\left(\frac{G'}{G}\right)^2$$
$$+\left((10A_0^2A_1^3 + 5A_1^4B_1)K_1 + 2K_2A_1 + (50\lambda^2 A_1 + 40\mu A_1)K_3\right)\left(\frac{G'}{G}\right)^3$$
$$+(5K_1A_0A_1^4 + 60K_3\lambda A_1)\left(\frac{G'}{G}\right)^4 + (K_1A_1^5 + 24K_3A_1)\left(\frac{G'}{G}\right)^5$$
$$+\begin{pmatrix}(4A_0^4B_1 + 30A_0^2A_1B_1^2 + 6A_1^2B_1^3 + 4A_1^2B_1^3A_0^4)K_1 + (\lambda B_1\lambda^2 + 2\mu B_1)K_2\\ +(B_1\lambda^4 + 22\mu\lambda^2B_1 + 16B_1\mu^2)K_3 - cB_1\end{pmatrix}\left(\frac{G'}{G}\right)^{-1}$$
$$+((10A_0^3B_1^2 + 20A_0A_1B_1^3)K_1 + (2\mu B_1\lambda + \lambda B_1\mu)K_2$$
$$+(15B_1\lambda^3\mu + 60\mu^2\lambda B_1)K_3)\left(\frac{G'}{G}\right)^{-2}$$
$$+\left((10A_0^2B_1^3 + 5A_1B_1^4)K_1 + 2K_2B_1\mu^2 + (50B_1\mu^2\lambda^2 + 40B_1\mu^3)K_3\right)\left(\frac{G'}{G}\right)^{-3}$$
$$+(5K_1B_1^4A_0 + 60K_3B_1\mu^3\lambda)\left(\frac{G'}{G}\right)^{-4} + (K_1B_1^5 + 24K_3B_1\mu^4)\left(\frac{G'}{G}\right)^{-5} = 0.$$

This will allow us to write

$$-cA_0 + (4A_1^2B_1^2A_0^5 + 25A_0A_1^2B_1^2 + 20A_0^3A_1B_1)K_1 + (\lambda\mu A_1 + \lambda B_1)K_2$$
$$+(\lambda^3\mu A_1 + 8\lambda\mu^2A_1 + B_1\lambda^3 + 8\mu\lambda B_1)K_3 = 0$$
$$(4A_0^4A_1 + 30A_1^2A_0^2B_1 + 4A_1^3B_1^2A_0^4 + 6A_1^3B_1^2)K_1 + (\lambda^2A_1 + 2A_1\mu)K_2$$
$$+(\lambda^4A_1 + 22\lambda^2\mu A_1 + 16\mu^2A_1)K_3 - cA_1 = 0$$
$$(10A_0^3A_1^2 + 20A_0A_1^3B_1)K_1 + 3K_2\lambda A_1 + (15\lambda^3A_1 + 60\lambda\mu A_1)K_3 = 0$$
$$(10A_0^2A_1^3 + 5A_1^4B_1)K_1 + 2K_2A_1 + (50\lambda^2A_1 + 40\mu A_1)K_3 = 0$$
$$5K_1A_0A_1^4 + 60K_3\lambda A_1 = 0$$
$$K_1A_1^5 + 24K_3A_1 = 0$$
$$(4A_0^4B_1 + 30A_0^2A_1B_1^2 + 6A_1^2B_1^3 + 4A_1^2B_1^3A_0^4)K_1 + (\lambda B_1\lambda^2 + 2\mu B_1)K_2$$
$$+(B_1\lambda^4 + 22\mu\lambda^2B_1 + 16B_1\mu^2)K_3 - cB_1 = 0$$
$$(10A_0^3B_1^2 + 20A_0A_1B_1^3)K_1 + (2\mu B_1\lambda + \lambda B_1\mu)K_2$$
$$+(15B_1\lambda^3\mu + 60\mu^2\lambda B_1)K_3 = 0$$
$$(10A_0^2B_1^3 + 5A_1B_1^4)K_1 + 2K_2B_1\mu^2 + (50B_1\mu^2\lambda^2 + 40B_1\mu^3)K_3 = 0$$
$$5K_1B_1^4A_0 + 60K_3B_1\mu^3\lambda = 0$$
$$K_1B_1^5 + 24K_3B_1\mu^4 = 0$$

Solving this system by Maple, we obtain

$$A_0 = \frac{\sqrt{15}}{3000c_1^3a_4c\sqrt{c_1a_4}}\begin{pmatrix}9a_2^2c_1^4 + 18a_2a_3c_1^2c_3^2 + 18a_2a_3c_1^2c_2^2 + 9a_3^2c_3^4\\ +18a_3^2c_2^2c_3^2 + 9a_3^2c_2^4 + 150ca_4c_1^2\end{pmatrix}^{\frac{1}{4}}$$
$$\begin{pmatrix}-12a_2 * c_1^2 - 12a_3c_3^2 - 12a_3c_2^2\\ +4\sqrt{\begin{matrix}18a_2a_3c_1^2c_3^2 + 18a_2a_3c_1^2c_2^2 + 9a_3^2c_3^4\\ +18a_3^2c_2^2c_3^2 + 9a_3^2c_2^4 + 150ca_4c_1^2\end{matrix}}\end{pmatrix}\left(-\frac{120a_4(c_1)^3}{a_1}\right)^{\frac{1}{4}}$$

$$\left(\frac{1}{2}a_2c_1^2+\frac{1}{2}a_3c_2^2+\frac{1}{2}a_3c_3^2+\frac{1}{6}\sqrt{\begin{matrix}18a_2a_3c_1^2c_3^2+18a_2a_3c_1^2c_2^2+9a_3^2c_3^4+18a_3^2c_2^2c_3^2\\+9a_3^2c_2^4+150ca_4c_1^2\end{matrix}}\right)$$

$$A_1=\left(-\frac{120a_4(c_1)^3}{a_1}\right)^{\frac{1}{4}}, B_1=0,$$

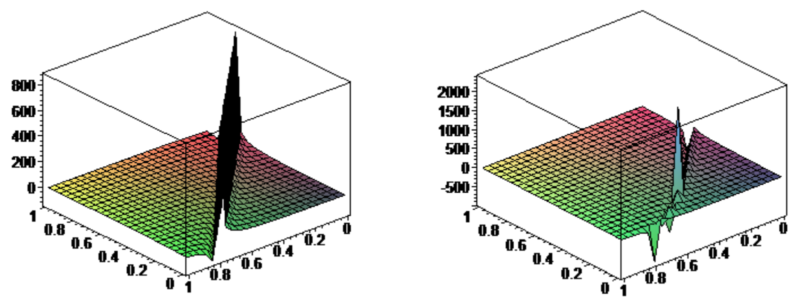
$$\lambda=\frac{1}{c_1\sqrt{15c_1a_4}}\begin{pmatrix}9a_2^2c_1^4+18a_2a_3c_1^2c_3^2+18a_2a_3c_1^2c_2^2+9a_3^2c_3^4+18a_3^2c_2^2c_3^2\\+9a_3^2c_2^4+150ca_4c_1^2\end{pmatrix}^{\frac{1}{4}},$$

$$\mu=\frac{1}{240a_4c_1^3}\begin{pmatrix}-12a_2c_1^2-12a_3c_3^2-12a_3c_2^2\\+4\sqrt{\begin{matrix}18a_2a_3c_1^2c_3^2+18a_2a_3c_1^2c_2^2+9a_3^2c_3^4+18a_3^2c_2^2c_3^2\\+9a_3^2c_2^4+150ca_4c_1^2\end{matrix}}\end{pmatrix},$$

Consequently, we obtain the traveling wave solutions

$$u(t,x_1,x_2,x_3)=A_0+A_1\left(-\frac{\lambda}{2}+\frac{\sqrt{\Delta}}{2}\left(\frac{\begin{matrix}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{matrix}}{\begin{matrix}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{matrix}}\right)\right) \tag{4.6}$$

Figure 1 represents the solution (4.6) of Eq. (4.1) when $x_3=x_2=0, c=1, k_1=-1, k_2=1, k_3=1, a_1=-\frac{1}{4}, a_2=-\frac{1}{2}, a_3=5, a_4=-1, \Theta_1=1$ and $\Theta_2=2$.



(a) (b)

Figure 1
3D plot of the solution (4.6) with $x_3=x_2=0$
in (a): $\alpha_1=\beta=1$ and in (b): $\alpha_1=0.98, \beta=0,75$

4.2 *Solutions for* $a_4 = 0, a_2 \neq 0, a_3 \neq 0$ *and* $\gamma = 2$

For $a_4 = 0, a_2 \neq 0, a_3 \neq 0$ and $\gamma = 2$, we have $m = 1$, then the (G'/G)-expansion method admits the use of the finite expansion

$$U(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right) + B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1}. \tag{4.7}$$

Putting (4.7) in equation (4.4),we have

$$\begin{aligned}
&-cA_0 + K_1A_0^3 + 6K_1A_0A_1B_1 + \lambda\mu K_2A_1 + \lambda K_2B_1\\
&(-cA_1 + (3A_0^2A_1 + 3A_1^2B_1)K_1 + (\lambda^2A_1 + 2\mu A_1)K_2)\left(\frac{G'}{G}\right)\\
&+(3K_1A_0A_1^2 + 3\lambda K_2A_1)\left(\frac{G'}{G}\right)^2 + (K_1A_1^3 + 2K_2A_1)\left(\frac{G'}{G}\right)^3\\
&+(-cB_1 + (3A_0^2B_1 + 3A_1B_1^2)K_1 + (\lambda^2B_1 + 2\mu B_1)K_2)\left(\frac{G'}{G}\right)^{-1}\\
&+(3K_1A_0B_1^2 + 3\lambda\mu K_2B_1)\left(\frac{G'}{G}\right)^{-2} + (K_1B_1^3 + 2\mu^2K_2B_1)\left(\frac{G'}{G}\right)^{-3} = 0
\end{aligned}$$

This allows us to write

$$\begin{aligned}
&-cA_0 + K_1A_0^3 + 6K_1A_0A_1B_1 + \lambda\mu K_2A_1 + \lambda K_2B_1 = 0\\
&-cA_1 + (3A_0^2A_1 + 3A_1^2B_1)K_1 + (\lambda^2A_1 + 2\mu A_1)K_2 = 0\\
&3K_1A_0A_1^2 + 3\lambda K_2A_1 = 0\\
&K_1A_1^3 + 2K_2A_1 = 0\\
&-cB_1 + (3A_0^2B_1 + 3A_1B_1^2)K_1 + (\lambda^2B_1 + 2\mu B_1)K_2 = 0\\
&3K_1A_0B_1^2 + 3\lambda\mu K_2B_1 = 0\\
&K_1B_1^3 + 2\mu^2K_2B_1 = 0
\end{aligned}$$

Solving this system by Maple, we obtain the following solutions.

Case 1:

$$\lambda = \lambda, \mu = \mu, A_0 = \sqrt{\frac{5c}{a_1c_1}}, A_1 = B_1 = 0,$$

$$u(t, x_1, x_2, x_3) = \sqrt{\frac{5c}{a_1c_1}} \tag{4.8}$$

Case 2:

$$\lambda = 0, \mu = \frac{c}{2c_1(a_2c_1^2 + a_3c_2^2 + a_3c_3^2)}, A_0 = 0,$$

$$A_1 = \sqrt{-\frac{10}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2)}, B_1 = 0,$$

$$u(t,x_1,x_2,x_3)=\frac{\sqrt{\Delta}}{2}A_1\left(\frac{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right) \tag{4.9}$$

Case 3:

$$\begin{aligned}&\lambda=\sqrt{\frac{-2c}{a_2c_1^3+a_3c_1c_2^2+a_3c_1c_3^2}},\mu=0,A_0=\frac{-5(a_2c_1^2+a_3c_2^2+a_3c_3^2)}{a_1},\\&A_1=\sqrt{-\frac{10}{a_1}(a_2c_1^2+a_3c_2^2+a_3c_3^2)},B_1=0,\end{aligned} \tag{4.10}$$

$$u(t,x_1,x_2,x_3)=A_0-\frac{\lambda}{2}A_1+\frac{\sqrt{\Delta}}{2}A_1\left(\frac{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right)$$

Case 4:

$$\lambda=0,\mu=\frac{c}{2c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)},A_0=A_1=0,B_1=\frac{c}{c_1}\sqrt{\frac{-5}{2a_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}},$$

$$u(t,x_1,x_2,x_3)=\frac{2B_1}{\sqrt{\Delta}}\left(\frac{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right) \tag{4.11}$$

Case 5:

$$\lambda=\sqrt{\frac{4\mu c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)-2c}{c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}},\mu=\mu,A_0=\frac{-5\lambda(a_2c_1^2+a_3c_2^2+a_3c_3^2)}{a_1\sqrt{-\frac{10}{a_1}(a_2c_1^2+a_3c_2^2+a_3c_3^2)}},$$

$$A_1 = 0, B_1 = \mu\sqrt{-\frac{10}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2)},$$

$$u(t,x_1,x_2,x_3) = A_0 + B_1\left(\frac{\sqrt{\Delta}}{2}\left(\frac{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right)-\frac{\lambda}{2}\right)^{-1} \tag{4.12}$$

Case 6:

$$\lambda = 0, \mu = \frac{-c}{4c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, A_0 = 0, A_1 = \sqrt{-\frac{10}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2)},$$

$$B_1 = \frac{5c}{2a_1c_1\sqrt{-\frac{10}{a_1}(a_2c_1^2+a_3c_2^2+a_3c_3^2)}},$$

$$\begin{aligned}u(t,x_1,x_2,x_3) = {}& \frac{A_1\sqrt{\Delta}}{2}\left(\frac{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right)\\ &+\frac{2B_1}{\sqrt{\Delta}}\left(\frac{\begin{array}{c}\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}{\begin{array}{c}\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\\+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)\end{array}}\right)\end{aligned} \tag{4.13}$$

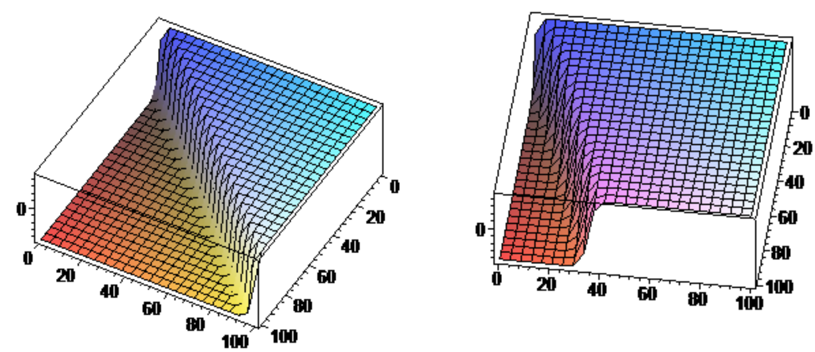
Case 7:

$$\lambda = 0, \mu = \frac{c}{8c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, A_0 = 0, A_1 = \sqrt{-\frac{10}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2)},$$

$$B_1=\frac{5c}{4a_1c_1\sqrt{-\frac{10}{a_1}\left(a_2c_1^2+a_3c_2^2+a_3c_3^2\right)}},$$

$$\begin{aligned}u(t,x_1,x_2,x_3)=\frac{A_1\sqrt{\Delta}}{2}&\left(\frac{\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)}{\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)}\right)\\
&+\frac{2B_1}{\sqrt{\Delta}}\left(\frac{\Theta_1\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)+\Theta_2\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)}{\Theta_1\sinh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)+\Theta_2\cosh\left(\frac{\sqrt{\Delta}}{2}\left(\frac{k_1x_1^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{k_2x_2^{\alpha_2}}{\Gamma(1+\alpha_2)}+\frac{k_3x_3^{\alpha_3}}{\Gamma(1+\alpha_3)}-\frac{ct^{\beta}}{\Gamma(1+\beta)}\right)\right)}\right)\end{aligned}\tag{4.14}$$

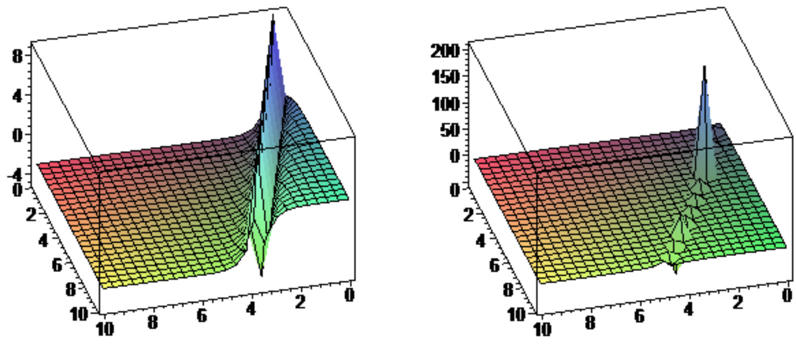
Figure 2 concerns the solution (4.9) of Eq. (4.1) when $x_1=x_2=0, c=-1, k_1=1, k_2=1, k_3=1, a_1=-1, a_2=1, a_3=1, a_4=0, \Theta_1=2$ and $\Theta_2=1$.



(a) (b)

Figure 2
3D plot of the solution (4.9) with $x_1=x_2=0$
in (a): $\alpha_1=\beta=1$ and in (b): $\alpha_1=0.98, \beta=0,75$

Figure 3 is the solution (4.13) of Eq. (4.1) when $x_1=x_2=0, c=3, k_1=1, k_2=1, k_3=1, a_1=-1, a_2=1, a_3=5, a_4=0, \Theta_1=2$ and $\Theta_2=1$.



(a) (b)

Figure 3

3D plot of the solution (4.13) with $x_1 = x_2 = 0$
in (a): $\alpha_1 = \beta = 1$ and in (b): $\alpha_1 = 0.98, \beta = 0,75$

4.3 *Solutions for* $a_4 = 0, a_2 \neq 0, a_3 \neq 0$ *and* $\gamma = 1$

For $a_4 = 0, a_2 \neq 0, a_3 \neq 0$ and $\gamma = 1$, we have $m = 2$, then the (G'/G)-expansion method admits the use of the finite expansion

$$U(\xi) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right) + A_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 + B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} + B_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2} \qquad (4.15)$$

Putting (4.15) in equation (4.4),we have

$$\begin{gathered}
-cA_0 + (A_0^2 + 2A_1B_1 + 2A_2B_2)K_1 \\
+(-cA_1 + (2A_0A_1 + 2A_2B_1)K_1 - (\lambda A_1 + 2A_2\mu)K_2)\left(\frac{G'}{G}\right) \\
+(-cA_2 + (A_1^2 + 2A_0A_2)K_1 - (A_1 + 2A_2\lambda)K_2)\left(\frac{G'}{G}\right)^2 \\
+(2A_1A_2K_1 - 2A_2K_2)\left(\frac{G'}{G}\right)^3 + A_2^2K_1\left(\frac{G'}{G}\right)^4 \\
+(-cB_1 + (2A_0B_1 + 2A_1B_2)K_1 + (\lambda B_1 + 2B_2)K_2)\left(\frac{G'}{G}\right)^{-1} \\
+(-cB_2 + (2A_0B_2 + B_1^2)K_1 + (\mu B_1 + 2B_2\lambda)K_2)\left(\frac{G'}{G}\right)^{-2} \\
+(2B_1B_2K_1 + 2B_2\mu K_2)\left(\frac{G'}{G}\right)^{-3} + B_2^2K_1\left(\frac{G'}{G}\right)^{-4} = 0
\end{gathered}$$

$$\begin{gathered}
-cA_0 + (A_0^2 + 2A_1B_1 + 2A_2B_2)K_1 = 0 \\
-cA_1 + (2A_0A_1 + 2A_2B_1)K_1 - (\lambda A_1 + 2A_2\mu)K_2 = 0 \\
-cA_2 + (A_1^2 + 2A_0A_2)K_1 - (A_1 + 2A_2\lambda)K_2 = 0 \\
+2A_1A_2K_1 - 2A_2K_2 = 0 \\
A_2^2K_1 = 0 \\
-cB_1 + (2A_0B_1 + 2A_1B_2)K_1 + (\lambda B_1 + 2B_2)K_2 = 0 \\
-cB_2 + (2A_0B_2 + B_1^2)K_1 + (\mu B_1 + 2B_2\lambda)K_2 = 0
\end{gathered}$$

$$2B_1B_2K_1 + 2B_2\mu K_2 = 0$$
$$B_2^2K_1 = 0$$

Solving this system by Maple, we obtain the following solutions.

Case 1:

$$\lambda = \lambda, \mu = \mu, A_0 = \frac{5c}{a_1c_1}, A_1 = A_2 = B_1 = B_2 = 0,$$
$$u(t, x_1, x_2, x_3) = \frac{5c}{a_1c_1} \tag{4.16}$$

Case 2:

$$\lambda = \frac{c}{c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, \mu = \mu, A_0 = A_1 = A_2 = 0, B_1$$
$$= -\frac{5}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2), B_2 = 0,$$
$$u(t, x_1, x_2, x_3) = B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} \tag{4.17}$$

Case 3:

$$\lambda = \frac{-c}{c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, \mu = \mu, A_0 = \frac{5c}{a_1c_1}, A_1 = A_2 = 0,$$
$$B_1 = -\frac{5}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2), B_2 = 0,$$
$$u(t, x_1, x_2, x_3) = A_0 + B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} \tag{4.17}$$

Case 4:

$$\lambda = \frac{-c}{c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, \mu = \mu, A_0 = A_2 = 0,$$
$$A_1 = \frac{5}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2), B_1 = B_2 = 0,$$
$$u(t, x_1, x_2, x_3) = A_1\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{4.18}$$

Case 5:

$$\lambda = \frac{c}{c_1(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, \mu = \mu, A_0 = A_2 = 0,$$
$$A_1 = \frac{5}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2), B_1 = B_2 = 0,$$
$$u(t, x_1, x_2, x_3) = A_1\left(\frac{G'(\xi)}{G(\xi)}\right) \tag{4.19}$$

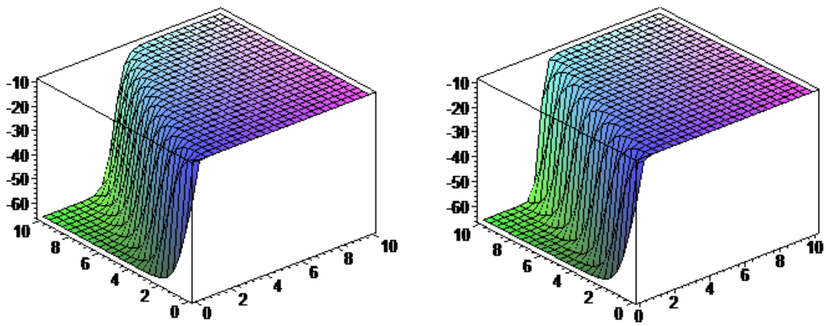
Case 6:

$$\lambda = 0, \mu = \frac{-c^2}{8c_1^2(a_2^2c_1^4+2a_2a_3c_1^2c_2^2++2a_2a_3c_1^2c_3^2+a_3^2c_2^4+2a_3^2c_2^2c_3^2+a_3^2c_3^4)},$$
$$A_0 = \frac{5c}{2a_1c_1}, A_1 = \frac{5}{a_1}(a_2c_1^2 + a_3c_2^2 + a_3c_3^2),$$

$$A_2 = 0, B_1 = \frac{5c^2}{8a_1c_1^2(a_2c_1^2+a_3c_2^2+a_3c_3^2)}, B_2 = 0,$$

$$u(t,x_1,x_2,x_3) = A_0 + A_1\left(\frac{G'(\xi)}{G(\xi)}\right) + B_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1} \tag{4.20}$$

Figure 4 presents the solution (4.17) of Eq. (4.1) when $x_2 = x_3 = 0, c = 15, k_1 = 1, k_2 = 1, k_3 = 1, a_1 = -1, a_2 = 1, a_3 = 2, a_4 = 0, \Theta_1 = 2$ and $\Theta_2 = 1$.

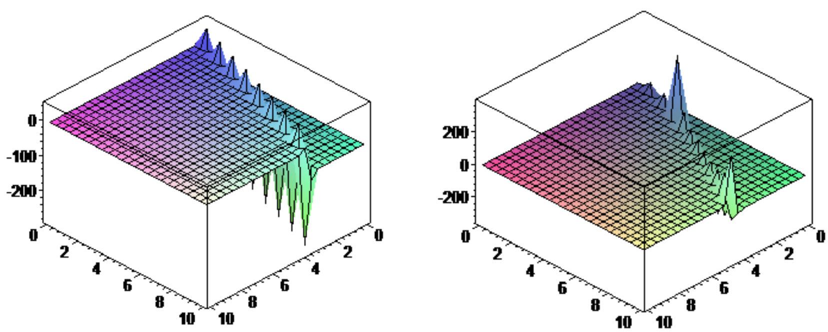


(a) (b)

Figure 4

3D plot of the solution (4.17) with $x_2 = x_3 = 0$
in (a): $\alpha_1 = \beta = 1$ and in (b): $\alpha_1 = 0.98, \beta = 0,75$

Figure 5 presents the solution (4.21) of Eq. (4.1) when $x_2 = x_3 = 0, c = 3, k_1 = 1, k_2 = 1, k_3 = 1, a_1 = -1, a_2 = 1, a_3 = 2, a_4 = 0, \Theta_1 = 2$ and $\Theta_2 = 1$.



(a) (b)

Figure 5

3D plot of the solution (4.21) with $x_2 = x_3 = 0$
in (a): $\alpha_1 = \beta = 1$ and in (b): $\alpha_1 = 0.98, \beta = 0,75$

5. Conclusion

The (G'/G)-expansion method was used to solve the FZKE equation. By applying this method, we have, in fact, derived new traveling wave solutions, which enrich the set of known solutions for this important nonlinear fractional equation. The proposed extension demonstrates its efficiency to handle fractional differential equations, providing a systematic approach to obtaining exact solutions. Several cases have been presented to illustrate the applicability and effectiveness of the method, showcasing its potential for solving other nonlinear fractional models in mathematical physics. The results obtained highlight the broader utility of the (G'/G)-expansion method in the study of fractional dynamical systems.

Future work could explore further generalizations of this method and its application to other complex nonlinear fractional equations.

References

[1] L. Akinyemi, M. Senol, and S. N. Huseen, "Modified homotopy methods for generalized fractional perturbed Zakharov–Kuznetsov equation in dusty plasma," *Advances in Difference Equations*, vol. 2021, no. 1, p. 45 (2021).

[2] A. Anber and Z. Dahmani, "Two numerical methods for solving conformable fractional KGE equation," *Journal of Interdisciplinary Mathematics,* vol. 27, no. 6, pp. 1361–1370 (2024).

[3] M. N. Alam and C. Tunc, "Constructions of the optical solitons and other solitons to the conformable fractional Zakharov–Kuznetsov equation with power law nonlinearity," *Journal of Taibah University for Science*, vol. 14, no. 1, pp. 94–100 (2020).

[4] M. Al Ghabshi, E. V. Krishnan, and M. Alquran, "Exact solutions of a Klein–Gordon system by (G′/G)-expansion method and Weierstrass elliptic function method," *Nonlinear Dynamics and Systems Theory,* vol. 19, no. 3, pp. 386–395 (2019).

[5] M. Alquran and A. Qawasmeh, "Soliton solutions of shallow water wave equations by means of (G′/G)-expansion method," *Journal of Applied Analysis and Computation*, vol. 4, no. 3, pp. 221–229 (2014).

[6] A. H. Ganie, F. Mofarreh, and A. Khan, "A fractional analysis of Zakharov–Kuznetsov equations with the Liouville–Caputo operator," *Axioms*, vol. 12, no. 6, p. 609 (2023).

[7] G. Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann–Liouville derivative for non-differentiable functions," *Applied Mathematics Letters*, vol. 22, pp. 378–385 (2009).

[8] M. Irshad, M. Hussan, A. Haider, H. M. Younas, Z. Ali, and I. Jahangir, "Some new invariant solutions of nonlinear fifth order partial

differential equation via Lie approach," *Partial Differential Equations in Applied Mathematics*, vol. 13, p. 101010 (2025).

[9] Q. Y. Jin, T. Xia, and J. Wang, "The exact solution of the space-time fractional modified KdV–Zakharov–Kuznetsov equation," *Journal of Applied Mathematics and Physics*, vol. 5, no. 4, pp. 844–852 (2017).

[10] D. Lu, M. Suleman, J. Ul Rahman, S. Noeiaghdam, and G. Murtaza, "Numerical simulation of fractional Zakharov–Kuznetsov equation for description of temporal discontinuity using projected differential transform method," *Complexity*, vol. 2021, no. 1, Art. no. 9998610 (2021).

[11] E. M. Eskandari, N. Taghizadeh, and M. Nazhand Fomani, "Multiple soliton solutions of the space-time fractional Zakharov–Kuznetsov equation by the multiple exp-function," ESS Open Archive (2022).

[12] W. W. Mohammed, F. M. Al-Askar, C. Cesarano, and M. El-Morshedy, "Solitary wave solutions of the fractional-stochastic quantum Zakharov–Kuznetsov equation arising in quantum magneto plasma," *Mathematics*, vol. 11, no. 2, p. 488 (2023).

[13] T. A. Nofal, K. A. Gepreel, and M. H. Farag, "Analytic solutions for nonlinear partial fractional differential equations," *International Journal of Scientific and Engineering Research*, vol. 5, no. 12, p. 1519 (2014).

[14] I. Podlubny, Fractional Differential Equations. San Diego, CA, USA: Academic Press (1999).

[15] S. Rashid, K. T. Kubra, and J. L. G. Guirao, "Construction of an approximate analytical solution for multi-dimensional fractional Zakharov–Kuznetsov equation via Aboodh Adomian decomposition method," *Symmetry*, vol. 13, no. 8, p. 1542 (2021).

[16] V. E. Zakharov and E. A. Kuznetsov, "On three-dimensional solitons," *Soviet Physics JETP*, vol. 39, pp. 285–288 (1974).

Received March, 2025